

## 3.5 Exponential and Logarithmic Models

### Introduction

There are many examples of exponential and logarithmic models in real life. In Section 3.1, you used the formula

$$A = Pe^{rt} \quad \text{Exponential model}$$

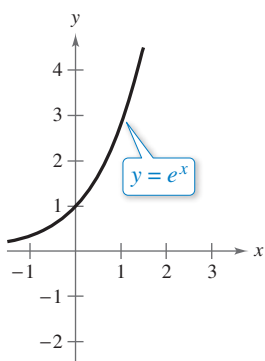
to find the balance in an account when the interest was compounded continuously. In Section 3.2, Example 10, you used the human memory model

$$f(t) = 75 - 6 \ln(t + 1). \quad \text{Logarithmic model}$$

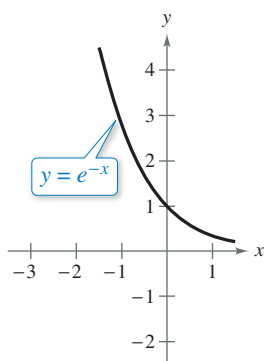
The five most common types of mathematical models involving exponential functions or logarithmic functions are listed below.

- Exponential growth model:**  $y = ae^{bx}$ ,  $b > 0$
- Exponential decay model:**  $y = ae^{-bx}$ ,  $b > 0$
- Gaussian model:**  $y = ae^{-(x-b)^2/c}$
- Logistic growth model:**  $y = \frac{a}{1 + be^{-rx}}$
- Logarithmic models:**  $y = a + b \ln x$ ,  $y = a + b \log_{10} x$

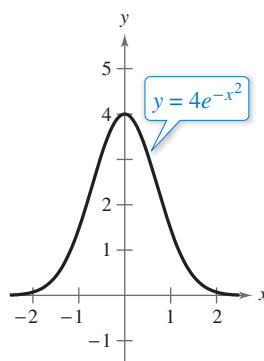
The basic shapes of the graphs of these models are shown in Figure 3.24.



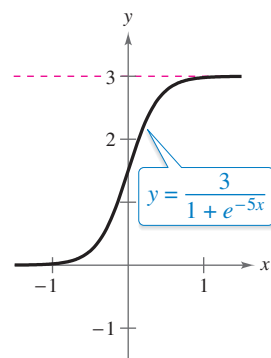
Exponential Growth Model



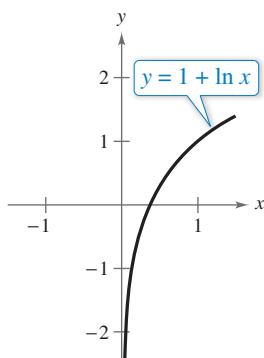
Exponential Decay Model



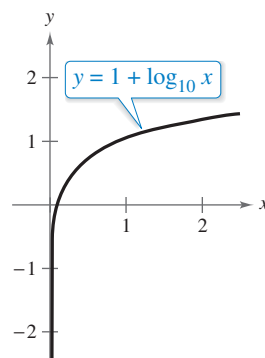
Gaussian Model



Logistic Growth Model



Natural Logarithmic Model



Common Logarithmic Model

Figure 3.24

You can often gain quite a bit of insight into a situation modeled by an exponential or logarithmic function by identifying and interpreting the function's asymptotes.

### What you should learn

- ▶ Recognize the five most common types of models involving exponential or logarithmic functions.
- ▶ Use exponential growth and decay functions to model and solve real-life problems.
- ▶ Use Gaussian functions to model and solve real-life problems.
- ▶ Use logistic growth functions to model and solve real-life problems.
- ▶ Use logarithmic functions to model and solve real-life problems.

### Why you should learn it

Exponential decay models are used in carbon dating. For instance, in Exercise 39 on page 230, you will use an exponential decay model to estimate the age of a piece of ancient charcoal.



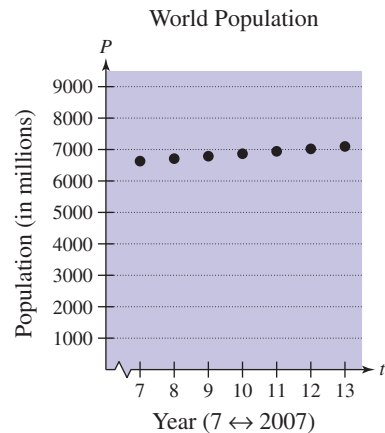
## Exponential Growth and Decay

### EXAMPLE 1 Demography

See *LarsonPrecalculus.com* for an interactive version of this type of example.

The table shows estimates of the world population (in millions) from 2007 through 2013. A scatter plot of the data is shown in the figure at the right. (Source: U.S. Census Bureau)

Year	Population, $P$
2007	6631
2008	6710
2009	6788
2010	6866
2011	6943
2012	7021
2013	7098



An exponential growth model that approximates these data is given by

$$P = 6128e^{0.0113t}$$

where  $P$  is the population (in millions) and  $t = 7$  represents 2007. Compare the values given by the model with the estimates shown in the table. According to this model, when will the world population reach 7.6 billion?

#### Algebraic Solution

The following table compares the two sets of population figures.

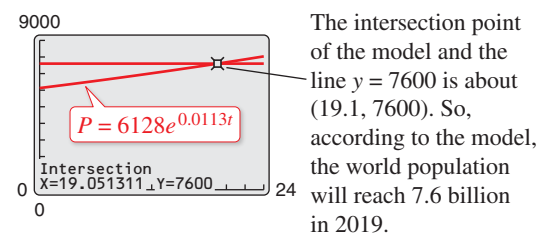
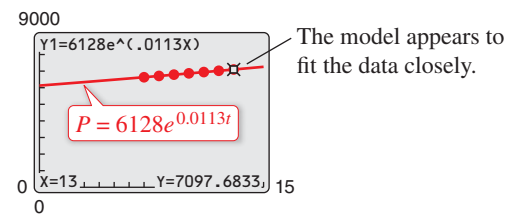
Year	2007	2008	2009	2010	2011	2012	2013
Population	6631	6710	6788	6866	6943	7021	7098
Model	6632	6708	6784	6861	6939	7018	7098

From the table, it appears that the model is a good fit for the data. To find when the world population will reach 7.6 billion, let  $P = 7600$  in the model and solve for  $t$ .

$$\begin{aligned}
 6128e^{0.0113t} &= P && \text{Write original equation.} \\
 6128e^{0.0113t} &= 7600 && \text{Substitute 7600 for } P. \\
 e^{0.0113t} &\approx 1.24021 && \text{Divide each side by 6128.} \\
 \ln e^{0.0113t} &\approx \ln 1.24021 && \text{Take natural log of each side.} \\
 0.0113t &\approx 0.21528 && \text{Inverse Property} \\
 t &\approx 19.1 && \text{Divide each side by 0.0113.}
 \end{aligned}$$

According to the model, the world population will reach 7.6 billion in 2019.

#### Graphical Solution



**✓ Checkpoint** Audio-video solution in English & Spanish at *LarsonPrecalculus.com*.

In Example 1, when will the world population reach 9 billion?

An exponential model increases (or decreases) by the same percent each year. What is the annual percent increase for the model in Example 1?

In Example 1, you were given the exponential growth model. Sometimes you must find such a model. One technique for doing this is shown in Example 2.

### EXAMPLE 2 Modeling Population Growth

In a research experiment, a population of fruit flies is increasing according to the law of exponential growth. After 2 days there are 100 flies, and after 4 days there are 300 flies. How many flies will there be after 5 days?

#### Solution

Let  $y$  be the number of flies at time  $t$  (in days). From the given information, you know that  $y = 100$  when  $t = 2$  and  $y = 300$  when  $t = 4$ . Substituting this information into the model  $y = ae^{bt}$  produces

$$100 = ae^{2b} \quad \text{and} \quad 300 = ae^{4b}.$$

To solve for  $b$ , solve for  $a$  in the first equation.

$$100 = ae^{2b} \quad \Rightarrow \quad a = \frac{100}{e^{2b}} \quad \text{Solve for } a \text{ in the first equation.}$$

Then substitute the result into the second equation.

$$300 = ae^{4b} \quad \text{Write second equation.}$$

$$300 = \left(\frac{100}{e^{2b}}\right)e^{4b} \quad \text{Substitute } \frac{100}{e^{2b}} \text{ for } a.$$

$$300 = 100e^{2b} \quad \text{Simplify.}$$

$$3 = e^{2b} \quad \text{Divide each side by 100.}$$

$$\ln 3 = \ln e^{2b} \quad \text{Take natural log of each side.}$$

$$\ln 3 = 2b \quad \text{Inverse Property}$$

$$\frac{1}{2} \ln 3 = b \quad \text{Solve for } b.$$

Using  $b = \frac{1}{2} \ln 3$  and the equation you found for  $a$ , you can determine that

$$a = \frac{100}{e^{2[(1/2)\ln 3]}} \quad \text{Substitute } \frac{1}{2} \ln 3 \text{ for } b.$$

$$= \frac{100}{e^{\ln 3}} \quad \text{Simplify.}$$

$$= \frac{100}{3} \quad \text{Inverse Property}$$

$$\approx 33.33. \quad \text{Simplify.}$$


So, with  $a \approx 33.33$  and  $b = \frac{1}{2} \ln 3 \approx 0.5493$ , the exponential growth model is

$$y = 33.33e^{0.5493t}$$

as shown in Figure 3.25. After 5 days, the population will be

$$y = 33.33e^{0.5493(5)} \approx 520 \text{ flies.}$$

**✓ Checkpoint**  Audio-video solution in English & Spanish at [LarsonPrecalculus.com](http://LarsonPrecalculus.com).

The number of bacteria in a culture is increasing according to the law of exponential growth. After 1 hour there are 100 bacteria, and after 2 hours there are 200 bacteria. How many bacteria will there be after 3 hours? 



Entomologist

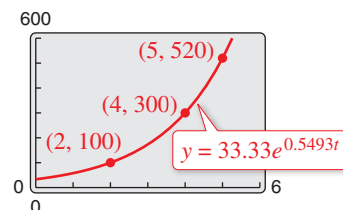


Figure 3.25

In living organic material, the ratio of the content of radioactive carbon isotopes (carbon 14) to the content of nonradioactive carbon isotopes (carbon 12) is about 1 to  $10^{12}$ . When organic material dies, its carbon 12 content remains fixed, whereas its radioactive carbon 14 begins to decay with a half-life of about 5700 years. To estimate the age of dead organic material, scientists use the following formula, which denotes the ratio of carbon 14 to carbon 12 present at any time  $t$  (in years).

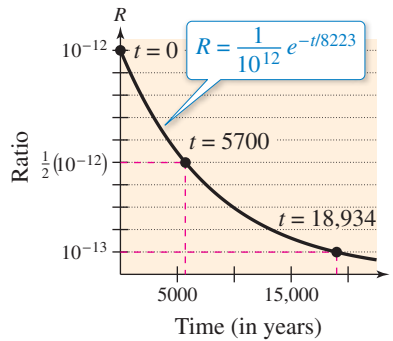


Figure 3.26

$$R = \frac{1}{10^{12}}e^{-t/8223} \quad \text{Carbon dating model}$$

The graph of  $R$  is shown in Figure 3.26. Note that  $R$  decreases as  $t$  increases.

### EXAMPLE 3 Carbon Dating

The ratio of carbon 14 to carbon 12 in a newly discovered fossil is

$$R = \frac{1}{10^{13}}$$

Estimate the age of the fossil.

#### Algebraic Solution

In the carbon dating model, substitute the given value of  $R$  and solve for  $t$ .

$$\frac{1}{10^{12}}e^{-t/8223} = R \quad \text{Write original model.}$$

$$\frac{e^{-t/8223}}{10^{12}} = \frac{1}{10^{13}} \quad \text{Substitute } \frac{1}{10^{13}} \text{ for } R.$$

$$e^{-t/8223} = \frac{1}{10} \quad \text{Multiply each side by } 10^{12}.$$

$$\ln e^{-t/8223} = \ln \frac{1}{10} \quad \text{Take natural log of each side.}$$

$$-\frac{t}{8223} \approx -2.3026 \quad \text{Inverse Property}$$

$$t \approx 18,934 \quad \text{Multiply each side by } -8223.$$

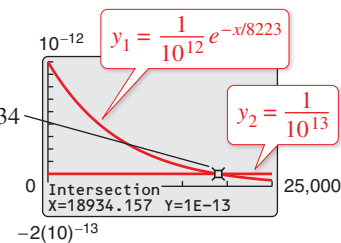
So, to the nearest thousand years, you can estimate the age of the fossil to be 19,000 years.

#### Graphical Solution

Use a graphing utility to graph the formula for the ratio of carbon 14 to carbon 12 at any time  $t$  as

$$y_1 = \frac{1}{10^{12}}e^{-x/8223}$$

In the same viewing window, graph  $y_2 = 1/(10^{13})$ , as shown in the figure.



Use the *intersect* feature to estimate that  $x \approx 18,934$  when  $y = 1/(10^{13})$ .

So, to the nearest thousand years, you can estimate the age of the fossil to be 19,000 years.

**✓ Checkpoint** Audio-video solution in English & Spanish at [LarsonPrecalculus.com](http://LarsonPrecalculus.com).

Estimate the age of a newly discovered fossil for which the ratio of carbon 14 to carbon 12 is  $R = 1/10^{14}$ .

The carbon dating model in Example 3 assumed that the carbon 14 to carbon 12 ratio was one part in 10,000,000,000,000. Suppose an error in measurement occurred and the actual ratio was only one part in 8,000,000,000,000. The fossil age corresponding to the actual ratio would then be approximately 17,000 years. Try checking this result.

## Gaussian Models

As mentioned at the beginning of this section, Gaussian models are of the form

$$y = ae^{-(x-b)^2/c}$$

This type of model is commonly used in probability and statistics to represent populations that are **normally distributed**. For *standard* normal distributions, the model takes the form

$$y = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

The graph of a Gaussian model is called a **bell-shaped curve**. Try graphing the standard normal distribution curve with a graphing utility. Can you see why it is called a bell-shaped curve?

The average value for a population can be found from the bell-shaped curve by observing where the maximum  $y$ -value of the function occurs. The  $x$ -value corresponding to the maximum  $y$ -value of the function represents the average value of the independent variable—in this case,  $x$ .

