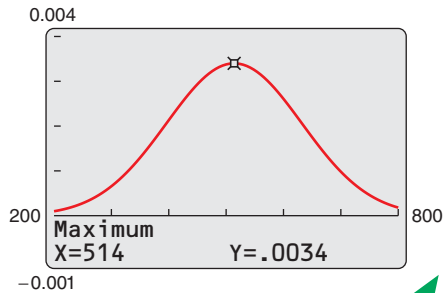


3

Exponential and Logarithmic Functions



Section 3.5, Example 4
SAT Scores

- 3.1 Exponential Functions and Their Graphs
- 3.2 Logarithmic Functions and Their Graphs
- 3.3 Properties of Logarithms
- 3.4 Solving Exponential and Logarithmic Equations
- 3.5 Exponential and Logarithmic Models
- 3.6 Nonlinear Models



3.1 Exponential Functions and Their Graphs

Exponential Functions

So far, this text has dealt mainly with **algebraic functions**, which include polynomial functions and rational functions. In this chapter, you will study two types of nonalgebraic functions—*exponential functions* and *logarithmic functions*. These functions are examples of **transcendental functions**.

What you should learn

- ▶ Recognize and evaluate exponential functions with base a .
- ▶ Graph exponential functions with base a .
- ▶ Recognize, evaluate, and graph exponential functions with base e .
- ▶ Use exponential functions to model and solve real-life problems.

Why you should learn it

Exponential functions are useful in modeling data that represent quantities that increase or decrease quickly. For instance, Exercise 82 on page 191 shows how an exponential function is used to model the depreciation of a new vehicle.



Definition of Exponential Function

The **exponential function f with base a** is denoted by

$$f(x) = a^x$$

where $a > 0$, $a \neq 1$, and x is any real number.

Note that in the definition of an exponential function, the base $a = 1$ is excluded because it yields

$$f(x) = 1^x = 1. \quad \text{Constant function}$$

This is a constant function, not an exponential function.

You have already evaluated a^x for integer and rational values of x . For example, you know that

$$4^3 = 64 \quad \text{and} \quad 4^{1/2} = 2.$$

However, to evaluate 4^x for any real number x , you need to interpret forms with *irrational* exponents. For the purposes of this text, it is sufficient to think of $a^{\sqrt{2}}$ (where $\sqrt{2} \approx 1.41421356$) as the number that has the successively closer approximations

$$a^{1.4}, a^{1.41}, a^{1.414}, a^{1.4142}, a^{1.41421}, \dots$$

Example 1 shows how to use a calculator to evaluate exponential functions.

EXAMPLE 1 Evaluating Exponential Functions

Use a calculator to evaluate each function at the indicated value of x .

<i>Function</i>	<i>Value</i>
a. $f(x) = 2^x$	$x = -3.1$
b. $f(x) = 2^{-x}$	$x = \pi$
c. $f(x) = 0.6^x$	$x = \frac{3}{2}$
d. $f(x) = 1.05^{2x}$	$x = 12$

Solution

<i>Function Value</i>	<i>Graphing Calculator Keystrokes</i>	<i>Display</i>
a. $f(-3.1) = 2^{-3.1}$	2 \wedge (−) 3.1 ENTER	0.1166291
b. $f(\pi) = 2^{-\pi}$	2 \wedge (−) π ENTER	0.1133147
c. $f(\frac{3}{2}) = (0.6)^{3/2}$.6 \wedge () 3 \div 2 ENTER	0.4647580
d. $f(12) = (1.05)^{2(12)}$	1.05 \wedge () 2 \times 12 ENTER	3.2250999

Technology Tip

When evaluating exponential functions with a calculator, remember to enclose fractional exponents in parentheses. Because the calculator follows the order of operations, parentheses are crucial in order to obtain the correct result.

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Use a calculator to evaluate $f(x) = 8^{-x}$ at $x = \sqrt{2}$.

Graphs of Exponential Functions

The graphs of all exponential functions have similar characteristics, as shown in Examples 2, 3, and 4.

EXAMPLE 2 Graphs of $y = a^x$

In the same coordinate plane, sketch the graphs of $f(x) = 2^x$ and $g(x) = 4^x$.

Solution

The table below lists some values for each function. By plotting these points and connecting them with smooth curves, you obtain the graphs shown in Figure 3.1. Note that both graphs are increasing. Moreover, the graph of $g(x) = 4^x$ is increasing more rapidly than the graph of $f(x) = 2^x$.

x	-3	-2	-1	0	1	2
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
4^x	$\frac{1}{64}$	$\frac{1}{16}$	$\frac{1}{4}$	1	4	16

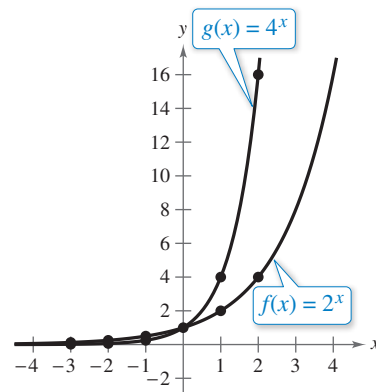


Figure 3.1

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In the same coordinate plane, sketch the graphs of $f(x) = 3^x$ and $g(x) = 9^x$.

EXAMPLE 3 Graphs of $y = a^{-x}$

In the same coordinate plane, sketch the graphs of $F(x) = 2^{-x}$ and $G(x) = 4^{-x}$.

Solution

The table below lists some values for each function. By plotting these points and connecting them with smooth curves, you obtain the graphs shown in Figure 3.2. Note that both graphs are decreasing. Moreover, the graph of $G(x) = 4^{-x}$ is decreasing more rapidly than the graph of $F(x) = 2^{-x}$.

x	-2	-1	0	1	2	3
2^{-x}	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
4^{-x}	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$

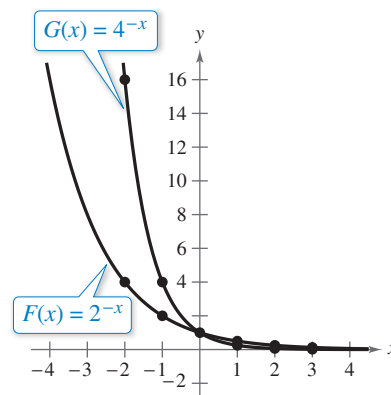


Figure 3.2

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In the same coordinate plane, sketch the graphs of $f(x) = 3^{-x}$ and $g(x) = 9^{-x}$. 

The properties of exponents can also be applied to real-number exponents. For review, these properties are listed below.

- $a^x a^y = a^{x+y}$
- $\frac{a^x}{a^y} = a^{x-y}$
- $a^{-x} = \frac{1}{a^x} = \left(\frac{1}{a}\right)^x$
- $a^0 = 1$
- $(ab)^x = a^x b^x$
- $(a^x)^y = a^{xy}$
- $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$
- $|a^2| = |a|^2 = a^2$

In Example 3, note that the functions $F(x) = 2^{-x}$ and $G(x) = 4^{-x}$ can be rewritten with positive exponents as

$$F(x) = 2^{-x} = \left(\frac{1}{2}\right)^x \quad \text{and} \quad G(x) = 4^{-x} = \left(\frac{1}{4}\right)^x.$$

Comparing the functions in Examples 2 and 3, observe that

$$F(x) = 2^{-x} = f(-x) \quad \text{and} \quad G(x) = 4^{-x} = g(-x).$$

Consequently, the graph of F is a reflection (in the y -axis) of the graph of f , as shown in Figure 3.3. The graphs of G and g have the same relationship, as shown in Figure 3.4.