### EXAMPLE 4 SAT Scores

In 2013, the Scholastic Aptitude Test (SAT) mathematics scores for college-bound seniors roughly followed the normal distribution

$$y = 0.0034e^{-(x-514)^2/27,848}, \quad 200 \leq x \leq 800$$

where  $x$  is the SAT score for mathematics. Use a graphing utility to graph this function and estimate the average SAT mathematics score. (Source: College Board)

#### Solution

The graph of the function is shown in Figure 3.27. On this bell-shaped curve, the maximum value of the curve represents the average score. Using the *maximum* feature of the graphing utility, you can see that the average mathematics score for college-bound seniors in 2013 was 514.

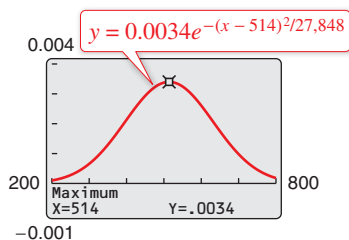


Figure 3.27

**✓ Checkpoint**  Audio-video solution in English & Spanish at [LarsonPrecalculus.com](http://LarsonPrecalculus.com).

In 2013, the Scholastic Aptitude Test (SAT) critical reading scores for college-bound seniors roughly followed the normal distribution

$$y = 0.0035e^{-(x-496)^2/26,450}$$

where  $x$  is the SAT score for critical reading. Use a graphing utility to graph this function and estimate the average SAT critical reading score. (Source: College Board)

In Example 4, note that 50% of the seniors who took the test received scores lower than 514 (see Figure 3.28).

#### Additional Example

Radioactive iodine is a by-product of some types of nuclear reactors. Its half-life is about 60 days. That is, after 60 days, a given amount of radioactive iodine will have decayed to half the original amount. Suppose a contained nuclear accident occurs and gives off an initial amount  $C$  of radioactive iodine.

- Write an exponential decay model for the amount of radioactive iodine present at any time  $t$  following the accident.
- How long will it take for the radioactive iodine to decay to 20% of the original amount?

#### Solution

- Knowing that half the original amount remains after 60 days, you can use the exponential decay model  $y = ae^{-bt}$  to obtain

$$\frac{1}{2}C = Ce^{-b(60)}$$

$$\frac{1}{2} = e^{-60b}$$

$$-\ln 2 = -60b$$

$$b = \frac{\ln 2}{60} \approx 0.0116.$$

So, the exponential model is  $y = Ce^{-0.0116t}$ .

- The time required for the radioactive iodine to decay to 20% of the original amount is

$$Ce^{-0.0116t} = (0.2)C$$

$$e^{-0.0116t} = 0.2$$

$$-0.0116t = \ln 0.2$$

$$t = \frac{\ln 0.2}{-0.0116} \approx 139 \text{ days.}$$

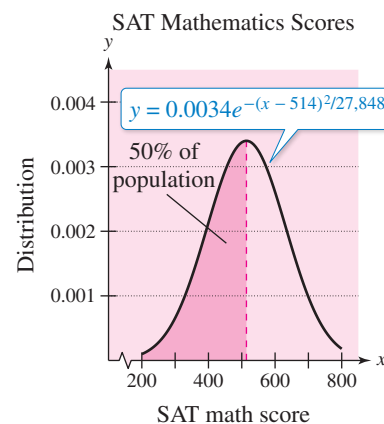


Figure 3.28

## Logistic Growth Models

Some populations initially have rapid growth, followed by a declining rate of growth, as indicated by the graph in Figure 3.29. One model for describing this type of growth pattern is the **logistic curve** given by the function

$$y = \frac{a}{1 + be^{-rx}}$$

where  $y$  is the population size and  $x$  is the time. An example is a bacteria culture that is initially allowed to grow under ideal conditions, and then under less favorable conditions that inhibit growth. A logistic growth curve is also called a **sigmoidal curve**.

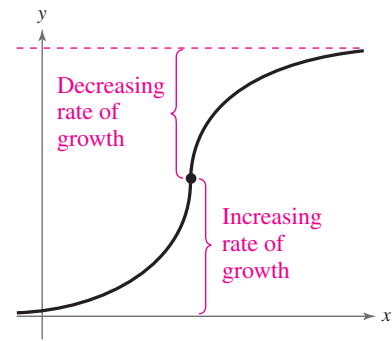


Figure 3.29 Logistic Curve

### EXAMPLE 5 Spread of a Virus

On a college campus of 5000 students, one student returns from vacation with a contagious flu virus. The spread of the virus is modeled by

$$y = \frac{5000}{1 + 4999e^{-0.8t}}, \quad t \geq 0$$

where  $y$  is the total number of students infected after  $t$  days. The college will cancel classes when 40% or more of the students are infected.

- How many students are infected after 5 days?
- After how many days will the college cancel classes?

#### Algebraic Solution

- a. After 5 days, the number of students infected is

$$y = \frac{5000}{1 + 4999e^{-0.8(5)}} = \frac{5000}{1 + 4999e^{-4}} \approx 54.$$

- b. Classes are canceled when the number of infected students is  $(0.40)(5000) = 2000$ .

$$2000 = \frac{5000}{1 + 4999e^{-0.8t}}$$

$$1 + 4999e^{-0.8t} = 2.5$$

$$e^{-0.8t} = \frac{1.5}{4999}$$

$$\ln e^{-0.8t} = \ln \frac{1.5}{4999}$$

$$-0.8t = \ln \frac{1.5}{4999}$$

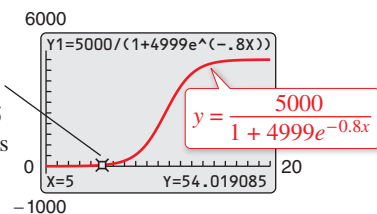
$$t = -\frac{1}{0.8} \ln \frac{1.5}{4999}$$

$$t \approx 10.14$$

So, after about 10 days, at least 40% of the students will be infected, and classes will be canceled.

#### Graphical Solution

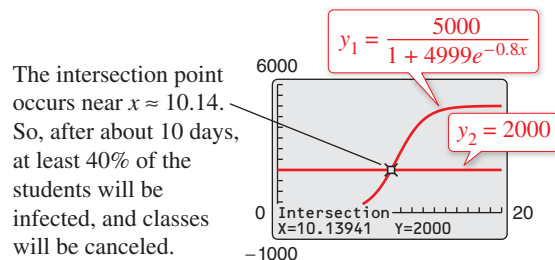
- a. Use the *value* feature to estimate that  $y \approx 54$  when  $x = 5$ . So, after 5 days, about 54 students will be infected.



- b. Classes are canceled when the number of infected students is  $(0.40)(5000) = 2000$ . Use a graphing utility to graph

$$y_1 = \frac{5000}{1 + 4999e^{-0.8x}} \quad \text{and} \quad y_2 = 2000$$

in the same viewing window. Use the *intersect* feature of the graphing utility to find the point of intersection of the graphs, as shown in the figure.



**✓ Checkpoint** Audio-video solution in English & Spanish at [LarsonPrecalculus.com](http://LarsonPrecalculus.com).

In Example 5, after how many days are 250 students infected?



## Logarithmic Models

On the Richter scale, the magnitude  $R$  of an earthquake of intensity  $I$  is given by

$$R = \log_{10} \frac{I}{I_0}$$

where  $I_0 = 1$  is the minimum intensity used for comparison. Intensity is a measure of the wave energy of an earthquake.

### EXAMPLE 6 Magnitudes of Earthquakes

In 2014, Edgefield, South Carolina, experienced an earthquake that measured 4.1 on the Richter scale. Also in 2014, Nago, Japan, experienced an earthquake that measured 6.5 on the Richter scale.

(Source: U.S. Geological Survey)

- Find the intensity of each earthquake.
- Compare the two intensities.

#### Solution

- Because  $I_0 = 1$  and  $R = 4.1$ , the intensity of the earthquake in South Carolina can be found as shown.

$$R = \log_{10} \frac{I}{I_0} \quad \text{Write original model.}$$

$$4.1 = \log_{10} \frac{I}{1} \quad \text{Substitute 1 for } I_0 \text{ and 4.1 for } R.$$

$$10^{4.1} = 10^{\log_{10} I} \quad \text{Exponentiate each side.}$$

$$10^{4.1} = I \quad \text{Inverse Property}$$

So, the intensity was about  $10^{4.1} \approx 12,589$ . For the earthquake in Japan,  $R = 6.5$ . The intensity of this earthquake was

$$R = \log_{10} \frac{I}{I_0} \quad \text{Write original model.}$$

$$6.5 = \log_{10} \frac{I}{1} \quad \text{Substitute 1 for } I_0 \text{ and 6.5 for } R.$$

$$10^{6.5} = 10^{\log_{10} I} \quad \text{Exponentiate each side.}$$

$$10^{6.5} = I. \quad \text{Inverse Property}$$


So, the intensity was about  $10^{6.5} \approx 3,162,278$ .

- Note that an increase of 2.4 units on the Richter scale (from 4.1 to 6.5) represents an increase in intensity by a factor of

$$\begin{aligned} \frac{10^{6.5}}{10^{4.1}} &= 10^{2.4} \\ &\approx 251 \end{aligned}$$

In other words, the intensity of the earthquake in Japan was about 251 times as great as the intensity of the earthquake in South Carolina.

 **Checkpoint**  Audio-video solution in English & Spanish at [LarsonPrecalculus.com](http://LarsonPrecalculus.com).

Find the intensities of earthquakes whose magnitudes are (a)  $R = 6.0$  and (b)  $R = 7.9$ . 



Seismologist

### 3.5 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

#### Vocabulary and Concept Check

1. Match each equation with its model.

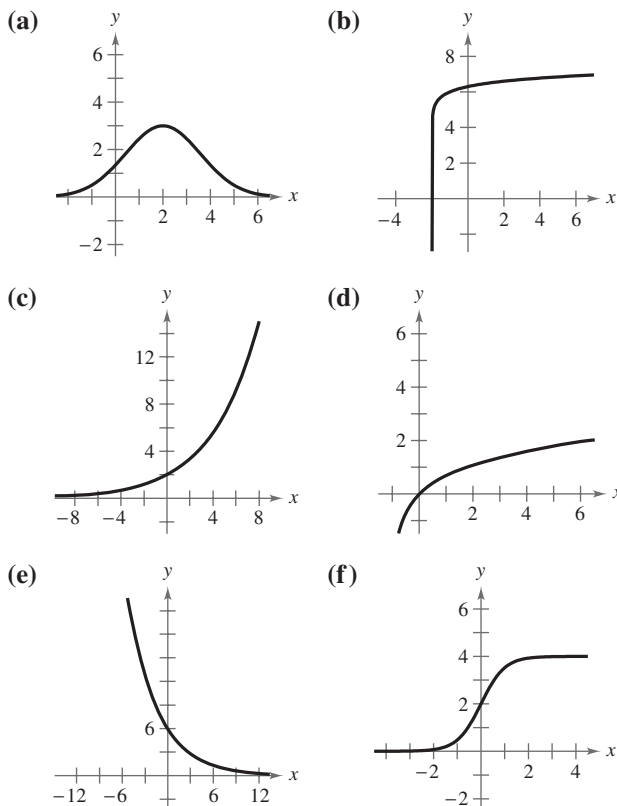
- |                               |                                    |
|-------------------------------|------------------------------------|
| (a) Exponential growth model  | (i) $y = ae^{-bx}, b > 0$          |
| (b) Exponential decay model   | (ii) $y = a + b \ln x$             |
| (c) Logistic growth model     | (iii) $y = \frac{a}{1 + be^{-rx}}$ |
| (d) Gaussian model            | (iv) $y = ae^{bx}, b > 0$          |
| (e) Natural logarithmic model | (v) $y = a + b \log_{10} x$        |
| (f) Common logarithmic model  | (vi) $y = ae^{-(x-b)^2/c}$         |

In Exercises 2 and 3, fill in the blank.

- Gaussian models are commonly used in probability and statistics to represent populations that are \_\_\_\_\_ distributed.
- Logistic growth curves are also called \_\_\_\_\_ curves.
- Which model in Exercise 1 has a graph called a bell-shaped curve?
- Does the model  $y = 120e^{-0.25x}$  represent exponential growth or exponential decay?
- Which model in Exercise 1 has a graph with two horizontal asymptotes?

#### Procedures and Problem Solving

**Identifying Graphs of Models** In Exercises 7–12, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]



- |                               |                                 |
|-------------------------------|---------------------------------|
| 7. $y = 2e^{x/4}$             | 8. $y = 6e^{-x/4}$              |
| 9. $y = 6 + \log_{10}(x + 2)$ | 10. $y = 3e^{-(x-2)^2/5}$       |
| 11. $y = \ln(x + 1)$          | 12. $y = \frac{4}{1 + e^{-2x}}$ |

**Using a Compound Interest Formula** In Exercises 13–20, complete the table for a savings account in which interest is compounded continuously.

	Initial Investment	Annual % Rate	Time to Double	Amount After 10 Years
13.	\$10,000	4%	<input type="text"/>	<input type="text"/>
14.	\$2000	2%	<input type="text"/>	<input type="text"/>
15.	\$7500	<input type="text"/>	21 years	<input type="text"/>
16.	\$1000	<input type="text"/>	12 years	<input type="text"/>
17.	\$5000	<input type="text"/>	<input type="text"/>	\$5665.74
18.	\$300	<input type="text"/>	<input type="text"/>	\$385.21
19.	<input type="text"/>	4.5%	<input type="text"/>	\$100,000.00
20.	<input type="text"/>	3.5%	<input type="text"/>	\$2500.00

21. **Tripling an Investment** Complete the table for the time  $t$  (in years) necessary for  $P$  dollars to triple when interest is compounded continuously at rate  $r$ . Create a scatter plot of the data.

$r$	2%	4%	6%	8%	10%	12%
$t$						



- 22. Tripling an Investment** Complete the table for the time  $t$  (in years) necessary for  $P$  dollars to triple when interest is compounded annually at rate  $r$ . Create a scatter plot of the data.

$r$	2%	4%	6%	8%	10%	12%
$t$						

- 23. Finance** When \$1 is invested in an account over a 10-year period, the amount  $A$  in the account after  $t$  years is given by

$$A = 1 + 0.075t \quad \text{or} \quad A = e^{0.07t}$$

depending on whether the account pays simple interest at  $7\frac{1}{2}\%$  or continuous compound interest at  $7\%$ . Use a graphing utility to graph each function in the same viewing window. Which grows at a greater rate?

- 24. Finance** When \$1 is invested in an account over a 10-year period, the amount  $A$  in the account after  $t$  years is given by

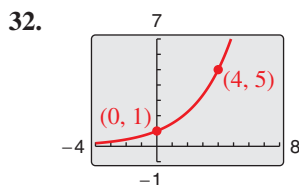
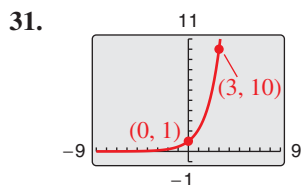
$$A = 1 + 0.06t \quad \text{or} \quad A = \left(1 + \frac{0.055}{365}\right)^{365t}$$

depending on whether the account pays simple interest at  $6\%$  or compound interest at  $5\frac{1}{2}\%$  compounded daily. Use a graphing utility to graph each function in the same viewing window. Which grows at a greater rate?

- Radioactive Decay** In Exercises 25–30, complete the table for the radioactive isotope.

Isotope	Half-Life (years)	Initial Quantity	Amount After 1000 Years
25. $^{226}\text{Ra}$	1600	10 g	<input type="text"/>
26. $^{14}\text{C}$	5730	3 g	<input type="text"/>
27. $^{231}\text{Pa}$	32,800	<input type="text"/>	1.5 g
28. $^{239}\text{Pu}$	24,100	<input type="text"/>	0.4 g
29. $^{241}\text{Am}$	432.2	26.4 g	<input type="text"/>
30. $^{238}\text{Pu}$	87.74	<input type="text"/>	0.1 g

- Writing a Model** In Exercises 31–34, find the exponential model  $y = ae^{bx}$  that fits the points shown in the graph or table.



33. 

$x$	0	5
$y$	4	1

34. 

$x$	0	3
$y$	5	$\frac{1}{2}$

- 35. Demography** The populations  $P$  (in thousands) of Antioch, California, from 2006 through 2012 can be modeled by  $P = 90e^{0.013t}$ , where  $t$  is the year, with  $t = 6$  corresponding to 2006. (Source: U.S. Census Bureau)

- (a) According to the model, was the population of Antioch increasing or decreasing from 2006 through 2012? Explain your reasoning.
- (b) What were the populations of Antioch in 2006, 2009, and 2012?
- (c) According to the model, when will the population of Antioch be approximately 116,000?

### 36. MODELING DATA

The table shows the populations (in millions) of five countries in 2013 and the projected populations (in millions) for 2025. (Source: U.S. Census Bureau)

Country	2013	2025
Australia	22.3	25.1
Canada	34.6	37.6
Hungary	9.9	9.6
Philippines	105.7	128.9
Turkey	80.7	90.5

- (a) Find the exponential growth or decay model,  $y = ae^{bt}$  or  $y = ae^{-bt}$ , for the population of each country, where  $t$  is the year, with  $t = 13$  corresponding to 2013. Use the model to predict the population of each country in 2040.
- (b) You can see that the populations of Canada and the Philippines are growing at different rates. What constant in the equation  $y = ae^{bt}$  is determined by these different growth rates? Discuss the relationship between the different growth rates and the magnitude of the constant.
- (c) The population of Turkey is increasing while the population of Hungary is decreasing. What constant in the equation  $y = ae^{bt}$  reflects this difference? Explain.

- 37. Demography** The populations  $P$  (in thousands) of Cameron County, Texas, from 2006 through 2012 can be modeled by

$$P = 339.2e^{kt}$$

where  $t$  is the year, with  $t = 6$  corresponding to 2006. In 2011, the population was 412,600. (Source: U.S. Census Bureau)

- (a) Find the value of  $k$  for the model. Round your result to four decimal places.
- (b) Use your model to predict the population in 2018.

**38. Demography** The populations  $P$  (in thousands) of Pineville, North Carolina, from 2006 through 2012 can be modeled by  $P = 5.4e^{kt}$ , where  $t$  is the year, with  $t = 6$  corresponding to 2006. In 2008, the population was 7000. (Source: U.S. Census Bureau)

- Find the value of  $k$  for the model. Round your result to four decimal places.
- Use your model to predict the population in 2018.

**39. Why you should learn it** (p. 221) Carbon 14 ( $^{14}\text{C}$ ) dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, then the amount of  $^{14}\text{C}$  absorbed by a tree that grew several centuries ago should be the same as the amount of  $^{14}\text{C}$  absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal, given that the half-life of  $^{14}\text{C}$  is about 5700 years?



**40. Radioactive Decay** The half-life of radioactive radium ( $^{226}\text{Ra}$ ) is 1600 years. What percent of a present amount of radioactive radium will remain after 100 years?

#### 41. MODELING DATA

A new 2014 luxury sedan that sold for \$39,780 has a book value  $V$  of \$25,459 after 2 years.

- Find a linear model for the value  $V$  of the sedan.
- Find an exponential model for the value  $V$  of the sedan. Round the numbers in the model to four decimal places.
- Use a graphing utility to graph the two models in the same viewing window.
- Which model represents a greater depreciation rate in the first year?
- For what years is the value of the sedan greater using the linear model? the exponential model?

#### 42. MODELING DATA

A new laptop computer that sold for \$1200 in 2014 has a book value  $V$  of \$650 after 2 years.

- Find a linear model for the value  $V$  of the laptop.
- Find an exponential model for the value  $V$  of the laptop. Round the numbers in the model to four decimal places.
- Use a graphing utility to graph the two models in the same viewing window.
- Which model represents a greater depreciation rate in the first year?
- For what years is the value of the laptop greater using the linear model? the exponential model?

**43. Psychology** The IQ scores for adults roughly follow the normal distribution  $y = 0.0266e^{-(x-100)^2/450}$ ,  $70 \leq x \leq 115$ , where  $x$  is the IQ score.

- Use a graphing utility to graph the function.
- Use the graph in part (a) to estimate the average IQ score.

**44. Marketing** The sales  $S$  (in thousands of units) of a cleaning solution after  $x$  hundred dollars is spent on advertising are given by  $S = 10(1 - e^{-kx})$ . When \$500 is spent on advertising, 2500 units are sold.

- Complete the model by solving for  $k$ .
- Estimate the number of units that will be sold when advertising expenditures are raised to \$700.

**45. Forestry** A conservation organization releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a carrying capacity of 1000 animals and that the growth of the herd will follow the logistic curve

$$p(t) = \frac{1000}{1 + 9e^{-0.1656t}}$$

where  $t$  is measured in months.

- What is the population after 5 months?
- After how many months will the population reach 500?
- Use a graphing utility to graph the function. Use the graph to determine the values of  $p$  at which the horizontal asymptotes occur. Identify the asymptote that is most relevant in the context of the problem and interpret its meaning.

**46. Biology** The number  $Y$  of yeast organisms in a culture is given by the model

$$Y = \frac{663}{1 + 72e^{-0.547t}}$$

where  $t$  represents the time (in hours).

- Use a graphing utility to graph the model.
- Use the model to predict the populations for the 19th hour and the 30th hour.
- According to this model, what is the limiting value of the population?
- Why do you think this population of yeast follows a logistic growth model instead of an exponential growth model?

**Geology** In Exercises 47 and 48, use the Richter scale (see page 227) for measuring the magnitudes of earthquakes.

**47.** Find the intensities  $I$  of the following earthquakes measuring  $R$  on the Richter scale (let  $I_0 = 1$ ). (Source: U.S. Geological Survey)

- Falkland Islands in 2013,  $R = 7.0$
- Sea of Okhotsk in 2013,  $R = 8.3$
- Kunisaki-shi, Japan in 2014,  $R = 6.3$

48. Find the magnitudes  $R$  of the following earthquakes of intensity  $I$  (let  $I_0 = 1$ ).

(a)  $I = 39,811,000$

(b)  $I = 12,589,000$

(c)  $I = 251,200$

**Audiology** In Exercises 49–52, use the following information for determining sound intensity. The level of sound  $\beta$  (in decibels) with an intensity  $I$  is

$$\beta = 10 \log_{10} \frac{I}{I_0}$$

where  $I_0$  is an intensity of  $10^{-12}$  watt per square meter, corresponding roughly to the faintest sound that can be heard by the human ear. In Exercises 49 and 50, find the level of each sound  $\beta$ .

49. (a)  $I = 10^{-10}$  watt per  $\text{m}^2$  (quiet room)

(b)  $I = 10^{-5}$  watt per  $\text{m}^2$  (busy street corner)

(c)  $I \approx 10^0$  watt per  $\text{m}^2$  (threshold of pain)

50. (a)  $I = 10^{-4}$  watt per  $\text{m}^2$  (door slamming)

(b)  $I = 10^{-3}$  watt per  $\text{m}^2$  (loud car horn)

(c)  $I = 10^{-2}$  watt per  $\text{m}^2$  (siren at 30 meters)

51. As a result of the installation of a muffler, the noise level of an engine was reduced from 88 to 72 decibels. Find the percent decrease in the intensity level of the noise due to the installation of the muffler.

52. As a result of the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Find the percent decrease in the intensity level of the noise due to the installation of these materials.

**Chemistry** In Exercises 53–56, use the acidity model

$$\text{pH} = -\log[\text{H}^+]$$

where acidity (pH) is a measure of the hydrogen ion concentration  $[\text{H}^+]$  (in moles of hydrogen per liter) of a solution.

53. Find the pH when  $[\text{H}^+] = 2.3 \times 10^{-5}$ .

54. Compute  $[\text{H}^+]$  for a solution for which  $\text{pH} = 5.8$ .

55. A grape has a pH of 3.5, and baking soda has a pH of 8.0. The hydrogen ion concentration of the grape is how many times that of the baking soda?

56. The pH of a solution is decreased by one unit. The hydrogen ion concentration is increased by what factor?

57. **Finance** The total interest  $u$  paid on a home mortgage of  $P$  dollars at interest rate  $r$  for  $t$  years is given by

$$u = P \left[ \frac{rt}{1 - \left( \frac{1}{1 + r/12} \right)^{12t}} - 1 \right].$$

Consider a \$230,000 home mortgage at 3%.

(a) Use a graphing utility to graph the total interest function.

(b) Approximate the length of the mortgage when the total interest paid is the same as the amount of the mortgage. Is it possible that a person could pay twice as much in interest charges as the amount of the mortgage?

58. **Finance** A \$200,000 home mortgage for 30 years at 4.25% has a monthly payment of \$983.88. Part of the monthly payment goes toward the interest charge on the unpaid balance, and the remainder of the payment is used to reduce the principal. The amount that goes toward the interest is given by

$$u = M - \left( M - \frac{Pr}{12} \right) \left( 1 + \frac{r}{12} \right)^{12t}$$

and the amount that goes toward reduction of the principal is given by

$$v = \left( M - \frac{Pr}{12} \right) \left( 1 + \frac{r}{12} \right)^{12t}.$$

In these formulas,  $P$  is the amount of the mortgage,  $r$  is the interest rate,  $M$  is the monthly payment, and  $t$  is the time (in years).

(a) Use a graphing utility to graph each function in the same viewing window. (The viewing window should show all 30 years of mortgage payments.)

(b) In the early years of the mortgage, the larger part of the monthly payment goes for what purpose? Approximate the time when the monthly payment is evenly divided between interest and principal reduction.

(c) Repeat parts (a) and (b) for a repayment period of 20 years ( $M = \$1238.47$ ). What can you conclude?

59. **Forensics** At 8:30 A.M., a coroner was called to the home of a person who had died during the night. In order to estimate the time of death, the coroner took the person's temperature twice. At 9:00 A.M. the temperature was 85.7°F, and at 11:00 A.M. the temperature was 82.8°F. From these two temperatures the coroner was able to determine that the time elapsed since death and the body temperature were related by the formula

$$t = -10 \ln \frac{T - 70}{98.6 - 70}$$

where  $t$  is the time (in hours elapsed since the person died) and  $T$  is the temperature (in degrees Fahrenheit) of the person's body. Assume that the person had a normal body temperature of 98.6°F at death and that the room temperature was a constant 70°F. Use the formula to estimate the time of death of the person. (This formula is derived from a general cooling principle called Newton's Law of Cooling.)

**60. Culinary Arts** You take a five-pound package of steaks out of a freezer at 11 A.M. and place it in a refrigerator. Will the steaks be thawed in time to be grilled at 6 P.M.? Assume that the refrigerator temperature is 40°F and the freezer temperature is 0°F. Use the formula (derived from Newton's Law of Cooling)

$$t = -5.05 \ln \frac{T - 40}{0 - 40}$$

where  $t$  is the time in hours (with  $t = 0$  corresponding to 11 A.M.) and  $T$  is the temperature of the package of steaks (in degrees Fahrenheit).

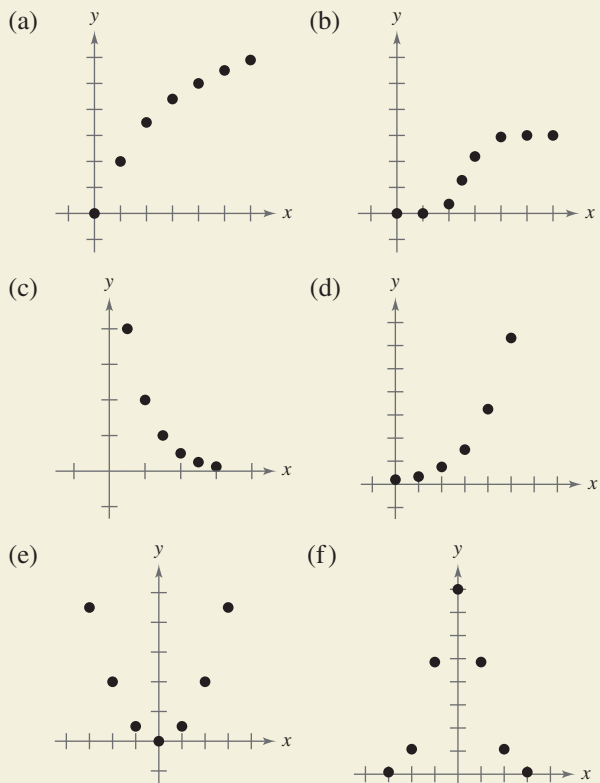
**Conclusions**

**True or False?** In Exercises 61 and 62, determine whether the statement is true or false. Justify your answer.

- 61. The domain of a logistic growth function cannot be the set of real numbers.
- 62. The graph of a logistic growth function will always have an  $x$ -intercept.
- 63. **Think About It** Can the graph of a Gaussian model ever have an  $x$ -intercept? Explain.



**64. HOW DO YOU SEE IT?** For each graph, state whether an exponential, Gaussian, logarithmic, logistic, or quadratic model will fit the data best. Explain your reasoning. Then describe a real-life situation that could be represented by the data.