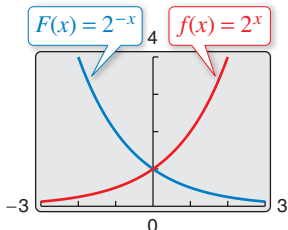


Figure 3.3

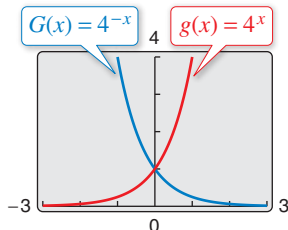


Figure 3.4

The graphs in Figures 3.3 and 3.4 are typical of the graphs of the exponential functions

$$f(x) = a^x \quad \text{and} \quad f(x) = a^{-x}.$$

They have one y -intercept and one horizontal asymptote (the x -axis), and they are continuous. The basic characteristics of these exponential functions are summarized below.

Encourage your students to use their graphing utilities to explore different aspects of exponential functions (such as those listed at the left and in Example 4) on their own.

Library of Parent Functions: Exponential Function

The parent exponential function

$$f(x) = a^x, \quad a > 0, \quad a \neq 1$$

is different from all the functions you have studied so far because the variable x is an *exponent*. A distinguishing characteristic of an exponential function is its rapid increase as x increases (for $a > 1$). Many real-life phenomena with patterns of rapid growth (or decline) can be modeled by exponential functions. The basic characteristics of the exponential function are summarized below and on the inside cover of this text.

Graph of $f(x) = a^x, a > 1$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

Intercept: $(0, 1)$

Increasing on $(-\infty, \infty)$

x -axis is a horizontal asymptote

$(a^x \rightarrow 0 \text{ as } x \rightarrow -\infty)$

Continuous

Graph of $f(x) = a^{-x}, a > 1$

Domain: $(-\infty, \infty)$

Range: $(0, \infty)$

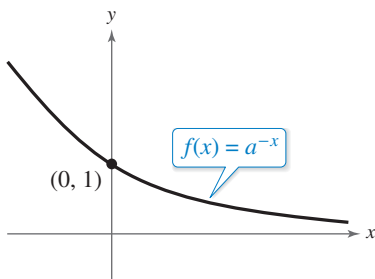
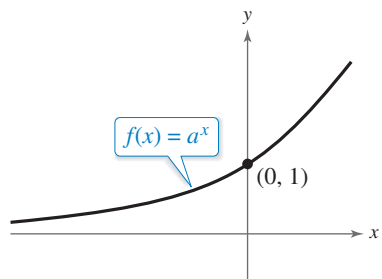
Intercept: $(0, 1)$

Decreasing on $(-\infty, \infty)$

x -axis is a horizontal asymptote

$(a^{-x} \rightarrow 0 \text{ as } x \rightarrow \infty)$

Continuous



Explore the Concept

Use a graphing utility to graph $y = a^x$ for $a = 3, 5,$ and 7 in the same viewing window. (Use a viewing window in which $-2 \leq x \leq 1$ and $0 \leq y \leq 2$.) How do the graphs compare with each other? Which graph is on the top in the interval $(-\infty, 0)$? Which is on the bottom? Which graph is on the top in the interval $(0, \infty)$? Which is on the bottom? Repeat this experiment with the graphs of $y = b^x$ for $b = \frac{1}{3}, \frac{1}{5},$ and $\frac{1}{7}$. (Use a viewing window in which $-1 \leq x \leq 2$ and $0 \leq y \leq 2$.) What can you conclude about the shape of the graph of $y = b^x$ and the value of b ?

In the next example, notice how the graph of $y = a^x$ can be used to sketch the graphs of functions of the form $f(x) = b \pm a^{x+c}$.

EXAMPLE 4 Library of Parent Functions: $f(x) = a^x$

See LarsonPrecalculus.com for an interactive version of this type of example.

Each of the following graphs is a transformation of the graph of $f(x) = 3^x$.

- Because $g(x) = 3^{x+1} = f(x+1)$, the graph of g can be obtained by shifting the graph of f one unit to the *left*, as shown in Figure 3.5.
- Because $h(x) = 3^x - 2 = f(x) - 2$, the graph of h can be obtained by shifting the graph of f *downward* two units, as shown in Figure 3.6.
- Because $k(x) = -3^x = -f(x)$, the graph of k can be obtained by *reflecting* the graph of f in the x -axis, as shown in Figure 3.7.
- Because $j(x) = 3^{-x} = f(-x)$, the graph of j can be obtained by *reflecting* the graph of f in the y -axis, as shown in Figure 3.8.

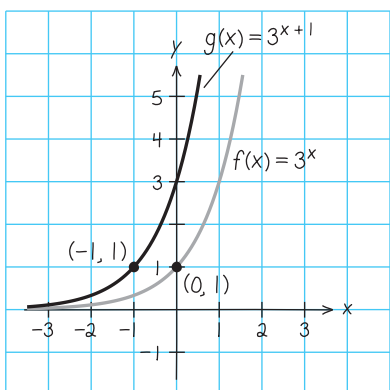


Figure 3.5

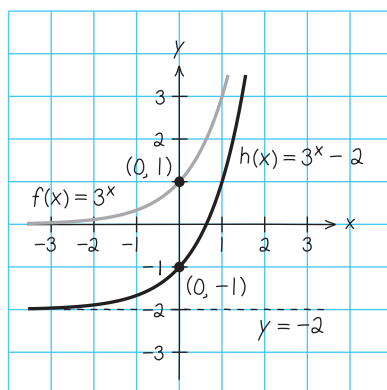


Figure 3.6

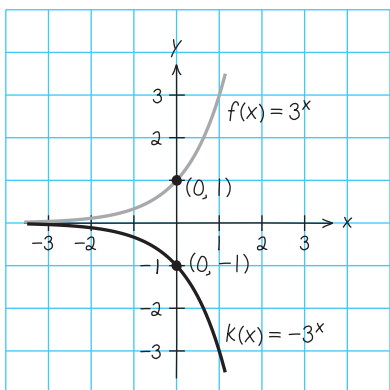


Figure 3.7

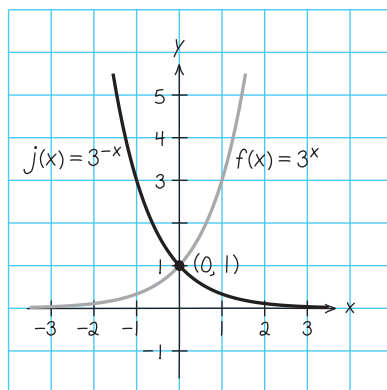


Figure 3.8

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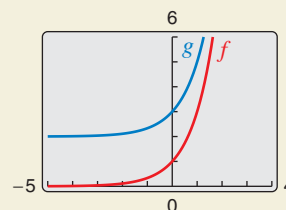
Use the graph of $f(x) = 4^x$ to describe the transformation that yields the graph of each function.

- $g(x) = 4^{x-2}$
- $h(x) = 4^x + 3$
- $k(x) = 4^{-x} - 3$

Notice that the transformations in Figures 3.5, 3.7, and 3.8 keep the x -axis ($y = 0$) as a horizontal asymptote, but the transformation in Figure 3.6 yields a new horizontal asymptote of $y = -2$. Also, be sure to note how the y -intercept is affected by each transformation.

What's Wrong?