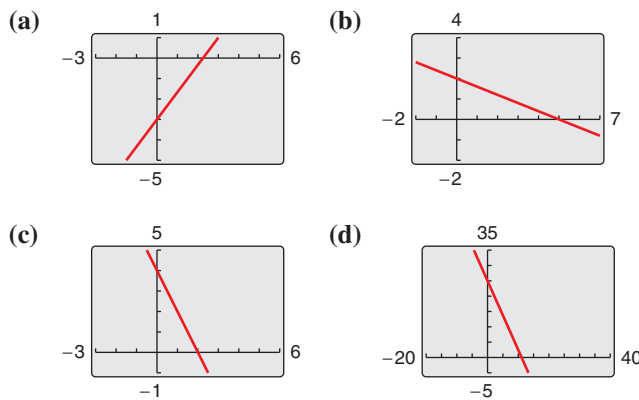


**Identifying Models** In Exercises 65–68, identify the type of model you studied in this section that has the given characteristic.

- 65. The maximum value of the function occurs at the average value of the independent variable.
- 66. A horizontal asymptote of its graph represents the limiting value of a population.
- 67. Its graph shows a steadily increasing rate of growth.
- 68. The only asymptote of its graph is a vertical asymptote.

**Cumulative Mixed Review**

**Identifying Graphs of Linear Equations** In Exercises 69–72, match the equation with its graph and identify any intercepts. [The graphs are labeled (a), (b), (c), and (d).]



- 69.  $4x - 3y - 9 = 0$
- 70.  $2x + 5y - 10 = 0$
- 71.  $y = 25 - 2.25x$
- 72.  $\frac{x}{2} + \frac{y}{4} = 1$

**Applying the Leading Coefficient Test** In Exercises 73–76, use the Leading Coefficient Test to determine the right-hand and left-hand behavior of the graph of the polynomial function.

- 73.  $f(x) = 2x^3 - 3x^2 + x - 1$
- 74.  $f(x) = 5 - x^2 - 4x^4$
- 75.  $g(x) = -1.6x^5 + 4x^2 - 2$
- 76.  $g(x) = 7x^6 + 9.1x^5 - 3.2x^4 + 25x^3$

**Using Synthetic Division** In Exercises 77 and 78, divide using synthetic division.

- 77.  $(2x^3 - 8x^2 + 3x - 9) \div (x - 4)$
- 78.  $(x^4 - 3x + 1) \div (x + 5)$

**79. Make a Decision** To work an extended application analyzing the sales per share for Kohl's Corporation from 1995 through 2012, visit this textbook's website at [LarsonPrecalculus.com](http://LarsonPrecalculus.com). (Data Source: Kohl's Corp.)

## 3.6 Nonlinear Models

### Classifying Scatter Plots

In Section 1.7, you saw how to fit linear models to data, and in Section 2.8, you saw how to fit quadratic models to data. In real life, many relationships between two variables are represented by different types of growth patterns. You can use a scatter plot to get an idea of which type of model will best fit a set of data.

#### EXAMPLE 1 Classifying Scatter Plots

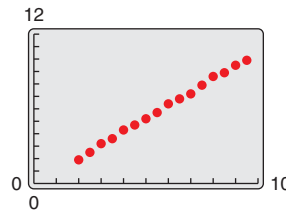
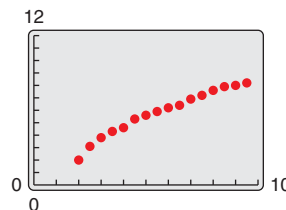
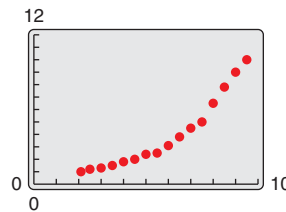
See *LarsonPrecalculus.com* for an interactive version of this type of example.

Decide whether each set of data could best be modeled by a linear model,  $y = ax + b$ , an exponential model,  $y = ab^x$ , or a logarithmic model,  $y = a + b \ln x$ .

- (2, 1), (2.5, 1.2), (3, 1.3), (3.5, 1.5), (4, 1.8), (4.5, 2), (5, 2.4), (5.5, 2.5), (6, 3.1), (6.5, 3.8), (7, 4.5), (7.5, 5), (8, 6.5), (8.5, 7.8), (9, 9), (9.5, 10)
- (2, 2), (2.5, 3.1), (3, 3.8), (3.5, 4.3), (4, 4.6), (4.5, 5.3), (5, 5.6), (5.5, 5.9), (6, 6.2), (6.5, 6.4), (7, 6.9), (7.5, 7.2), (8, 7.6), (8.5, 7.9), (9, 8), (9.5, 8.2)
- (2, 1.9), (2.5, 2.5), (3, 3.2), (3.5, 3.6), (4, 4.3), (4.5, 4.7), (5, 5.2), (5.5, 5.7), (6, 6.4), (6.5, 6.8), (7, 7.2), (7.5, 7.9), (8, 8.6), (8.5, 8.9), (9, 9.5), (9.5, 9.9)


#### Solution

- From the figure, it appears that the data can best be modeled by an exponential function.
- From the figure, it appears that the data can best be modeled by a logarithmic function.
- From the figure, it appears that the data can best be modeled by a linear function.



 **Checkpoint**  Audio-video solution in English & Spanish at *LarsonPrecalculus.com*.

Decide whether the set of data could best be modeled by a linear model,  $y = ax + b$ , an exponential model,  $y = ab^x$ , or a logarithmic model,  $y = a + b \ln x$ .

- (0.5, 12), (1, 8), (1.5, 5.7), (2, 4), (2.5, 3.1), (3, 2), (3.5, 1.4), (4, 1), (4.5, 0.8), (5, 0.6), (5.5, 0.4), (6, 0.25), (6.5, 0.18), (7, 0.13), (7.5, 0.09), (8, 0.06) 

#### What you should learn

- ▶ Classify scatter plots.
- ▶ Use scatter plots and a graphing utility to find models for data and choose the model that best fits a set of data.
- ▶ Use a graphing utility to find exponential and logistic models for data.

#### Why you should learn it

Many real-life applications can be modeled by nonlinear equations. For instance, in Exercise 34 on page 240, you are asked to find a nonlinear model that relates air pressure to altitude.



## Fitting Nonlinear Models to Data

Once you have used a scatter plot to determine the type of model that would best fit a set of data, there are several ways that you can actually find the model. Each method is best used with a computer or calculator, rather than with hand calculations.

### EXAMPLE 2 Fitting a Model to Data

Fit the data from Example 1(a) to an exponential model and a power model. Identify the coefficient of determination and determine which model fits the data better.

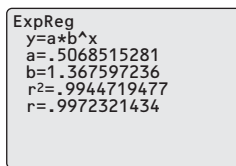
(2, 1), (2.5, 1.2), (3, 1.3), (3.5, 1.5), (4, 1.8), (4.5, 2), (5, 2.4), (5.5, 2.5),  
 (6, 3.1), (6.5, 3.8), (7, 4.5), (7.5, 5), (8, 6.5), (8.5, 7.8), (9, 9), (9.5, 10)

You can use the following set of data in an extra model-fitting example as a brief activity during class.

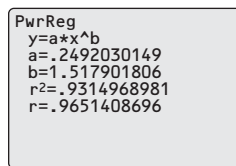
(1.1, 0.46), (1.3, 0.91), (1.4, 1.25),  
 (1.5, 1.71), (1.8, 3.60), (1.9, 4.52),  
 (2.1, 7.11), (2.3, 10.34), (2.4, 12.43)

### Solution

Begin by entering the data into a graphing utility. Then use the *regression* feature of the graphing utility to find exponential and power models for the data, as shown in Figure 3.30.



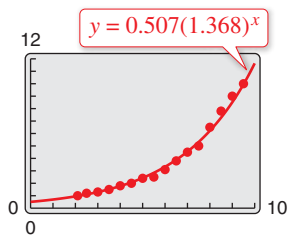
Exponential Model



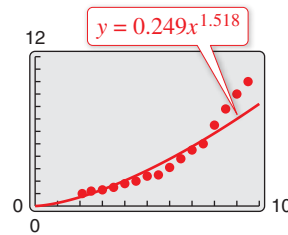
Power Model

Figure 3.30

So, an exponential model for the data is  $y = 0.507(1.368)^x$ , and a power model for the data is  $y = 0.249x^{1.518}$ . Plot the data and each model in the same viewing window, as shown in Figure 3.31. To determine which model fits the data better, compare the coefficients of determination for each model. The model whose  $r^2$ -value is closest to 1 is the model that better fits the data. In this case, the better-fitting model is the exponential model.



Exponential Model



Power Model

Figure 3.31

**✓ Checkpoint** Audio-video solution in English & Spanish at [LarsonPrecalculus.com](http://LarsonPrecalculus.com).

Fit the data from the Checkpoint for Example 1 to an exponential model and a power model. Identify the coefficient of determination and determine which model fits the data better.

(0.5, 12), (1, 8), (1.5, 5.7), (2, 4), (2.5, 3.1), (3, 2), (3.5, 1.4), (4, 1), (4.5, 0.8),  
 (5, 0.6), (5.5, 0.4), (6, 0.25), (6.5, 0.18), (7, 0.13), (7.5, 0.09), (8, 0.06)

Deciding which model best fits a set of data is a question that is studied in detail in statistics. Recall from Section 1.7 that the model that best fits a set of data is the one whose *sum of squared differences* is the least. In Example 2, the sums of squared differences are 0.90 for the exponential model and 14.30 for the power model.

**EXAMPLE 3** Fitting a Model to Data

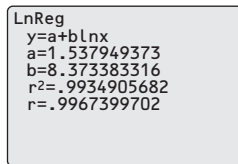
The table shows the yield  $y$  (in milligrams) of a chemical reaction after  $x$  minutes. Use a graphing utility to find a logarithmic model and a linear model for the data and identify the coefficient of determination for each model. Determine which model fits the data better.

Minutes, $x$	Yield, $y$
1	1.5
2	7.4
3	10.2
4	13.4
5	15.8
6	16.3
7	18.2
8	18.3

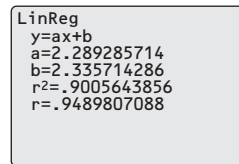
Spreadsheet at LarsonPrecalculus.com

**Solution**

Begin by entering the data into a graphing utility. Then use the *regression* feature of the graphing utility to find logarithmic and linear models for the data, as shown in Figure 3.32.



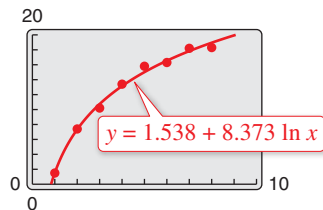
Logarithmic Model



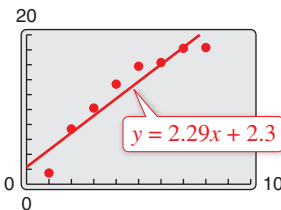
Linear Model

Figure 3.32

So, a logarithmic model for the data is  $y = 1.538 + 8.373 \ln x$ , and a linear model for the data is  $y = 2.29x + 2.3$ . Plot the data and each model in the same viewing window, as shown in Figure 3.33. To determine which model fits the data better, compare the coefficients of determination for each model. The model whose coefficient of determination is closer to 1 is the model that better fits the data. In this case, the better-fitting model is the logarithmic model.



Logarithmic Model



Linear Model

Figure 3.33

**Explore the Concept**

Use a graphing utility to find a quadratic model for the data in Example 3. Do you think this model fits the data better than the logarithmic model in Example 3? Explain your reasoning.



**✓ Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use a graphing utility to find a logarithmic model and a linear model for the set of data. Identify the coefficient of determination for each model and determine which model fits the data better.

(0.5, 1.7), (1, 2.5), (1.5, 3), (2, 3.2), (2.5, 3.5), (3, 3.7),  
 (3.5, 3.8), (4, 3.9), (4.5, 3.9), (5, 4.1), (5.5, 4.2), (6, 4.3),  
 (6.5, 4.5), (7, 4.6), (7.5, 4.8), (8, 5)

In Example 3, the sum of the squared differences for the logarithmic model is 1.59 and the sum of the squared differences for the linear model is 24.31.

## Modeling With Exponential and Logistic Functions

### EXAMPLE 4 Fitting an Exponential Model to Data

The table shows the amount  $y$  (in grams) of a radioactive substance remaining after  $x$  days. Use a graphing utility to find a model for the data. How much of the substance remains after 15 days?

#### Solution

Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 3.34.

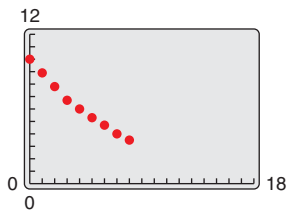


Figure 3.34

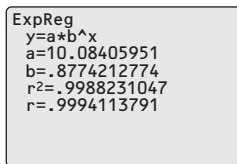


Figure 3.35

Day, $x$	Amount, $y$
0	10.0
1	8.9
2	7.8
3	6.7
4	6.0
5	5.3
6	4.7
7	4.0
8	3.5

Spreadsheet at  
LarsonPrecalculus.com

From the scatter plot, it appears that an exponential model is a good fit. Use the *regression* feature of the graphing utility to find the exponential model, as shown in Figure 3.35. Change the model to a natural exponential model, as follows.

$$\begin{aligned}
 y &= 10.08(0.877)^x && \text{Write original model.} \\
 &= 10.08e^{(\ln 0.877)x} && b = e^{\ln b} \\
 &\approx 10.08e^{-0.131x} && \text{Simplify.}
 \end{aligned}$$

Graph the data and the natural exponential model

$$y = 10.08e^{-0.131x}$$

in the same viewing window, as shown in Figure 3.36. From the model, you can see that the amount of the substance decreases by about 13% each day. Also, the amount of the substance remaining after 15 days is

$$\begin{aligned}
 y &= 10.08e^{-0.131x} && \text{Write natural exponential model.} \\
 &= 10.08e^{-0.131(15)} && \text{Substitute 15 for } x. \\
 &\approx 1.4 \text{ grams.} && \text{Use a calculator.}
 \end{aligned}$$

You can also use the *value* feature of the graphing utility to approximate the amount remaining after 15 days, as shown in Figure 3.37.

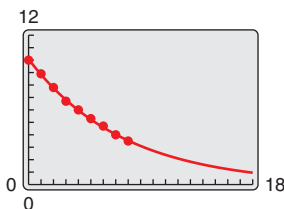


Figure 3.36

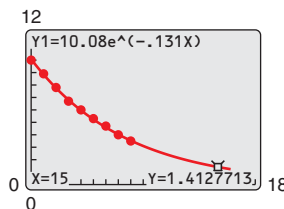


Figure 3.37

### Remark

You can change an exponential model of the form

$$y = ab^x$$

to one of the form

$$y = ae^{cx}$$

by rewriting  $b$  in the form

$$b = e^{\ln b}.$$

For instance,

$$y = 3(2^x)$$

can be written as

$$\begin{aligned}
 y &= 3(2^x) \\
 &= 3e^{(\ln 2)x} \\
 &\approx 3e^{0.693x}.
 \end{aligned}$$

**Checkpoint** Audio-video solution in English & Spanish at [LarsonPrecalculus.com](http://LarsonPrecalculus.com).

In Example 4, the initial amount of the substance is 10 grams. According to the natural exponential model, after how many days is the amount of the substance 5 grams? Interpret this value in the context of the problem.



The next example demonstrates how to use a graphing utility to fit a logistic model to data.

### EXAMPLE 5 Fitting a Logistic Model to Data

To estimate the amount of defoliation caused by the gypsy moth during a given year, a forester counts the number  $x$  of egg masses on  $\frac{1}{40}$  of an acre (circle of radius 18.6 feet) in the fall. The percent of defoliation  $y$  the next spring is shown in the table. (Source: USDA, Forest Service)

Egg masses, $x$	Percent of defoliation, $y$
0	12
25	44
50	81
75	96
100	99

- Use the *regression* feature of a graphing utility to find a logistic model for the data.
- How closely does the model represent the data?

#### Graphical Solution

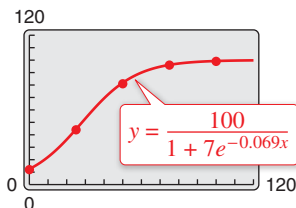
- Enter the data into a graphing utility. Using the *regression* feature of the graphing utility, you can find the logistic model, as shown in the figure.

```
Logistic
y=c/(1+ae^(-bx))
a=7.163373551
b=.0689483064
c=99.74649743
```

You can approximate this model to be

$$y = \frac{100}{1 + 7e^{-0.069x}}$$

- You can use the graphing utility to graph the actual data and the model in the same viewing window. In the figure, it appears that the model is a good fit for the actual data.



Forester

#### Numerical Solution

- Enter the data into a graphing utility. Using the *regression* feature of the graphing utility, you can approximate the logistic model to be

$$y = \frac{100}{1 + 7e^{-0.069x}}$$

- You can see how well the model fits the data by comparing the actual values of  $y$  with the values of  $y$  given by the model, which are labeled  $y^*$  in the table below.

$x$	0	25	50	75	100
$y$	12	44	81	96	99
$y^*$	12.5	44.5	81.8	96.2	99.3

In the table, you can see that the model appears to be a good fit for the actual data.

**Checkpoint** Audio-video solution in English & Spanish at [LarsonPrecalculus.com](http://LarsonPrecalculus.com).

Use a graphing utility to graph the logistic model in Example 5. Use the graph to determine the horizontal asymptotes. Interpret the asymptotes in the context of the real-life situation.

### 3.6 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

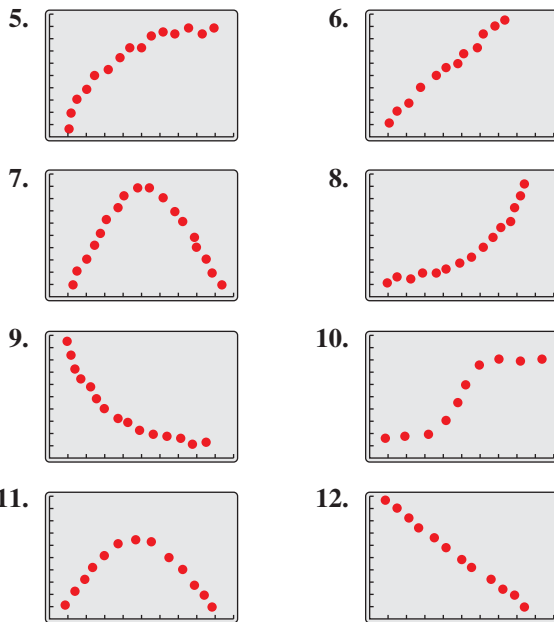
#### Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. A power model has the form \_\_\_\_\_ .
2. An exponential model of the form  $y = ab^x$  can be rewritten as a natural exponential model of the form \_\_\_\_\_ .
3. What type of visual display can you create to get an idea of which type of model will best fit the data set?
4. A power model for a set of data has a coefficient of determination of  $r^2 \approx 0.901$  and an exponential model for the data has a coefficient of determination of  $r^2 \approx 0.967$ . Which model fits the data better?

#### Procedures and Problem Solving

**Classifying Scatter Plots** In Exercises 5–12, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, an exponential model, a logarithmic model, or a logistic model.



**Classifying Scatter Plots** In Exercises 13–18, use a graphing utility to create a scatter plot of the data. Decide whether the data could best be modeled by a linear model, an exponential model, or a logarithmic model.

13. (1, 2.0), (1.5, 3.5), (2, 4.0), (4, 5.8), (6, 7.0), (8, 7.8)
14. (1, 5.8), (1.5, 6.0), (2, 6.5), (4, 7.6), (6, 8.9), (8, 10.0)
15. (1, 4.4), (1.5, 4.7), (2, 5.5), (4, 9.9), (6, 18.1), (8, 33.0)
16. (1, 11.0), (1.5, 9.6), (2, 8.2), (4, 4.5), (6, 2.5), (8, 1.4)
17. (1, 7.5), (1.5, 7.0), (2, 6.8), (4, 5.0), (6, 3.5), (8, 2.0)
18. (1, 5.0), (1.5, 6.0), (2, 6.4), (4, 7.8), (6, 8.6), (8, 9.0)

**Finding an Exponential Model** In Exercises 19–22, use the *regression* feature of a graphing utility to find an exponential model  $y = ab^x$  for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.

19. (0, 5), (1, 6), (2, 7), (3, 9), (4, 13)
20. (0, 4.0), (2, 6.9), (4, 18.0), (6, 32.3), (8, 59.1), (10, 118.5)
21. (0, 8.3), (1, 6.1), (2, 4.6), (3, 3.8), (4, 3.6)
22. (-3, 102.2), (0, 80.5), (3, 67.8), (6, 58.2), (10, 55.0)

**Finding a Logarithmic Model** In Exercises 23–26, use the *regression* feature of a graphing utility to find a logarithmic model  $y = a + b \ln x$  for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.

23. (1, 2.0), (2, 3.0), (3, 3.5), (4, 4.0), (5, 4.1), (6, 4.2), (7, 4.5)
24. (1, 8.5), (2, 11.4), (4, 12.8), (6, 13.6), (8, 14.2), (10, 14.6)
25. (1, 11), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2)
26. (3, 14.6), (6, 11.0), (9, 9.0), (12, 7.6), (15, 6.5)

**Finding a Power Model** In Exercises 27–30, use the *regression* feature of a graphing utility to find a power model  $y = ax^b$  for the data and identify the coefficient of determination. Use the graphing utility to plot the data and graph the model in the same viewing window.

27. (1, 2.0), (2, 3.4), (5, 6.7), (6, 7.3), (10, 12.0)
28. (0.5, 1.0), (2, 12.5), (4, 33.2), (6, 65.7), (8, 98.5), (10, 150.0)
29. (1, 10.0), (2, 4.0), (3, 0.7), (4, 0.1)
30. (2, 450), (4, 385), (6, 345), (8, 332), (10, 312)

**31. MODELING DATA**

The table shows the yearly sales  $S$  (in millions of dollars) of Whole Foods Market for the years 2006 through 2013. (Source: Whole Foods Market)

DATA		Year	Sales, $S$
Spreadsheet at LarsonPrecalculus.com		2006	5,607.4
		2007	6,591.8
		2008	7,953.9
		2009	8,031.6
		2010	9,005.8
		2011	10,108.0
		2012	11,699.0
		2013	12,917.0

- Use the *regression* feature of a graphing utility to find an exponential model and a power model for the data and identify the coefficient of determination for each model. Let  $t$  represent the year, with  $t = 6$  corresponding to 2006.
- Use the graphing utility to graph each model with the data.
- Use the coefficients of determination to determine which model fits the data better.

**32. MODELING DATA**

The table shows the numbers of single beds  $B$  (in thousands) on North American cruise ships from 2007 through 2012. (Source: Cruise Lines International Association)

DATA		Year	Beds, $B$
Spreadsheet at LarsonPrecalculus.com		2007	260.0
		2008	270.7
		2009	284.8
		2010	307.7
		2011	321.2
		2012	333.7

- Use the *regression* feature of a graphing utility to find a linear model, an exponential model, and a logarithmic model for the data and identify the coefficient of determination for each model. Let  $t$  represent the year, with  $t = 7$  corresponding to 2007.
- Which model is the best fit for the data? Explain.
- Use the model you chose in part (b) to predict the number of beds in 2017. Is the number reasonable?

**33. MODELING DATA**

The populations  $P$  (in thousands) of Luxembourg for the years 1999 through 2013 are shown in the table, where  $t$  represents the year, with  $t = 9$  corresponding to 1999. (Source: European Commission Eurostat)

DATA		Year	Population, $P$
Spreadsheet at LarsonPrecalculus.com		1999	427.4
		2000	433.6
		2001	439.0
		2002	444.1
		2003	448.3
		2004	455.0
		2005	461.2
		2006	469.1
		2007	476.2
		2008	483.8
		2009	493.5
		2010	502.1
		2011	511.8
		2012	524.9
		2013	537.0

- Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.
- Use the *regression* feature of the graphing utility to find a power model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.
- Use the *regression* feature of the graphing utility to find an exponential model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.
- Use the *regression* feature of the graphing utility to find a logarithmic model for the data and to identify the coefficient of determination. Plot the model and the data in the same viewing window.
- Which model is the best fit for the data? Explain.
- Use each model to predict the populations of Luxembourg for the years 2014 through 2018.
- Which model is the best choice for predicting the future population of Luxembourg? Explain.
- Were your choices of models the same for parts (e) and (g)? If not, explain why your choices were different.

**34. Why you should learn it** (p. 233) The atmospheric pressure decreases with increasing altitude. At sea level, the average air pressure is approximately 1.03323 kilograms per square centimeter, and this pressure is called one atmosphere. Variations in weather conditions cause changes in the atmospheric pressure of up to  $\pm 5$  percent. The ordered pairs  $(h, p)$  give the pressures  $p$  (in atmospheres) for various altitudes  $h$  (in kilometers). (*Spreadsheet at LarsonPrecalculus.com*)



**DATA**  $(0, 1), (10, 0.25), (20, 0.06), (5, 0.55), (15, 0.12), (25, 0.02)$

- Use the *regression* feature of a graphing utility to attempt to find the logarithmic model  $p = a + b \ln h$  for the data. Explain why the result is an error message.
- Use the *regression* feature of the graphing utility to find the logarithmic model  $h = a + b \ln p$  for the data.
- Use the graphing utility to plot the data and graph the logarithmic model in the same viewing window.
- Use the model to estimate the altitude at which the pressure is 0.75 atmosphere.
- Use the graph in part (c) to estimate the pressure at an altitude of 13 kilometers.

### 35. MODELING DATA

The table shows the annual sales  $S$  (in billions of dollars) of Starbucks for the years from 2009 through 2013. (*Source: Starbucks Corp.*)

Year	Sales, $S$
2009	9.77
2010	10.71
2011	11.70
2012	13.30
2013	14.89

- Use the *regression* feature of a graphing utility to find an exponential model for the data. Let  $t$  represent the year, with  $t = 9$  corresponding to 2009.
- Rewrite the model from part (a) as a natural exponential model.
- Use the natural exponential model to predict the annual sales of Starbucks in 2018. Is the value reasonable?

### 36. MODELING DATA

A beaker of liquid at an initial temperature of  $78^\circ\text{C}$  is placed in a room at a constant temperature of  $21^\circ\text{C}$ . The temperature of the liquid is measured every 5 minutes for a period of  $\frac{1}{2}$  hour. The results are recorded in the table, where  $t$  is the time (in minutes) and  $T$  is the temperature (in degrees Celsius).

<b>DATA</b>	Time, $t$	Temperature, $T$
Spreadsheet at LarsonPrecalculus.com	0	$78.0^\circ$
	5	$66.0^\circ$
	10	$57.5^\circ$
	15	$51.2^\circ$
	20	$46.3^\circ$
	25	$42.5^\circ$
	30	$39.6^\circ$



- Use the *regression* feature of a graphing utility to find a linear model for the data. Use the graphing utility to plot the data and graph the model in the same viewing window. Do the data appear linear? Explain.
- Use the *regression* feature of the graphing utility to find a quadratic model for the data. Use the graphing utility to plot the data and graph the model in the same viewing window. Do the data appear quadratic? Even though the quadratic model appears to be a good fit, explain why it might not be a good model for predicting the temperature of the liquid when  $t = 60$ .
- The graph of the temperature of the room should be an asymptote of the graph of the model. Subtract the room temperature from each of the temperatures in the table. Use the *regression* feature of the graphing utility to find an exponential model for the revised data. Add the room temperature to this model. Use the graphing utility to plot the original data and graph the model in the same viewing window.
- Explain why the procedure in part (c) was necessary for finding the exponential model.

## 37. MODELING DATA

The table shows the percents  $P$  of women in different age groups (in years) who have been married at least once. (Source: U.S. Census Bureau)

DATA	Age group	Percent, $P$
Spreadsheet at LarsonPrecalculus.com	18–24	14.6
	25–29	49.0
	30–34	70.3
	35–39	79.9
	40–44	85.0
	45–49	87.0
	50–54	89.5
	55–59	91.1

- (a) Use the *regression* feature of a graphing utility to find a logistic model for the data. Let  $x$  represent the midpoint of the age group.
- (b) Use the graphing utility to graph the model with the original data. How closely does the model represent the data?

## 38. MODELING DATA

The table shows the lengths  $y$  (in centimeters) of yellowtail snappers caught off the coast of Brazil for different ages (in years). (Source: Brazilian Journal of Oceanography)

DATA	Age, $x$	Length, $y$
Spreadsheet at LarsonPrecalculus.com	1	11.21
	2	20.77
	3	28.94
	4	35.92
	5	41.87
	6	46.96
	7	51.30
	8	55.01
	9	58.17
	10	60.87
	11	63.18
	12	65.15
	13	66.84
	14	68.27
	15	69.50

- (a) Use the *regression* feature of a graphing utility to find a logistic model and a power model for the data.
- (b) Use the graphing utility to graph each model from part (a) with the data. Use the graphs to determine which model better fits the data.
- (c) Use the model from part (b) to predict the length of a 17-year-old yellowtail snapper.