You use a graphing utility to graph $f(x) = 3^x$ and $g(x) = 3^{x+2}$, as shown in the figure. You use the graph to conclude that the graph of g can be obtained by shifting the graph of f upward two units. What's wrong?



Explore the Concept

The following table shows some points on the graphs in Figure 3.5, where $Y_1 = f(x)$ and $Y_2 = g(x)$. Explain how you can use the table to describe the transformation.

X	Y ₁	Y ₂
-3	.03704	.11111
-2	.11111	.33333
-1	.33333	1
0	1	3
1	3	9
2	9	27
3	27	81

The Natural Base e

For many applications, the convenient choice for a base is the irrational number

$$e = 2.71828128 \dots$$

This number is called the **natural base**. The function $f(x) = e^x$ is called the **natural exponential function**, and its graph is shown in Figure 3.9. The graph of the natural exponential function has the same basic characteristics as the graph of the function $f(x) = a^x$ (see page 182). Be sure you see that for the natural exponential function $f(x) = e^x$, e is the constant 2.71828128 . . . , whereas x is the variable.

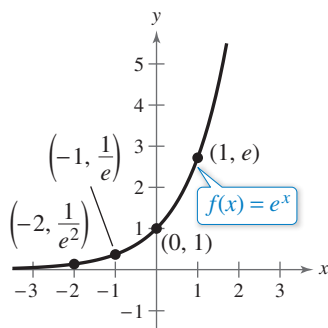


Figure 3.9 The Natural Exponential Function

In Example 5, you will see that, for large values of x , the number e can be approximated by the expression

$$\left(1 + \frac{1}{x}\right)^x.$$

Explore the Concept

Use your graphing utility to graph the functions

$$y_1 = 2^x$$

$$y_2 = e^x$$

$$y_3 = 3^x$$

in the same viewing window. From the relative positions of these graphs, make a guess as to the value of the real number e . Then try to find a number a such that the graphs of $y_2 = e^x$ and $y_4 = a^x$ are as close to each other as possible.

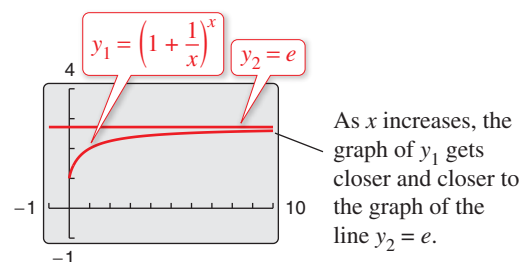
EXAMPLE 5 Approximation of the Number e

Evaluate the expression

$$\left(1 + \frac{1}{x}\right)^x$$

for several large values of x to see that the values approach $e \approx 2.71828128$ as x increases without bound.

Graphical Solution



Numerical Solution

Enter $y_1 = [1 + (1/x)]^x$.

Use the *table* feature (in *ask* mode) to evaluate y_1 for increasing values of x .

X	Y1
10	2.5937
100	2.7048
1000	2.7169
10000	2.7181
100000	2.7183
1E6	2.7183

X=2.71828046932

From the table, it seems reasonable to conclude that

$$\left(1 + \frac{1}{x}\right)^x \rightarrow e \text{ as } x \rightarrow \infty.$$

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Evaluate the expression $\left(1 + \frac{2}{x}\right)^x$ for several large values of x to see that the values approach $e^2 \approx 7.389056099$.

EXAMPLE 6 Evaluating the Natural Exponential Function

Use a calculator to evaluate the function $f(x) = e^x$ at each indicated value of x .

- $x = -2$
- $x = 0.25$
- $x = -0.4$
- $x = \frac{2}{3}$

Solution

Function Value	Graphing Calculator Keystrokes	Display
a. $f(-2) = e^{-2}$	e^x $(-)$ 2 ENTER	0.1353353
b. $f(0.25) = e^{0.25}$	e^x .25 ENTER	1.2840254
c. $f(-0.4) = e^{-0.4}$	e^x $(-)$.4 ENTER	0.6703200
d. $f(\frac{2}{3}) = e^{2/3}$	e^x $($ 2 \div 3 $)$ ENTER	1.9477340

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Use a calculator to evaluate the function $f(x) = e^x$ at each value of x .

- $x = 0.3$
- $x = -1.2$
- $x = 6.2$

EXAMPLE 7 Graphing Natural Exponential Functions

Sketch the graphs of $f(x) = 2e^{0.24x}$ and $g(x) = \frac{1}{2}e^{-0.58x}$.

Solution

To sketch these two graphs, you can use a calculator to construct a table of values, as shown below.

x	-3	-2	-1	0	1	2	3
$f(x)$	0.974	1.238	1.573	2.000	2.542	3.232	4.109
$g(x)$	2.849	1.595	0.893	0.500	0.280	0.157	0.088

After constructing the table, plot the points and connect them with smooth curves. Note that the graph in Figure 3.10 is increasing, whereas the graph in Figure 3.11 is decreasing. Use a graphing calculator to verify these graphs.

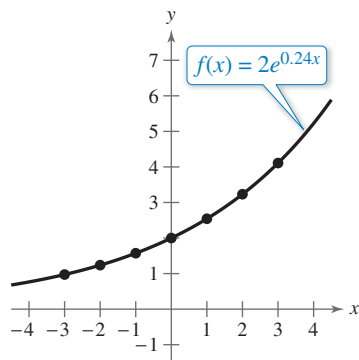


Figure 3.10

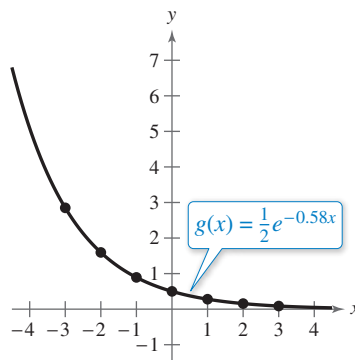


Figure 3.11

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Sketch the graph of $f(x) = 5e^{0.17x}$.

Explore the Concept

Use a graphing utility to graph $y = (1 + x)^{1/x}$. Describe the behavior of the graph near $x = 0$. Is there a y -intercept? How does the behavior of the graph near $x = 0$ relate to the result of Example 5? Use the *table* feature of the graphing utility to create a table that shows values of y for values of x near $x = 0$ to help you describe the behavior of the graph near this point.



Applications

One of the most familiar examples of exponential growth is an investment earning *continuously compounded interest*. Suppose a principal P is invested at an annual interest rate r , compounded once a year. If the interest is added to the principal at the end of the year, then the new balance P_1 is

$$P_1 = P + Pr = P(1 + r).$$

This pattern of multiplying the previous principal by $1 + r$ is then repeated each successive year, as shown in the table.

Time in years	Balance after each compounding
0	$P = P$
1	$P_1 = P(1 + r)$
2	$P_2 = P_1(1 + r) = P(1 + r)(1 + r) = P(1 + r)^2$
\vdots	\vdots
t	$P_t = P(1 + r)^t$

To accommodate more frequent (quarterly, monthly, or daily) compounding of interest, let n be the number of compoundings per year and let t be the number of years. (The product nt represents the total number of times the interest will be compounded.) Then the interest rate per compounding period is r/n , and the account balance after t years is

$$A = P\left(1 + \frac{r}{n}\right)^{nt}. \quad \text{Amount (balance) with } n \text{ compoundings per year}$$

When the number of compoundings n increases without bound, the process approaches what is called **continuous compounding**. In the formula for n compoundings per year, let $m = n/r$. This produces

$$A = P\left(1 + \frac{r}{n}\right)^{nt} = P\left(1 + \frac{1}{m}\right)^{mrt} = P\left[\left(1 + \frac{1}{m}\right)^m\right]^{rt}.$$

As m increases without bound, you know from Example 5 that

$$\left(1 + \frac{1}{m}\right)^m$$

approaches e . So, for continuous compounding, it follows that

$$P\left[\left(1 + \frac{1}{m}\right)^m\right]^{rt} \rightarrow P[e]^{rt}$$

and you can write $A = Pe^{rt}$. This result is part of the reason that e is the “natural” choice for a base of an exponential function.