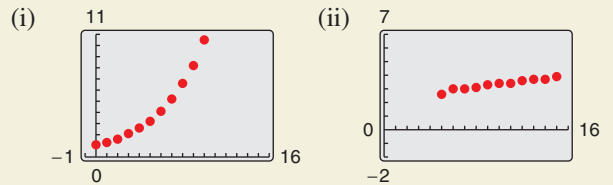
## Conclusions

**True or False?** In Exercises 39 and 40, determine whether the statement is true or false. Justify your answer.

39. The exponential model  $y = ae^{bx}$  represents a growth model when  $b > 0$ .
40. To change an exponential model of the form  $y = ab^x$  to one of the form  $y = ae^{cx}$ , rewrite  $b$  as  $b = \ln e^b$ .
41. **Writing** In your own words, explain how to fit a model to a set of data using a graphing utility.



**42. HOW DO YOU SEE IT?** Each graphing utility screen below shows a model that fits the set of data. The equations and  $r^2$ -values for the models are given. Determine the equation and coefficient of determination that represents each graph. Explain how you found your result.



$$y = 1.05e^{0.02288x}$$

$$y = 1.06 \ln x + 0.96$$

$$r^2 = 0.968$$

$$r^2 = 0.9991$$

## Cumulative Mixed Review

**Using the Slope-Intercept Form** In Exercises 43–46, find the slope and  $y$ -intercept of the equation of the line. Then sketch the line by hand.

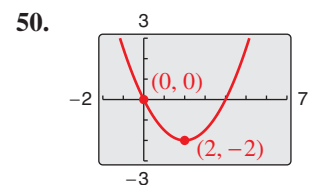
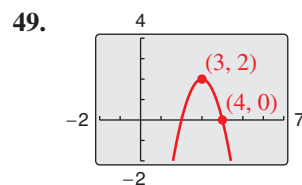
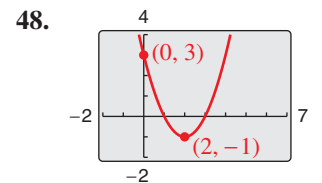
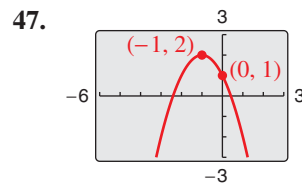
43.  $2x + 5y = 10$

44.  $3x - 2y = 9$

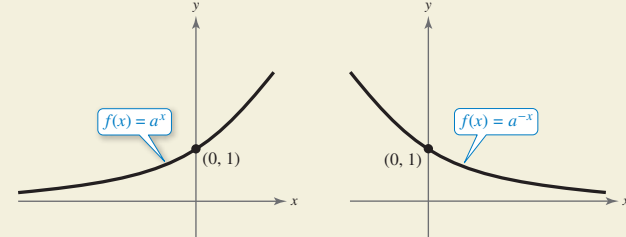
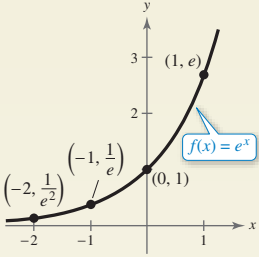
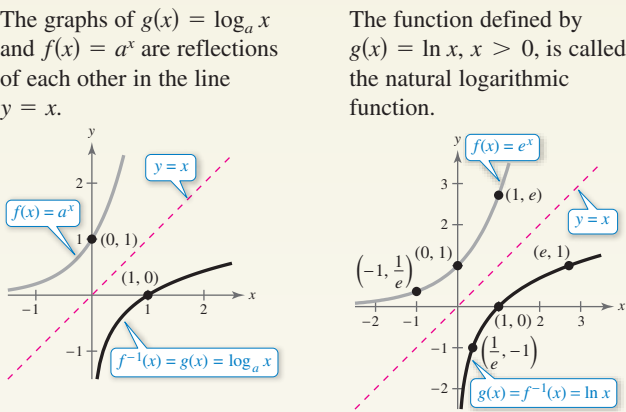
45.  $0.4x - 2.5y = 12.5$

46.  $1.2x + 3.5y = 10.5$

**Writing the Equation of a Parabola in Standard Form** In Exercises 47–50, write an equation of the parabola in standard form.



**3 Chapter Summary**

What did you learn?	Explanation and Examples	Review Exercises			
<p><b>Recognize and evaluate exponential functions with base <math>a</math> (p. 180).</b></p>	<p>The exponential function <math>f</math> with base <math>a</math> is denoted by <math>f(x) = a^x</math>, where <math>a &gt; 0</math>, <math>a \neq 1</math>, and <math>x</math> is any real number.</p>	1–4			
<p><b>Graph exponential functions with base <math>a</math> (p. 181).</b></p>		5–12			
<p><b>3.1 Recognize, evaluate, and graph exponential functions with base <math>e</math> (p. 184).</b></p>	<p>The function <math>f(x) = e^x</math> is called the natural exponential function.</p> 	13–18			
<p><b>Use exponential functions to model and solve real-life problems (p. 186).</b></p>	<p>Exponential functions are used in compound interest formulas (see Example 8) and in radioactive decay models (see Example 10).</p>	19–22			
<p><b>Recognize and evaluate logarithmic functions with base <math>a</math> (p. 192).</b></p>	<p>For <math>x &gt; 0</math>, <math>a &gt; 0</math>, and <math>a \neq 1</math>, <math>y = \log_a x</math> if and only if <math>x = a^y</math>. The function <math>f(x) = \log_a x</math> is called the logarithmic function with base <math>a</math>.</p>	23–36			
<p><b>3.2 Graph logarithmic functions with base <math>a</math> (p. 194), and recognize, evaluate, and graph natural logarithmic functions (p. 196).</b></p>	<p>The graphs of <math>g(x) = \log_a x</math> and <math>f(x) = a^x</math> are reflections of each other in the line <math>y = x</math>.</p> 	37–48			
<p><b>Use logarithmic functions to model and solve real-life problems (p. 198).</b></p>	<p>A logarithmic function is used in the human memory model. (See Example 10.)</p>	49, 50			
<p><b>3.3 Rewrite logarithms with different bases (p. 203).</b></p>	<p>Let <math>a</math>, <math>b</math>, and <math>x</math> be positive real numbers such that <math>a \neq 1</math> and <math>b \neq 1</math>. Then <math>\log_a x</math> can be converted to a different base as follows.</p> <table style="width: 100%; border: none;"> <tr> <td style="text-align: center; padding: 0 20px;"> <p>Base <math>b</math></p> <math display="block">\log_a x = \frac{\log_b x}{\log_b a}</math> </td> <td style="text-align: center; padding: 0 20px;"> <p>Base 10</p> <math display="block">\log_a x = \frac{\log_{10} x}{\log_{10} a}</math> </td> <td style="text-align: center; padding: 0 20px;"> <p>Base <math>e</math></p> <math display="block">\log_a x = \frac{\ln x}{\ln a}</math> </td> </tr> </table>	<p>Base <math>b</math></p> $\log_a x = \frac{\log_b x}{\log_b a}$	<p>Base 10</p> $\log_a x = \frac{\log_{10} x}{\log_{10} a}$	<p>Base <math>e</math></p> $\log_a x = \frac{\ln x}{\ln a}$	51–58
<p>Base <math>b</math></p> $\log_a x = \frac{\log_b x}{\log_b a}$	<p>Base 10</p> $\log_a x = \frac{\log_{10} x}{\log_{10} a}$	<p>Base <math>e</math></p> $\log_a x = \frac{\ln x}{\ln a}$			

	What did you learn?	Explanation and Examples	Review Exercises
3.3	Use properties of logarithms to evaluate, rewrite, expand, or condense logarithmic expressions (p. 204).	Let $a$ be a real positive number ( $a \neq 1$ ), $n$ be a real number, and $u$ and $v$ be positive real numbers. <b>1. Product Property:</b> $\log_a(uv) = \log_a u + \log_a v$ $\ln(uv) = \ln u + \ln v$ <b>2. Quotient Property:</b> $\log_a(u/v) = \log_a u - \log_a v$ $\ln(u/v) = \ln u - \ln v$ <b>3. Power Property:</b> $\log_a u^n = n \log_a u$ , $\ln u^n = n \ln u$	59–78
	Use logarithmic functions to model and solve real-life problems (p. 206).	Logarithmic functions can be used to find an equation that relates the periods of several planets and their distances from the sun. (See Example 7.)	79, 80
3.4	Solve simple exponential and logarithmic equations (p. 210).	Solve simple exponential or logarithmic equations using the One-to-One Properties and Inverse Properties of exponential and logarithmic functions.	81–94
	Solve more complicated exponential (p. 211) and logarithmic (p. 213) equations.	To solve more complicated equations, rewrite the equations so that the One-to-One Properties and Inverse Properties of exponential and logarithmic functions can be used. (See Examples 2–9.)	95–118
	Use exponential and logarithmic equations to model and solve real-life problems (p. 216).	Exponential and logarithmic equations can be used to find how long it will take to double an investment (see Example 12) and to find the year in which the average salary for public school teachers reached \$50,000 (see Example 13).	119, 120
3.5	Recognize the five most common types of models involving exponential or logarithmic functions (p. 221).	<b>1. Exponential growth model:</b> $y = ae^{bx}$ , $b > 0$ <b>2. Exponential decay model:</b> $y = ae^{-bx}$ , $b > 0$ <b>3. Gaussian model:</b> $y = ae^{-(x-b)^2/c}$ <b>4. Logistic growth model:</b> $y = \frac{a}{1 + be^{-rx}}$ <b>5. Logarithmic models:</b> $y = a + b \ln x$ , $y = a + b \log_{10} x$	121–126
	Use exponential growth and decay functions to model and solve real-life problems (p. 222).	An exponential growth function can be used to model the world population (see Example 1) and an exponential decay function can be used to estimate the age of a fossil (see Example 3).	127
	Use Gaussian functions (p. 225), logistic growth functions (p. 226), and logarithmic functions (p. 227) to model and solve real-life problems.	A Gaussian function can be used to model SAT mathematics scores for college-bound seniors (see Example 4). A logistic growth function can be used to model the spread of a flu virus (see Example 5). A logarithmic function can be used to find the intensity of an earthquake using its magnitude (see Example 6).	128–130
3.6	Classify scatter plots (p. 233), and use scatter plots and a graphing utility to find models for data and choose the model that best fits a set of data (p. 234).	You can use a scatter plot and a graphing utility to choose a model that best fits a set of data that represents the yield of a chemical reaction. (See Example 3.)	131, 132
	Use a graphing utility to find exponential and logistic models for data (p. 236).	An exponential model can be used to estimate the amount of a radioactive substance remaining after a given day (see Example 4), and a logistic model can be used to estimate the percent of defoliation caused by the gypsy moth (see Example 5).	133, 134

### 3 Review Exercises

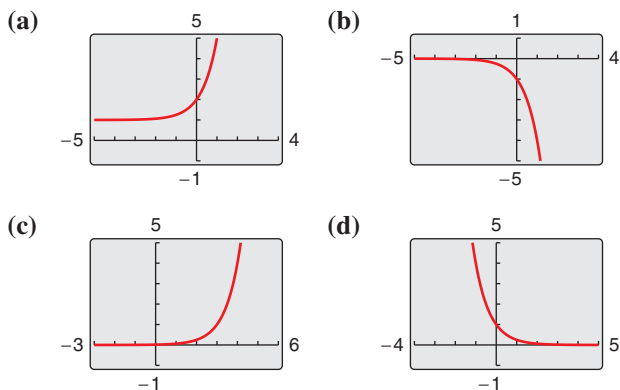
See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

#### 3.1

**Evaluating Exponential Functions** In Exercises 1–4, use a calculator to evaluate the function at the indicated value of  $x$ . Round your result to four decimal places.

1.  $f(x) = 1.45^x$ ,  $x = 2\pi$
2.  $f(x) = 7^x$ ,  $x = \sqrt{11}$
3.  $g(x) = 60^{2x}$ ,  $x = -0.5$
4.  $g(x) = 25^{-3x}$ ,  $x = \frac{3}{2}$

**Library of Parent Functions** In Exercises 5–8, use the graph of  $y = 4^x$  to match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]



5.  $f(x) = 4^{x-3}$
6.  $f(x) = 4^{-x}$
7.  $f(x) = -4^x$
8.  $f(x) = 4^x + 1$

**Graphs of  $y = a^x$  and  $y = a^{-x}$**  In Exercises 9–12, graph the exponential function by hand. Identify any asymptotes and intercepts and determine whether the graph of the function is increasing or decreasing.

9.  $f(x) = 8^x$
10.  $f(x) = 0.3^x$
11.  $g(x) = 8^{-x}$
12.  $g(x) = 0.3^{-x}$

**Graphing Natural Exponential Functions** In Exercises 13–18, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function. Identify any asymptotes of the graph.

13.  $h(x) = e^{x-1}$
14.  $f(x) = e^{x+2}$
15.  $h(x) = -e^x + 3$
16.  $f(x) = -2e^{-x}$
17.  $f(x) = 4e^{-0.5x}$
18.  $f(x) = 2 + e^{x-3}$

**Finding the Balance for Compound Interest** In Exercises 19 and 20, complete the table to determine the balance  $A$  for \$10,000 invested at rate  $r$  for  $t$  years, compounded continuously.

$t$	1	10	20	30	40	50
$A$						

19.  $r = 8\%$
20.  $r = 3\%$

21. **Economics** A new SUV costs \$30,795. The value  $V$  of the SUV after  $t$  years is modeled by  $V(t) = 30,795\left(\frac{4}{5}\right)^t$ .
  - (a) Use a graphing utility to graph the function.
  - (b) Find the value of the SUV after 2 years.
  - (c) According to the model, when does the SUV depreciate most rapidly? Is this realistic? Explain.
22. **Radioactive Decay** Let  $Q$  represent the mass, in grams, of a quantity of plutonium 241 ( $^{241}\text{Pu}$ ), whose half-life is about 14 years. The quantity of plutonium present after  $t$  years is given by  $Q = 50\left(\frac{1}{2}\right)^{t/14}$ .
  - (a) Determine the initial quantity (when  $t = 0$ ).
  - (b) Determine the quantity present after 10 years.
  - (c) Use a graphing utility to graph the function over the interval  $t = 0$  to  $t = 50$ .

#### 3.2

**Rewriting Equations** In Exercises 23–32, write the logarithmic equation in exponential form, or write the exponential equation in logarithmic form.