Explore the Concept

Use the formula

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

to calculate the amount in an account when $P = \$3000$, $r = 6\%$, $t = 10$ years, and the interest is compounded (a) by the day, (b) by the hour, (c) by the minute, and (d) by the second. Does increasing the number of compoundings per year result in unlimited growth of the amount in the account? Explain.

You may want to point out to your students that the option of continuous compounding is not offered to consumers very often. However, it can be used to figure the maximum earnings (regardless of the number of compounding periods) that a principal may earn at a particular interest rate.

Formulas for Compound Interest

After t years, the balance A in an account with principal P and annual interest rate r (in decimal form) is given by the following formulas.

1. For n compoundings per year: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

2. For continuous compounding: $A = Pe^{rt}$

Remark

The interest rate r in the formulas for compound interest should be written as a decimal. For example, an interest rate of 2.5% would be written as $r = 0.025$.

EXAMPLE 8 Finding the Balance for Compound Interest

A total of \$9000 is invested at an annual interest rate of 2.5%, compounded annually. Find the balance in the account after 5 years.

Algebraic Solution

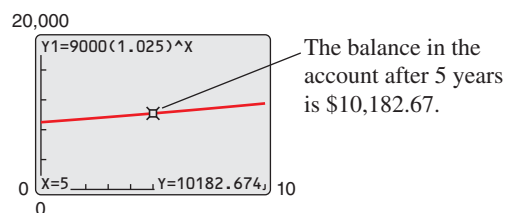
In this case, $P = 9000$, $r = 2.5\% = 0.025$, $n = 1$, and $t = 5$. Using the formula for compound interest with n compoundings per year, you have

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Formula for compound interest} \\ &= 9000\left(1 + \frac{0.025}{1}\right)^{1(5)} && \text{Substitute for } P, r, n, \text{ and } t. \\ &= 9000(1.025)^5 && \text{Simplify.} \\ &\approx \$10,182.67. && \text{Use a calculator.} \end{aligned}$$

So, the balance in the account after 5 years is \$10,182.67.

Graphical Solution

Substitute the values for P , r , and n into the formula for compound interest with n compoundings per year and simplify to obtain $A = 9000(1.025)^t$. Use a graphing utility to graph $A = 9000(1.025)^t$. Then use the *value* feature to approximate the value of A when $t = 5$, as shown in the figure.



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For the account in Example 8, find the balance after 10 years.

EXAMPLE 9 Finding Compound Interest

A total of \$12,000 is invested at an annual interest rate of 3%. Find the balance after 4 years when the interest is compounded (a) quarterly and (b) continuously.

Solution

a. For quarterly compoundings, $n = 4$. So, after 4 years at 3%, the balance is


$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} && \text{Formula for compound interest} \\ &= 12,000\left(1 + \frac{0.03}{4}\right)^{4(4)} && \text{Substitute for } P, r, n, \text{ and } t. \\ &\approx \$13,523.91. && \text{Use a calculator.} \end{aligned}$$

b. For continuous compounding, the balance is

$$\begin{aligned} A &= Pe^{rt} && \text{Formula for continuous compounding} \\ &= 12,000e^{0.03(4)} && \text{Substitute for } P, r, \text{ and } t. \\ &\approx \$13,529.96. && \text{Use a calculator.} \end{aligned}$$

Note that a continuous-compounding account yields more than a quarterly-compounding account.

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You invest \$6000 at an annual rate of 4%. Find the balance after 7 years when the interest is compounded (a) quarterly, (b) monthly, and (c) continuously. 

Example 9 illustrates the following general rule. For a given principal, interest rate, and time, the more often the interest is compounded per year, the greater the balance will be. Moreover, the balance obtained by continuous compounding is greater than the balance obtained by compounding n times per year.



Financial Analyst

EXAMPLE 10 Radioactive Decay

Let y represent a mass, in grams, of radioactive strontium (^{90}Sr), whose half-life is about 29 years. The quantity of strontium present after t years is $y = 10\left(\frac{1}{2}\right)^{t/29}$.

- a. What is the initial mass (when $t = 0$)?
- b. How much of the initial mass is present after 80 years?

Algebraic Solution

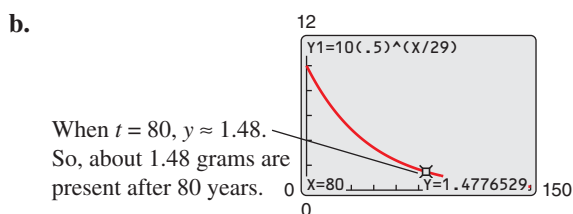
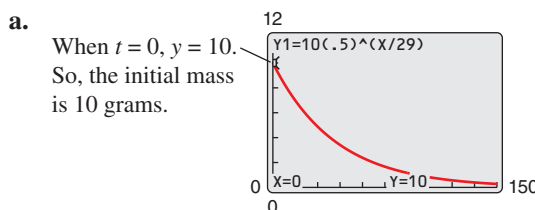
a. $y = 10\left(\frac{1}{2}\right)^{t/29}$ Write original equation.
 $= 10\left(\frac{1}{2}\right)^{0/29}$ Substitute 0 for t .
 $= 10$ Simplify.

So, the initial mass is 10 grams.

b. $y = 10\left(\frac{1}{2}\right)^{t/29}$ Write original equation.
 $= 10\left(\frac{1}{2}\right)^{80/29}$ Substitute 80 for t .
 $\approx 10\left(\frac{1}{2}\right)^{2.759}$ Simplify.
 ≈ 1.48 Use a calculator.

So, about 1.48 grams are present after 80 years.

Graphical Solution



✓ Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

In Example 10, how much of the initial mass is present after 160 years?

EXAMPLE 11 Population Growth

The approximate number of fruit flies in an experimental population after t hours is given by $Q(t) = 20e^{0.03t}$, where $t \geq 0$.

- a. Find the initial number of fruit flies in the population.
- b. How large is the population of fruit flies after 72 hours?
- c. Graph Q .

Solution

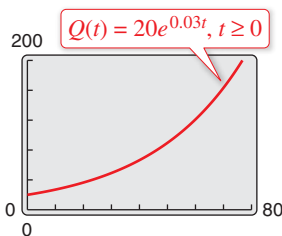
a. To find the initial population, evaluate $Q(t)$ when $t = 0$.

$$Q(0) = 20e^{0.03(0)} = 20e^0 = 20(1) = 20 \text{ flies}$$

b. After 72 hours, the population size is

$$Q(72) = 20e^{0.03(72)} = 20e^{2.16} \approx 173 \text{ flies.}$$

c. The graph of Q is shown in the figure.