23.  $\log_5 625 = 4$
24.  $\log_9 81 = 2$
25.  $\log_{64} 2 = \frac{1}{6}$
26.  $\log_{10}\left(\frac{1}{100}\right) = -2$
27.  $4^3 = 64$
28.  $3^5 = 243$
29.  $e^{3.2} = 24.532 \dots$
30.  $\ln 5 = 1.6094 \dots$
31.  $\left(\frac{1}{2}\right)^{-3} = 8$
32.  $\left(\frac{2}{3}\right)^{-2} = \frac{9}{4}$

**Evaluating Logarithms** In Exercises 33–36, use the definition of a logarithmic function to evaluate the function at the indicated value of  $x$  without using a calculator.

Function	Value
33. $f(x) = \log_6 x$	$x = 216$
34. $f(x) = \log_7 x$	$x = 1$
35. $f(x) = \log_4 x$	$x = \frac{1}{4}$
36. $f(x) = \log_{10} x$	$x = 0.00001$

**Sketching the Graph of a Logarithmic Function** In Exercises 37–40, find the domain, vertical asymptote, and  $x$ -intercept of the logarithmic function, and sketch its graph by hand.

37.  $g(x) = -\log_{10} x + 5$
38.  $g(x) = \log_{10}(x - 3)$
39.  $f(x) = \log_{10}(x - 1) + 6$
40.  $f(x) = \log_{10}(x + 2) - 3$

**Evaluating the Natural Logarithmic Function** In Exercises 41–44, use a calculator to evaluate the function  $f(x) = \ln x$  at the indicated value of  $x$ . Round your result to three decimal places.

41.  $x = 21.5$
42.  $x = 0.46$
43.  $x = \sqrt{6}$
44.  $x = \frac{5}{6}$



**Analyzing Graphs of Functions** In Exercises 45–48, use a graphing utility to graph the logarithmic function. Determine the domain and identify any vertical asymptote and  $x$ -intercept.

45.  $f(x) = \ln x + 3$       46.  $f(x) = \ln(x - 3)$   
 47.  $h(x) = \frac{1}{2} \ln x$       48.  $f(x) = 4 \ln x$

49. **Aeronautics** The time  $t$  (in minutes) for a small plane to climb to an altitude of  $h$  feet is given by

$$t = 50 \log_{10} [18,000 / (18,000 - h)]$$

where 18,000 feet is the plane's absolute ceiling.

- Determine the domain of the function appropriate for the context of the problem.
- Use a graphing utility to graph the function and identify any asymptotes.
- As the plane approaches its absolute ceiling, what can be said about the time required to further increase its altitude?
- Find the amount of time it will take for the plane to climb to an altitude of 4000 feet.

50. **Real Estate** The model

$$t = 12.542 \ln [x / (x - 1000)], \quad x > 1000$$

approximates the length of a home mortgage of \$150,000 at 8% in terms of the monthly payment. In the model,  $t$  is the length of the mortgage in years and  $x$  is the monthly payment in dollars.

- Use the model to approximate the length of a \$150,000 mortgage at 8% when the monthly payment is \$1254.68.
- Approximate the total amount paid over the term of the mortgage with a monthly payment of \$1254.68. What amount of the total is interest costs?

### 3.3

**Changing the Base** In Exercises 51–54, evaluate the logarithm using the change-of-base formula. Do each problem twice, once with common logarithms and once with natural logarithms. Round your results to three decimal places.

51.  $\log_4 9$       52.  $\log_{1/2} 10$   
 53.  $\log_{14} 15.6$       54.  $\log_3 0.28$

**Graphing a Logarithm with Any Base** In Exercises 55–58, use the change-of-base formula and a graphing utility to graph the function.

55.  $f(x) = \log_2(x - 2)$   
 56.  $f(x) = 2 - \log_3 x$   
 57.  $f(x) = -\log_{1/2}(x - 2)$   
 58.  $f(x) = \log_{1/3}(x - 1) + 1$

**Using Properties to Evaluate Logarithms** In Exercises 59–62, approximate the logarithm using the properties of logarithms, given the values  $\log_b 2 \approx 0.3562$ ,  $\log_b 3 \approx 0.5646$ , and  $\log_b 5 \approx 0.8271$ .

59.  $\log_b 9$       60.  $\log_b \frac{4}{9}$   
 61.  $\log_b \sqrt{5}$       62.  $\log_b 50$

**Simplifying a Logarithm** In Exercises 63–66, use the properties of logarithms to rewrite and simplify the logarithmic expression.

63.  $\ln(5e^{-2})$   
 64.  $\ln \sqrt{e^5}$   
 65.  $\log_{10} 200$   
 66.  $\log_{10} 0.002$

**Expanding Logarithmic Expressions** In Exercises 67–72, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

67.  $\log_5 5x^2$       68.  $\log_4 16xy^2$   
 69.  $\log_{10} \frac{5\sqrt{y}}{x^2}$       70.  $\ln \frac{\sqrt{x}}{4}$   
 71.  $\ln \frac{x+3}{xy}$       72.  $\ln \frac{xy^5}{\sqrt{z}}$

**Condensing Logarithmic Expressions** In Exercises 73–78, condense the expression to the logarithm of a single quantity.

73.  $\log_2 9 + \log_2 x$   
 74.  $\log_6 y - 2 \log_6 z$   
 75.  $\frac{1}{2} \ln(2x - 1) - 2 \ln(x + 1)$   
 76.  $5 \ln(x - 2) - \ln(x + 2) + 3 \ln x$   
 77.  $\ln 3 + \frac{1}{3} \ln(4 - x^2) - \ln x$   
 78.  $3[\ln x - 2 \ln(x^2 + 1)] + 2 \ln 5$

79. **Public Service** The number of miles  $s$  of roads cleared of snow in 1 hour is approximated by the model

$$s = 25 - \frac{13 \ln(h/12)}{\ln 3}, \quad 2 \leq h \leq 15$$

where  $h$  is the depth of the snow (in inches).

- Use a graphing utility to graph the function.
- Complete the table.

$h$	4	6	8	10	12	14
$s$						

- Using the graph of the function and the table, what conclusion can you make about the number of miles of roads cleared as the depth of the snow increases?

**80. Psychology** Students in a sociology class were given an exam and then retested monthly with an equivalent exam. The average scores for the class are given by the human memory model  $f(t) = 85 - 17 \log_{10}(t + 1)$ , where  $t$  is the time in months and  $0 \leq t \leq 10$ . When will the average score decrease to 68?

**3.4**

**Solving an Exponential or Logarithmic Equation** In Exercises 81–94, solve the equation without using a calculator.

- 81.  $10^x = 10,000$
- 82.  $7^x = 343$
- 83.  $6^x = \frac{1}{216}$
- 84.  $6^{x-2} = \frac{1}{1296}$
- 85.  $2^{x+1} = \frac{1}{16}$
- 86.  $4^{x/2} = 64$
- 87.  $\log_8 x = 4$
- 88.  $\log_x 729 = 6$
- 89.  $\log_2(x - 1) = 3$
- 90.  $\log_5(2x + 27) = 2$
- 91.  $\ln x = 4$
- 92.  $\ln x = -3$
- 93.  $\ln(x - 1) = 2$
- 94.  $\ln(2x + 1) = -4$

**Solving an Exponential Equation** In Exercises 95–104, solve the exponential equation algebraically. Round your result to three decimal places.

- 95.  $3e^{-5x} = 132$
- 96.  $14e^{3x+2} = 560$
- 97.  $2^x + 13 = 35$
- 98.  $6^x - 28 = -8$
- 99.  $-4(5^x) = -68$
- 100.  $2(12^x) = 190$
- 101.  $2e^{x-3} - 1 = 4$
- 102.  $-e^{x/2} + 1 = \frac{1}{2}$
- 103.  $e^{2x} - 7e^x + 10 = 0$
- 104.  $e^{2x} - 6e^x + 8 = 0$

**Solving a Logarithmic Equation** In Exercises 105–114, solve the logarithmic equation algebraically. Round your result to three decimal places.

- 105.  $\ln 3x = 6.4$
- 106.  $\ln 5x = 4.5$
- 107.  $\ln x - \ln 5 = 2$
- 108.  $\ln x - \ln 3 = 4$
- 109.  $\ln \sqrt{x+1} = 2$
- 110.  $\ln \sqrt{x+40} = 3$
- 111.  $\log_4(x+5) = \log_4(13-x) - \log_4(x-3)$
- 112.  $\log_5(x+2) - \log_5 x = \log_5(x+5)$
- 113.  $\log_{10}(1-x) = -1$
- 114.  $\log_{10}(-x-4) = 2$

**f Solving an Exponential or Logarithmic Equation** In Exercises 115–118, solve the equation algebraically. Round your result to three decimal places.

- 115.  $xe^x + e^x = 0$
- 116.  $2xe^{2x} + e^{2x} = 0$
- 117.  $x \ln x + x = 0$
- 118.  $\frac{1 + \ln x}{x^2} = 0$

**119. Finance** You deposit \$7550 in an account that pays 6.9% interest, compounded continuously. How long will it take for the money to double?

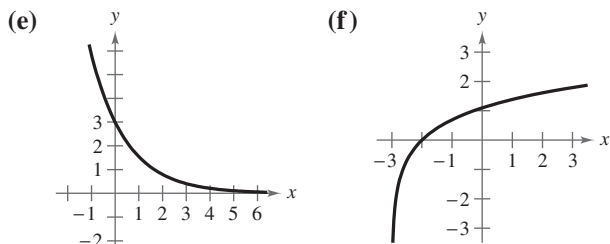
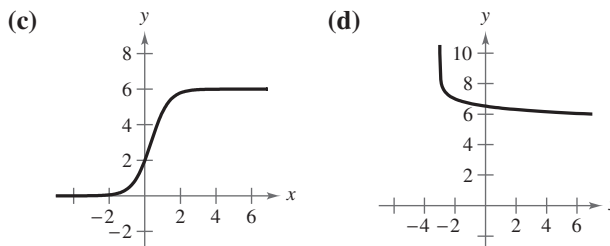
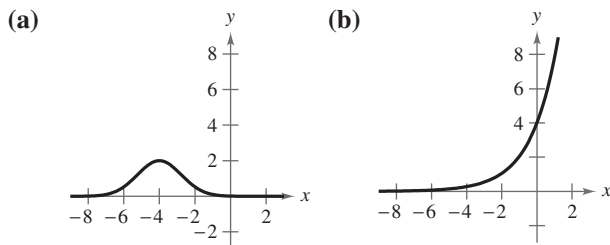
**120. Economics** The demand  $x$  for a 32-inch plasma television is modeled by

$$p = 5000 \left( 1 - \frac{4}{4 + e^{-0.0005x}} \right)$$

Find the demands  $x$  for prices of (a)  $p = \$500$  and (b)  $p = \$350$ .

**3.5**

**Identifying Graphs of Models** In Exercises 121–126, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]



- 121.  $y = 3e^{-2x/3}$
- 122.  $y = 4e^{2x/3}$
- 123.  $y = \ln(x + 3)$
- 124.  $y = 7 - \log_{10}(x + 3)$
- 125.  $y = 2e^{-(x+4)^2/3}$
- 126.  $y = \frac{6}{1 + 2e^{-2x}}$

**127. Demography** The populations  $P$  (in millions) of North Carolina from 2000 through 2012 can be modeled by  $P = 8.06e^{kt}$ , where  $t$  is the year, with  $t = 0$  corresponding to 2000. In 2001, the population was about 8,210,000. Find the value of  $k$  and use the result to predict the population in the year 2020. (Source: U.S. Census Bureau)

**128. Education** The scores for a biology test follow a normal distribution modeled by  $y = 0.0499e^{-(x-74)^2/128}$ , where  $x$  is the test score and  $40 \leq x \leq 100$ .  
 (a) Use a graphing utility to graph the function.  
 (b) Use the graph to estimate the average test score.

129. **Education** The average number  $N$  of words per minute that the students in a first grade class could read orally after  $t$  weeks of school is modeled by

$$N = \frac{62}{1 + 5.4e^{-0.24t}}$$

Find the numbers of weeks it took the class to read at average rates of (a) 45 words per minute and (b) 61 words per minute.

130. **Geology** On the Richter scale, the magnitude  $R$  of an earthquake of intensity  $I$  is modeled by

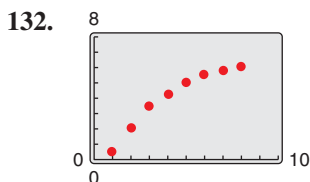
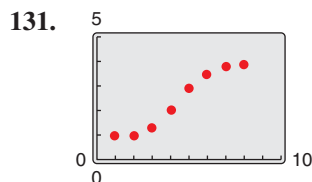
$$R = \log_{10} \frac{I}{I_0}$$

where  $I_0 = 1$  is the minimum intensity used for comparison. Find the intensities  $I$  of the following earthquakes measuring  $R$  on the Richter scale.

- (a)  $R = 7.1$     (b)  $R = 9.2$     (c)  $R = 3.8$

### 3.6

**Classifying Scatter Plots** In Exercises 131 and 132, determine whether the scatter plot could best be modeled by a linear model, an exponential model, a logarithmic model, or a logistic model.



### 133. MODELING DATA

Each ordered pair  $(t, N)$  represents the year  $t$  and the number  $N$  (in thousands) of female participants in high school athletic programs during 13 school years, with  $t = 1$  corresponding to the 2000–2001 school year. (*Spreadsheet at LarsonPrecalculus.com*) (Source: National Federation of State High School Associations)



(1, 2784), (2, 2807), (3, 2856), (4, 2865),  
(5, 2908), (6, 2953), (7, 3022), (8, 3057),  
(9, 3114), (10, 3173), (11, 3174), (12, 3208),  
(13, 3223)

- Use the *regression* feature of a graphing utility to find a linear model, an exponential model, and a power model for the data and identify the coefficient of determination for each model.
- Use the graphing utility to graph each model with the original data.
- Determine which model best fits the data. Explain.
- Use the model you chose in part (c) to predict the school year in which about 3,697,000 girls will participate.

### 134. MODELING DATA

You plant a tree when it is 1 meter tall and check its height  $h$  (in meters) every 10 years, as shown in the table.

Year	Height, $h$
0	1
10	3
20	7.5
30	14.5
40	19
50	20.5
60	21

- Use the *regression* feature of a graphing utility to find a logistic model for the data. Let  $x$  represent the year.
- Use the graphing utility to graph the model with the original data.
- How closely does the model represent the data?
- What is the limiting height of the tree?

### Conclusions

**True or False?** In Exercises 135–138, determine whether the equation or statement is true or false. Justify your answer.