Activities

1. Sketch the graphs of the functions $f(x) = e^x$ and $g(x) = 1 + e^x$ on the same coordinate system.
2. Determine the balance A at the end of 20 years when \$1500 is invested at 6.5% interest and the interest is compounded (a) quarterly and (b) continuously.
Answer: (a) \$5446.73 (b) \$5503.95
3. Determine the amount of money that should be invested at 9% interest, compounded monthly, to produce a final balance of \$30,000 in 15 years.
Answer: \$7816.48

✓ Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Rework Example 11 when the approximate number of fruit flies in the experimental population after t hours is given by $Q(t) = 10e^{0.02t}$.

3.1 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank(s).

- Exponential and logarithmic functions are examples of nonalgebraic functions, also called _____ functions.
- The exponential function $f(x) = e^x$ is called the _____ function, and the base e is called the _____ base.
- What type of transformation of the graph of $f(x) = 5^x$ is the graph of $f(x + 1)$?
- The formula $A = Pe^{rt}$ gives the balance A of an account earning what type of interest?

Procedures and Problem Solving

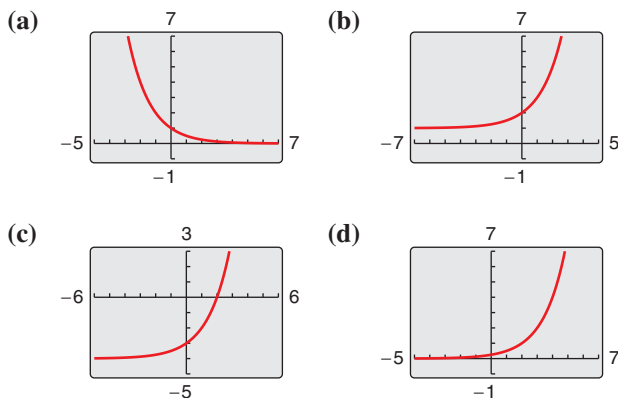
Evaluating Exponential Functions In Exercises 5–8, use a calculator to evaluate the function at the indicated value of x . Round your result to three decimal places.

Function	Value
5. $f(x) = 3.4^x$	$x = 6.8$
6. $f(x) = 1.2^x$	$x = \frac{1}{3}$
7. $g(x) = 5^x$	$x = -\pi$
8. $h(x) = 8.6^{-3x}$	$x = -\sqrt{2}$

Graphs of $y = a^x$ and $y = a^{-x}$ In Exercises 9–16, graph the exponential function by hand. Identify any asymptotes and intercepts and determine whether the graph of the function is increasing or decreasing.

- | | |
|--|--|
| 9. $g(x) = 5^x$ | 10. $f(x) = 10^x$ |
| 11. $f(x) = 5^{-x}$ | 12. $h(x) = 10^{-x}$ |
| 13. $h(x) = \left(\frac{5}{4}\right)^x$ | 14. $g(x) = \left(\frac{3}{2}\right)^x$ |
| 15. $g(x) = \left(\frac{5}{4}\right)^{-x}$ | 16. $f(x) = \left(\frac{3}{2}\right)^{-x}$ |

Library of Parent Functions In Exercises 17–20, use the graph of $y = 2^x$ to match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]



- | | |
|----------------------|----------------------|
| 17. $f(x) = 2^{x-2}$ | 18. $f(x) = 2^{-x}$ |
| 19. $f(x) = 2^x - 4$ | 20. $f(x) = 2^x + 1$ |

Library of Parent Functions In Exercises 21–30, use the graph of f to describe the transformation that yields the graph of g . Then sketch the graphs of f and g by hand.

- $f(x) = 3^x$, $g(x) = 3^{x-5}$
- $f(x) = 10^x$, $g(x) = 10^{x+4}$
- $f(x) = 6^x$, $g(x) = 6^x + 5$
- $f(x) = -2^x$, $g(x) = 5 - 2^x$
- $f(x) = \left(\frac{3}{5}\right)^x$, $g(x) = -\left(\frac{3}{5}\right)^{x+4}$
- $f(x) = 0.3^x$, $g(x) = -0.3^x + 5$
- $f(x) = 0.4^x$, $g(x) = 0.4^{x-2} - 3$
- $f(x) = \left(\frac{2}{3}\right)^x$, $g(x) = \left(\frac{2}{3}\right)^{x+2} - 3$
- $f(x) = \left(\frac{1}{4}\right)^x$, $g(x) = \left(\frac{1}{4}\right)^{-x} + 2$
- $f(x) = \left(\frac{1}{2}\right)^x$, $g(x) = \left(\frac{1}{2}\right)^{-(x+4)}$

Approximation of a Power with Base e In Exercises 31 and 32, show that the value of $f(x)$ approaches the value of $g(x)$ as x increases without bound (a) graphically and (b) numerically.

- $f(x) = 1 + \left(\frac{0.5}{x}\right)^x$, $g(x) = e^{0.5}$
- $f(x) = 1 + \left(\frac{3}{x}\right)^x$, $g(x) = e^3$

Evaluating the Natural Exponential Function In Exercises 33–36, use a calculator to evaluate the function at the indicated value of x . Round your result to the nearest thousandth.

Function	Value
33. $f(x) = e^x$	$x = 9.2$
34. $f(x) = e^{-x}$	$x = -\frac{3}{4}$
35. $g(x) = 50e^{4x}$	$x = 0.02$
36. $h(x) = -5.5e^{-x}$	$x = 200$

Graphing an Exponential Function In Exercises 37–52, use a graphing utility to construct a table of values for the function. Then sketch the graph of the function. Identify any asymptotes of the graph.

- 37. $f(x) = \left(\frac{5}{2}\right)^x$
- 38. $f(x) = \left(\frac{5}{2}\right)^{-x}$
- 39. $f(x) = 6^x$
- 40. $f(x) = 2^{x-1}$
- 41. $f(x) = 3^{x+2}$
- 42. $y = 2^{-x^2}$
- 43. $y = 3^{x-2} + 1$
- 44. $y = 4^{x+1} - 2$
- 45. $f(x) = e^{-x}$
- 46. $s(t) = 3e^{-0.2t}$
- 47. $f(x) = 3e^{x+4}$
- 48. $f(x) = 2e^{x-3}$
- 49. $f(x) = 2 + e^{x-5}$
- 50. $g(x) = e^{x+1} + 2$
- 51. $s(t) = 2e^{-0.12t}$
- 52. $g(x) = e^{0.5x} - 1$

Finding Asymptotes In Exercises 53–56, use a graphing utility to (a) graph the function and (b) find any asymptotes numerically by creating a table of values for the function.

- 53. $f(x) = \frac{8}{1 + e^{-0.5x}}$
- 54. $g(x) = \frac{8}{1 + e^{-0.5/x}}$
- 55. $f(x) = -\frac{6}{2 - e^{0.2x}}$
- 56. $f(x) = \frac{6}{2 - e^{0.2/x}}$

Finding Points of Intersection In Exercises 57–60, use a graphing utility to find the point(s) of intersection, if any, of the graphs of the functions. Round your result to three decimal places.

- 57. $y = 20e^{0.05x}$
- 58. $y = 100e^{0.01x}$
- $y = 1500$
- $y = 12,500$
- 59. $y = 2.5e^{0.3x}$
- 60. $y = 3.2e^{0.5x}$
- $y = 0.2$
- $y = 0.9$

Approximating Relative Extrema In Exercises 61–64, (a) use a graphing utility to graph the function, (b) use the graph to find the open intervals on which the function is increasing and decreasing, and (c) approximate any relative maximum or minimum values.

- 61. $f(x) = x^2e^{-x}$
- 62. $f(x) = 2x^2e^{x+1}$
- 63. $f(x) = x^3e^x$
- 64. $f(x) = x^3e^{-x+2}$

Finding the Balance for Compound Interest In Exercises 65–68, complete the table to determine the balance A for \$2500 invested at rate r for t years and compounded n times per year.

n	1	2	4	12	365	Continuous
A						

- 65. $r = 2\%$, $t = 10$ years
- 66. $r = 6\%$, $t = 10$ years
- 67. $r = 4\%$, $t = 20$ years
- 68. $r = 3\%$, $t = 40$ years

Finding the Balance for Compound Interest In Exercises 69–72, complete the table to determine the balance A for \$12,000 invested at rate r for t years, compounded continuously.

t	1	10	20	30	40	50
A						

- 69. $r = 4\%$
- 70. $r = 6\%$
- 71. $r = 3.5\%$
- 72. $r = 2.5\%$

Finding the Amount of an Annuity In Exercises 73–76, you build an annuity by investing P dollars every month at interest rate r , compounded monthly. Find the amount A accrued after n months using the formula

$$A = P \left[\frac{(1 + r/12)^n - 1}{r/12} \right], \text{ where } r \text{ is in decimal form.}$$

- 73. $P = \$25$, $r = 0.12$, $n = 48$ months
- 74. $P = \$100$, $r = 0.09$, $n = 60$ months
- 75. $P = \$200$, $r = 0.06$, $n = 72$ months
- 76. $P = \$75$, $r = 0.03$, $n = 24$ months

77. MODELING DATA

There are three options for investing \$500. The first earns 7% compounded annually, the second earns 7% compounded quarterly, and the third earns 7% compounded continuously.

- (a) Find equations that model the growth of each investment and use a graphing utility to graph each model in the same viewing window over a 20-year period.
- (b) Use the graph from part (a) to determine which investment yields the highest return after 20 years. What are the differences in earnings among the three investments?



78. Radioactive Decay Let Q represent a mass, in grams, of radioactive radium (^{226}Ra), whose half-life is 1600 years. The quantity of radium present after t years is given by $Q = 25\left(\frac{1}{2}\right)^{t/1600}$.

- (a) Determine the initial quantity (when $t = 0$).
- (b) Determine the quantity present after 1000 years.
- (c) Use a graphing utility to graph the function over the interval $t = 0$ to $t = 5000$.
- (d) When will the quantity of radium be 0 grams? Explain.

79. Radioactive Decay Let Q represent a mass, in grams, of carbon 14 (^{14}C), whose half-life is 5700 years. The quantity present after t years is given by $Q = 10\left(\frac{1}{2}\right)^{t/5700}$.

- Determine the initial quantity (when $t = 0$).
- Determine the quantity present after 2000 years.
- Sketch the graph of the function over the interval $t = 0$ to $t = 10,000$.

80. Algebraic-Graphical-Numerical Assume the annual rate of inflation is 4% for the next 10 years. The approximate cost C of goods or services during these years is $C(t) = P(1.04)^t$, where t is the time (in years) and P is the present cost. An oil change for your car presently costs \$26.88. Use the following methods to approximate the cost 10 years from now.

- Use a graphing utility to graph the function and then use the *value* feature.
- Use the *table* feature of the graphing utility to find a numerical approximation.
- Use a calculator to evaluate the cost function algebraically.

81. Population Growth The projected populations of California for the years 2020 through 2060 can be modeled by $P = 36.308e^{0.0065t}$, where P is the population (in millions) and t is the time (in years), with $t = 20$ corresponding to 2020. (Source: California Department of Finance)

- Use a graphing utility to graph the function for the years 2020 through 2060.
- Use the *table* feature of the graphing utility to create a table of values for the same time period as in part (a).
- According to the model, in what year will the population of California exceed 51 million?

82. Why you should learn it (p. 180) In early 2014, a new sedan had a manufacturer's suggested retail price of \$31,340. After t years, the sedan's value is given by



$$V(t) = 31,340\left(\frac{4}{5}\right)^t.$$

- Use a graphing utility to graph the function.
- Use the graphing utility to create a table of values that shows the value V for $t = 1$ to $t = 10$ years.
- According to the model, when will the sedan have no value?

Conclusions

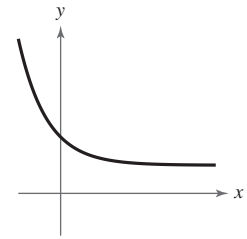
True or False? In Exercises 83 and 84, determine whether the statement is true or false. Justify your answer.

83. $f(x) = 1^x$ is not an exponential function.

84. $e = \frac{271,801}{99,990}$

85. Library of Parent Functions Determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)

- $y = e^x + 1$
- $y = -e^{-x} + 1$
- $y = e^{-x} - 1$
- $y = e^{-x} + 1$



86. Exploration Use a graphing utility to graph $y_1 = e^x$ and each of the functions $y_2 = x^2$, $y_3 = x^3$, $y_4 = \sqrt{x}$, and $y_5 = |x|$ in the same viewing window.

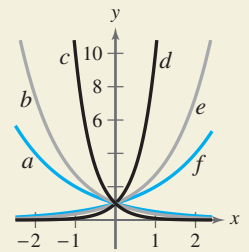
- Which function increases at the fastest rate for “large” values of x ?
- Use the result of part (a) to make a conjecture about the rates of growth of $y_1 = e^x$ and $y = x^n$, where n is a natural number and x is “large.”
- Use the results of parts (a) and (b) to describe what is implied when it is stated that a quantity is growing exponentially.

87. Think About It Graph $y = 3^x$ and $y = 4^x$. Use the graph to solve the inequality $3^x < 4^x$.



88. HOW DO YOU SEE IT?

The figure shows the graphs of $y = 2^x$, $y = e^x$, $y = 10^x$, $y = 2^{-x}$, $y = e^{-x}$, and $y = 10^{-x}$. Match each function with its graph. [The graphs are labeled (a) through (f).] Explain.



Think About It In Exercises 89–92, place the correct symbol ($<$ or $>$) between the two numbers.

- e^π π^e
- 2^{10} 10^2
- 5^{-3} 3^{-5}
- $4^{1/2}$ $\left(\frac{1}{2}\right)^4$

Cumulative Mixed Review

Inverse Functions In Exercises 93–96, determine whether the function has an inverse function. If it does, find f^{-1} .

- $f(x) = 5x - 7$
- $f(x) = -\frac{2}{3}x + \frac{5}{2}$
- $f(x) = \sqrt[3]{x+8}$
- $f(x) = \sqrt{x^2+6}$

97. Make a Decision To work an extended application analyzing the population per square mile in the United States, visit this textbook's website at LarsonPrecalculus.com. (Data Source: U.S. Census Bureau)

3.2 Logarithmic Functions and Their Graphs

Logarithmic Functions

In Section 1.6, you studied the concept of an inverse function. There, you learned that when a function is one-to-one—that is, when the function has the property that no horizontal line intersects its graph more than once—the function must have an inverse function. By looking back at the graphs of the exponential functions introduced in Section 3.1, you will see that every function of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$, passes the Horizontal Line Test and therefore must have an inverse function. This inverse function is called the **logarithmic function with base a** .

Definition of Logarithmic Function

For $x > 0$, $a > 0$, and $a \neq 1$,

$$y = \log_a x \text{ if and only if } x = a^y.$$

The function given by

$$f(x) = \log_a x \quad \text{Read as "log base } a \text{ of } x."$$

is called the **logarithmic function with base a** .