135.  $e^{x-1} = \frac{e^x}{e}$

136.  $\ln(x + y) = \ln(xy)$

- The domain of the function  $f(x) = \ln x$  is the set of all real numbers.
- The logarithm of the quotient of two numbers is equal to the difference of the logarithms of the numbers.
- Think About It** Without using a calculator, explain why you know that  $2\sqrt{2}$  is greater than 2, but less than 4.

### 140. Exploration

- Use a graphing utility to compare the graph of the function  $y = e^x$  with the graph of each function below. [ $n!$  (read as “ $n$  factorial”) is defined as  $n! = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot (n-1) \cdot n$ .]  

$$y_1 = 1 + \frac{x}{1!}, y_2 = 1 + \frac{x}{1!} + \frac{x^2}{2!}$$

$$y_3 = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$$
- Identify the pattern of successive polynomials given in part (a). Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of  $y = e^x$ . What do you think this pattern implies?

## 3 Chapter Test

See [CalcChat.com](http://CalcChat.com) for tutorial help and worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, use a graphing utility to construct a table of values for the function. Then sketch a graph of the function. Identify any asymptotes and intercepts.

1.  $f(x) = 10^{-x}$                       2.  $f(x) = -6^{x-2}$                       3.  $f(x) = 1 - e^{2x}$

In Exercises 4–6, evaluate the expression.

4.  $\log_7 7^{-0.89}$                       5.  $4.6 \ln e^2$                       6.  $5 - \log_{10} 1000$

In Exercises 7–9, find the domain, vertical asymptote, and  $x$ -intercept of the logarithmic function, and sketch its graph by hand.

7.  $f(x) = -\log_{10} x - 6$                       8.  $f(x) = \ln(x + 4)$                       9.  $f(x) = 1 + \ln(x - 6)$

In Exercises 10–12, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

10.  $\log_7 44$                       11.  $\log_{2/5} 1.3$                       12.  $\log_{12} 64$

In Exercises 13–15, use the properties of logarithms to expand the expression as a sum, difference, and/or multiple of logarithms.

13.  $\log_2 3a^4$                       14.  $\ln \frac{5\sqrt{x}}{6}$                       15.  $\ln \frac{x\sqrt{x+1}}{2e^4}$

In Exercises 16–18, condense the expression.

16.  $\log_3 13 + \log_3 y$                       17.  $4 \ln x - 4 \ln y$   
18.  $\ln x - \ln(x + 2) + \ln(2x - 3)$

In Exercises 19–22, solve the equation.

19.  $3^x = 81$                       20.  $5^{2x} = 2500$   
21.  $\log_7 x = 6$                       22.  $\log_{10}(x - 4) = 5$

In Exercises 23–26, solve the equation algebraically. Round your result to three decimal places.

23.  $\frac{1025}{8 + e^{4x}} = 5$                       24.  $-xe^{-x} + e^{-x} = 0$   
25.  $\log_{10} x - \log_{10}(8 - 5x) = 2$                       26.  $2x \ln x - x = 0$

27. The half-life of radioactive actinium ( $^{227}\text{Ac}$ ) is about 22 years. What percent of a present amount of radioactive actinium will remain after 15 years?

28. The table shows the annual revenues  $R$  (in billions of dollars) for Amazon.com from 2006 through 2013. (Source: [Amazon.com](http://Amazon.com))

(a) Use the *regression* feature of a graphing utility to find a logarithmic model, an exponential model, and a power model for the data. Let  $t$  represent the year, with  $t = 6$  corresponding to 2006.

(b) Use the graphing utility to graph each model with the original data.

(c) Determine which model best fits the data. Use the model to predict the revenue of Amazon.com in 2019.

DATA	Year	Revenue, $R$
	2006	10.711
	2007	14.836
	2008	19.166
	2009	24.509
	2010	34.204
	2011	48.077
	2012	61.093
	2013	74.452

Spreadsheet at  
[LarsonPrecalculus.com](http://LarsonPrecalculus.com)

Table for 28

## 1–3 Cumulative Test

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Take this test to review the material in Chapters 1–3. After you are finished, check your work against the answers given in the back of the book.

In Exercises 1–3, (a) write the slope-intercept form of the equation of the line that satisfies the given conditions and (b) find three additional points through which the line passes.

- The line passes through the points  $(-5, 8)$  and  $(-1, 4)$ .
- The line passes through the point  $(-\frac{1}{2}, 1)$  and has a slope of  $-2$ .
- The line has an undefined slope and passes through the point  $(\frac{3}{7}, \frac{1}{8})$ .

In Exercises 4 and 5, evaluate the function at each value of the independent variable and simplify.

4.  $f(x) = \frac{x}{x-2}$

(a)  $f(5)$    (b)  $f(2)$    (c)  $f(5 + 4s)$

5.  $f(x) = \begin{cases} 3x - 8, & x < 0 \\ x^2 + 4, & x \geq 0 \end{cases}$

(a)  $f(-8)$    (b)  $f(0)$    (c)  $f(4)$

- Does the graph at the right represent  $y$  as a function of  $x$ ? Explain.
- Use a graphing utility to graph the function  $f(x) = 2|x - 5| - |x + 5|$ . Then determine the open intervals over which the function is increasing, decreasing, or constant.
- Compare the graph of each function with the graph of  $f(x) = \sqrt{x}$ .  
(a)  $r(x) = \frac{1}{4}\sqrt{x}$    (b)  $h(x) = \sqrt{x} - 3$    (c)  $g(x) = -\sqrt{x + 3}$

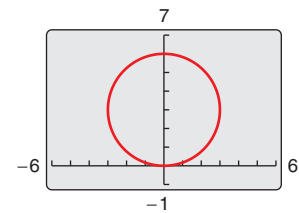


Figure for 6

In Exercises 9–12, evaluate the indicated function for

$f(x) = -x^2 + 3x - 10$    and    $g(x) = 4x + 1$ .

- $(f + g)(-4)$
- $(g - f)(\frac{3}{4})$
- $(g \circ f)(2)$
- $(fg)(-1)$

- Determine whether  $h(x) = 5x - 2$  has an inverse function. If so, find it.

In Exercises 14–16, sketch the graph of the function. Use a graphing utility to verify the graph.

- $f(x) = -(x - 2)^2 + 5$
- $f(x) = x^2 - 6x + 5$
- $f(x) = x^3 + 3x^2 - 4x - 12$
- Find all the zeros of  $f(x) = x^3 + 2x^2 + 4x + 8$ .
- Use a graphing utility to approximate any real zeros of  $g(x) = x^3 + 5x^2 + 4$  accurate to three decimal places.
- Divide  $(4x^2 + 14x - 9)$  by  $(x + 3)$  using long division.
- Divide  $(2x^3 - 5x^2 + 6x - 20)$  by  $(x - 6)$  using synthetic division.
- Multiply the complex number  $-8 + 3i$  by its complex conjugate.
- Find a polynomial function with real coefficients that has the zeros  $0$ ,  $-3$ , and  $1 + \sqrt{5}i$ .

In Exercises 23–26, sketch the graph of the rational function. Identify any asymptotes. Use a graphing utility to verify your graph.

23.  $f(x) = \frac{2x}{x-3}$

24.  $f(x) = \frac{5x}{x^2 + x - 6}$

25.  $f(x) = \frac{x^2 - 3x + 8}{x - 2}$

26.  $f(x) = \frac{4x^3}{x^3 + 5x}$

In Exercises 27–30, use a calculator to evaluate the expression. Round your answer to three decimal places.

27.  $(1.49)^{3.1}$

28.  $58\sqrt{5}$

29.  $e^{-8/5}$

30.  $4e^{2.56}$

In Exercises 31–34, sketch the graph of the function by hand. Use a graphing utility to verify your graph.

31.  $f(x) = -3^{x+4} - 5$

32.  $f(x) = -\left(\frac{1}{2}\right)^{-x} + 3$

33.  $f(x) = 4 + \log_{10}(x - 3)$

34.  $f(x) = \ln(4 - x)$

In Exercises 35–37, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

35.  $\log_5 16$

36.  $\log_9 7.3$

37.  $\log_2\left(\frac{3}{2}\right)$

38. Use the properties of logarithms to expand  $\ln \frac{x^2 - 4}{x^2 + 1}$ .

39. Write  $2 \ln x - \ln(x - 1) + \ln(x + 1)$  as a logarithm of a single quantity.

In Exercises 40–45, solve the equation algebraically. Round your result to three decimal places and verify your result graphically.

40.  $6e^{2x} = 72$

41.  $4^{x-5} + 21 = 30$

42.  $\log_2 x + \log_2 5 = 6$

43.  $2500e^{0.05x} = 500,000$

44.  $2x^2e^{2x} - 2xe^{2x} = 0$

45.  $\ln(2x - 5) - \ln x = 1$

46. A rectangular plot of land with a perimeter of 546 feet has a width of  $x$ .

(a) Write the area  $A$  of the plot as a function of  $x$ .

(b) Use a graphing utility to graph the area function. What is the domain of the function?

(c) Approximate the dimensions of the plot when the area is 15,000 square feet.

47. The table shows the annual sales  $S$  (in billions of dollars) for Wal-Mart from 2003 through 2012. (Source: Wal-Mart Stores, Inc.)

(a) Use the *regression* feature of a graphing utility to find a quadratic model, an exponential model, and a power model for the data and identify the coefficient of determination for each model. Let  $t$  represent the year, with  $t = 3$  corresponding to 2003.

(b) Use the graphing utility to graph each model with the original data.

(c) Determine which model best fits the data. Explain.

(d) Use the model you chose in part (c) to predict the annual sales of Wal-Mart in 2020. Is your answer reasonable? Explain.

	Year	Sales, $S$
Spreadsheet at LarsonPrecalculus.com	2003	258.681
	2004	287.989
	2005	315.654
	2006	348.650
	2007	378.799
	2008	405.607
	2009	408.214
	2010	421.849
	2011	446.950
	2012	469.162

Table for 47

**Proofs in Mathematics**

**Proofs in Mathematics**

Each of the following three properties of logarithms can be proved by using properties of exponential functions.

**Properties of Logarithms (p. 204)**

Let  $a$  be a positive real number such that  $a \neq 1$ , and let  $n$  be a real number. If  $u$  and  $v$  are positive real numbers, then the following properties are true.

	<i>Logarithm with Base <math>a</math></i>	<i>Natural Logarithm</i>
<b>1. Product Property:</b>	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
<b>2. Quotient Property:</b>	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
<b>3. Power Property:</b>	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

**Proof**

Let

$$x = \log_a u \quad \text{and} \quad y = \log_a v.$$

The corresponding exponential forms of these two equations are

$$a^x = u \quad \text{and} \quad a^y = v.$$

To prove the Product Property, multiply  $u$  and  $v$  to obtain

$$uv = a^x a^y = a^{x+y}.$$

The corresponding logarithmic form of  $uv = a^{x+y}$  is

$$\log_a(uv) = x + y.$$

So,

$$\log_a(uv) = \log_a u + \log_a v.$$

To prove the Quotient Property, divide  $u$  by  $v$  to obtain

$$\frac{u}{v} = \frac{a^x}{a^y} = a^{x-y}.$$

The corresponding logarithmic form of  $u/v = a^{x-y}$  is

$$\log_a \frac{u}{v} = x - y.$$

So,

$$\log_a \frac{u}{v} = \log_a u - \log_a v.$$

To prove the Power Property, substitute  $a^x$  for  $u$  in the expression  $\log_a u^n$ , as follows.

$$\begin{aligned} \log_a u^n &= \log_a (a^x)^n && \text{Substitute } a^x \text{ for } u. \\ &= \log_a a^{nx} && \text{Property of exponents} \\ &= nx && \text{Inverse Property of logarithms} \\ &= n \log_a u && \text{Substitute } \log_a u \text{ for } x. \end{aligned}$$

So,  $\log_a u^n = n \log_a u.$

**Slide Rules**

The slide rule was invented by William Oughtred (1574–1660) in 1625. The slide rule is a computational device with a sliding portion and a fixed portion. A slide rule enables you to perform multiplication by using the Product Property of logarithms. There are other slide rules that allow for the calculation of roots and trigonometric functions. Slide rules were used by mathematicians and engineers until the invention of the handheld calculator in 1972.



## Progressive Summary (Chapters 1–3)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, and 9. In each Progressive Summary, new topics encountered for the first time appear in blue.

Algebraic Functions	Transcendental Functions	Other Topics																														
<p><b>Polynomial, Rational, Radical</b></p>	<p><b>Exponential, Logarithmic</b></p>																															
<p>■ <b>Rewriting</b>                      Polynomial form ↔ Factored form                      Operations with polynomials                      Rationalize denominators                      Simplify rational expressions                      Operations with complex numbers</p>	<p>■ <b>Rewriting</b>                      Exponential form ↔ Logarithmic form                      Condense/expand logarithmic expressions</p>	<p>■ <b>Rewriting</b></p>																														
<p>■ <b>Solving</b></p> <table border="0"> <thead> <tr> <th><i>Equation</i></th> <th><i>Strategy</i></th> </tr> </thead> <tbody> <tr> <td>Linear . . . . .</td> <td>Isolate variable</td> </tr> <tr> <td>Quadratic . . . . .</td> <td>Factor, set to zero Extract square roots Complete the square Quadratic Formula</td> </tr> <tr> <td>Polynomial . . . . .</td> <td>Factor, set to zero Rational Zero Test</td> </tr> <tr> <td>Rational . . . . .</td> <td>Multiply by LCD</td> </tr> <tr> <td>Radical . . . . .</td> <td>Isolate, raise to power</td> </tr> <tr> <td>Absolute value . . .</td> <td>Isolate, form two equations</td> </tr> </tbody> </table>	<i>Equation</i>	<i>Strategy</i>	Linear . . . . .	Isolate variable	Quadratic . . . . .	Factor, set to zero Extract square roots Complete the square Quadratic Formula	Polynomial . . . . .	Factor, set to zero Rational Zero Test	Rational . . . . .	Multiply by LCD	Radical . . . . .	Isolate, raise to power	Absolute value . . .	Isolate, form two equations	<p>■ <b>Solving</b></p> <table border="0"> <thead> <tr> <th><i>Equation</i></th> <th><i>Strategy</i></th> </tr> </thead> <tbody> <tr> <td>Exponential . . . . .</td> <td>Take logarithm of each side</td> </tr> <tr> <td>Logarithmic . . . . .</td> <td>Exponentiate each side</td> </tr> </tbody> </table>	<i>Equation</i>	<i>Strategy</i>	Exponential . . . . .	Take logarithm of each side	Logarithmic . . . . .	Exponentiate each side	<p>■ <b>Solving</b></p>										
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