From the definition above, you can see that every logarithmic equation can be written in an equivalent exponential form and every exponential equation can be written in logarithmic form. So, the equations

$$y = \log_a x \quad \text{and} \quad x = a^y$$

are equivalent. For example, $2 = \log_3 9$ is equivalent to $9 = 3^2$, and $5^3 = 125$ is equivalent to $\log_5 125 = 3$.

When evaluating logarithms, remember that *a logarithm is an exponent*. This means that $\log_a x$ is the exponent to which a must be raised to obtain x . For instance, $\log_2 8 = 3$ because 2 must be raised to the third power to get 8.

EXAMPLE 1 Evaluating Logarithms

Evaluate each logarithm at the indicated value of x .

- $f(x) = \log_2 x$, $x = 32$
- $f(x) = \log_3 x$, $x = 1$
- $f(x) = \log_4 x$, $x = 2$
- $f(x) = \log_{10} x$, $x = \frac{1}{100}$

Solution

- $f(32) = \log_2 32 = 5$ because $2^5 = 32$.
- $f(1) = \log_3 1 = 0$ because $3^0 = 1$.
- $f(2) = \log_4 2 = \frac{1}{2}$ because $4^{1/2} = \sqrt{4} = 2$.
- $f\left(\frac{1}{100}\right) = \log_{10} \frac{1}{100} = -2$ because $10^{-2} = \frac{1}{10^2} = \frac{1}{100}$.

 **Checkpoint**  *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Evaluate each logarithm at the indicated value of x .

- $f(x) = \log_6 x$, $x = 1$
- $f(x) = \log_5 x$, $x = \frac{1}{125}$
- $f(x) = \log_{10} x$, $x = 10,000$

What you should learn

- ▶ Recognize and evaluate logarithmic functions with base a .
- ▶ Graph logarithmic functions with base a .
- ▶ Recognize, evaluate, and graph natural logarithmic functions.
- ▶ Use logarithmic functions to model and solve real-life problems.

Why you should learn it

Logarithmic functions are useful in modeling data that represent quantities that increase or decrease slowly. For instance, Exercise 114 on page 201 shows how to use a logarithmic function to model the minimum required ventilation rates in public school classrooms.



Remark

In this text, the parentheses in $\log_a(u)$ are sometimes omitted when u is an expression involving exponents, radicals, products, or quotients. For instance, $\log_{10}(2x)$ can be written as $\log_{10} 2x$. To evaluate $\log_{10} 2x$, find the logarithm of the product $2x$.

The logarithmic function with base 10 is called the **common logarithmic function**. On most calculators, this function is denoted by LOG . Example 2 shows how to use a calculator to evaluate common logarithmic functions. You will learn how to use a calculator to calculate logarithms to any base in the next section.

The logarithmic function can be one of the most difficult functions for students to understand. Remind students that a logarithm is an exponent. Converting back and forth between logarithmic form and exponential form supports this concept.

EXAMPLE 2 Evaluating Common Logarithms on a Calculator

Use a calculator to evaluate the function $f(x) = \log_{10} x$ at each value of x .

- a. $x = 10$ b. $x = \frac{1}{3}$ c. $x = 2.5$ d. $x = -2$

Solution

Function Value	Graphing Calculator Keystrokes	Display
a. $f(10) = \log_{10} 10$	$\text{LOG } 10 \text{ ENTER}$	1
b. $f(\frac{1}{3}) = \log_{10} \frac{1}{3}$	$\text{LOG } () 1 \div 3 \text{ ENTER}$	-0.4771213
c. $f(2.5) = \log_{10} 2.5$	$\text{LOG } 2.5 \text{ ENTER}$	0.3979400
d. $f(-2) = \log_{10}(-2)$	$\text{LOG } (-) 2 \text{ ENTER}$	ERROR

Note that the calculator displays an error message when you try to evaluate $\log_{10}(-2)$. In this case, there is no *real* power to which 10 can be raised to obtain -2 .

 **Checkpoint**  *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use a calculator to evaluate the function $f(x) = \log_{10} x$ at each value of x .

- a. $x = 275$ b. $x = 0.275$ c. $x = -\frac{1}{2}$ d. $x = \frac{1}{2}$ 

The properties of logarithms listed below follow directly from the definition of the logarithmic function with base a .

Properties of Logarithms

- $\log_a 1 = 0$ because $a^0 = 1$.
- $\log_a a = 1$ because $a^1 = a$.
- $\log_a a^x = x$ and $a^{\log_a x} = x$. Inverse Properties
- If $\log_a x = \log_a y$, then $x = y$. One-to-One Property


EXAMPLE 3 Using Properties of Logarithms

- a. Solve for x : $\log_2 x = \log_2 3$
 b. Solve for x : $\log_4 4 = x$
 c. Simplify: $\log_5 5^x$
 d. Simplify: $7^{\log_7 14}$

Solution

- a. Using the One-to-One Property (Property 4), you can conclude that $x = 3$.
 b. Using Property 2, you can conclude that $x = 1$.
 c. Using the Inverse Property (Property 3), it follows that $\log_5 5^x = x$.
 d. Using the Inverse Property (Property 3), it follows that $7^{\log_7 14} = 14$.

 **Checkpoint**  *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

- a. Solve for x : $\log_{10} x = \log_{10} 2$ b. Simplify: $20^{\log_{20} 3}$ 

Technology Tip

Some graphing utilities do not give an error message for $\log_{10}(-2)$. Instead, the graphing utility will display a complex number. For the purpose of this text, however, the domain of a logarithmic function is the set of positive *real* numbers.

Graphs of Logarithmic Functions

To sketch the graph of $y = \log_a x$, you can use the fact that the graphs of inverse functions are reflections of each other in the line $y = x$.

Show students graphically and algebraically that the logarithmic function is the inverse function of the exponential function.

EXAMPLE 4 Graphs of Exponential and Logarithmic Functions

In the same coordinate plane, sketch the graph of each function by hand.

- $f(x) = 2^x$
- $g(x) = \log_2 x$

Solution

- For $f(x) = 2^x$, construct a table of values. By plotting these points and connecting them with a smooth curve, you obtain the graph of f shown in Figure 3.12.

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

- Because $g(x) = \log_2 x$ is the inverse function of $f(x) = 2^x$, the graph of g is obtained by plotting the points $(f(x), x)$ and connecting them with a smooth curve. The graph of g is a reflection of the graph of f in the line $y = x$, as shown in Figure 3.12.

 **Checkpoint**  Audio-video solution in English & Spanish at LarsonPrecalculus.com.

In the same coordinate plane, sketch the graphs of (a) $f(x) = 8^x$ and (b) $g(x) = \log_8 x$.

Before you can confirm the result of Example 4 using a graphing utility, you need to know how to enter $\log_2 x$. You will learn how to do this using the *change-of-base formula* discussed in the next section.

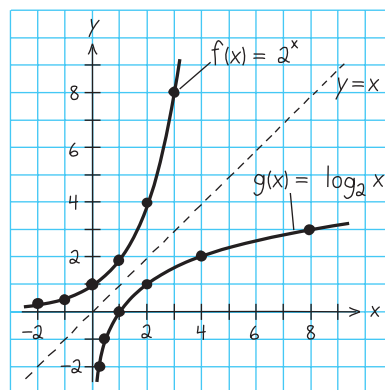


Figure 3.12

EXAMPLE 5 Sketching the Graph of a Logarithmic Function

Sketch the graph of the common logarithmic function $f(x) = \log_{10} x$ by hand.

Solution

Begin by constructing a table of values. Note that some of the values can be obtained without a calculator by using the Inverse Property of Logarithms. Others require a calculator. Next, plot the points and connect them with a smooth curve, as shown in Figure 3.13. Note that $x = 0$ (the y -axis) is a vertical asymptote of the graph.

	Without calculator				With calculator		
x	$\frac{1}{100}$	$\frac{1}{10}$	1	10	2	5	8
$f(x) = \log_{10} x$	-2	-1	0	1	0.301	0.699	0.903

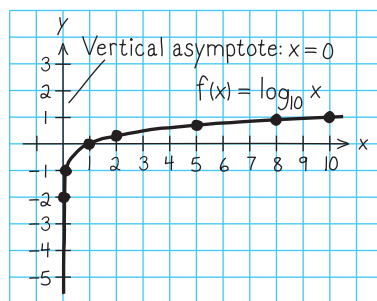


Figure 3.13

 **Checkpoint**  Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Without using a calculator, sketch the graph of $f(x) = \log_3 x$ by hand.

The nature of the graph in Figure 3.13 is typical of functions of the form $f(x) = \log_a x$, $a > 1$. They have one x -intercept and one vertical asymptote. Notice how slowly the graph rises for $x > 1$.

Library of Parent Functions: Logarithmic Function

The *parent logarithmic function* $f(x) = \log_a x$, $a > 0$, $a \neq 1$ is the inverse function of the exponential function. Its domain is the set of positive real numbers and its range is the set of all real numbers. This is the opposite of the exponential function. Moreover, the logarithmic function has the y -axis as a vertical asymptote, whereas the exponential function has the x -axis as a horizontal asymptote. Many real-life phenomena with slow rates of growth can be modeled by logarithmic functions. The basic characteristics of the logarithmic function are summarized below and on the inside cover of this text.

Graph of $f(x) = \log_a x$, $a > 1$

Domain: $(0, \infty)$

Range: $(-\infty, \infty)$

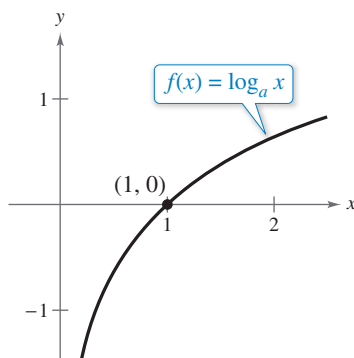
Intercept: $(1, 0)$

Increasing on $(0, \infty)$

y -axis is a vertical asymptote
($\log_a x \rightarrow -\infty$ as $x \rightarrow 0^+$)

Continuous

Reflection of graph of $f(x) = a^x$
in the line $y = x$



Explore the Concept

Use a graphing utility to graph $y = \log_{10} x$ and $y = 8$ in the same viewing window. Find a viewing window that shows the point of intersection. What is the point of intersection? Use the point of intersection to complete the equation $\log_{10} \square = 8$.



EXAMPLE 6 Library of Parent Functions $f(x) = \log_a x$

See LarsonPrecalculus.com for an interactive version of this type of example.

Each of the following functions is a transformation of the graph of $f(x) = \log_{10} x$.

- Because $g(x) = \log_{10}(x - 1) = f(x - 1)$, the graph of g can be obtained by shifting the graph of f one unit to the *right*, as shown in Figure 3.14.
- Because $h(x) = 2 + \log_{10} x = 2 + f(x)$, the graph of h can be obtained by shifting the graph of f two units *upward*, as shown in Figure 3.15.

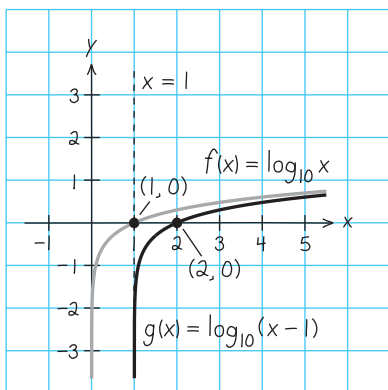


Figure 3.14

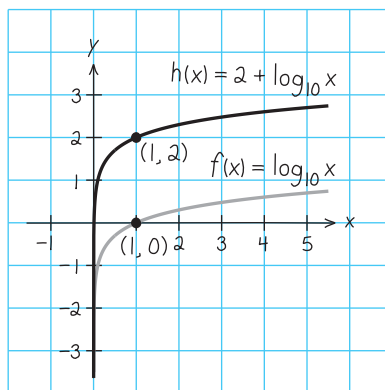


Figure 3.15

Notice that the transformation in Figure 3.15 keeps the y -axis as a vertical asymptote, but the transformation in Figure 3.14 yields the new vertical asymptote $x = 1$.

✓ Checkpoint  [Audio-video solution in English & Spanish at LarsonPrecalculus.com.](http://LarsonPrecalculus.com)

Use the graph of $f(x) = \log_{10} x$ to sketch the graph of each function by hand.

- $g(x) = -1 + \log_{10} x$
- $h(x) = \log_{10}(x + 3)$



Point out to your students that they can use their understanding of transformations to identify the vertical asymptotes of logarithmic functions. For instance, they know that $f(x - 2)$ shifts the graph of f two units to the right, so the vertical asymptote after the shift is $x = 2$.

The Natural Logarithmic Function

By looking back at the graph of the natural exponential function introduced in Section 3.1, you will see that $f(x) = e^x$ is one-to-one and so has an inverse function. This inverse function is called the **natural logarithmic function** and is denoted by the special symbol $\ln x$, read as “the natural log of x ” or “el en of x .”

The Natural Logarithmic Function

For $x > 0$,

$$y = \ln x \text{ if and only if } x = e^y.$$

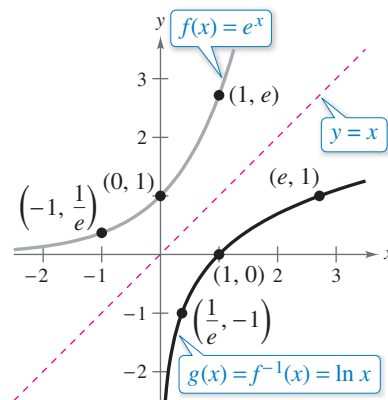
The function given by

$$f(x) = \log_e x = \ln x$$

is called the **natural logarithmic function**.

The equations $y = \ln x$ and $x = e^y$ are equivalent. Note that the natural logarithm $\ln x$ is written without a base. The base is understood to be e .

Because the functions $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions of each other, their graphs are reflections of each other in the line $y = x$. This reflective property is illustrated in Figure 3.16.



Reflection of graph of $f(x) = e^x$ in the line $y = x$

Figure 3.16

EXAMPLE 7 Evaluating the Natural Logarithmic Function

Use a calculator to evaluate the function $f(x) = \ln x$ at each value of x .

- $x = 2$
- $x = 0.3$
- $x = -1$
- $x = 1 + \sqrt{2}$

Solution

Function Value	Graphing Calculator Keystrokes	Display
a. $f(2) = \ln 2$	$\boxed{\text{LN}} \ 2 \ \boxed{\text{ENTER}}$	0.6931472
b. $f(0.3) = \ln 0.3$	$\boxed{\text{LN}} \ .3 \ \boxed{\text{ENTER}}$	-1.2039728
c. $f(-1) = \ln(-1)$	$\boxed{\text{LN}} \ \boxed{(-)} \ 1 \ \boxed{\text{ENTER}}$	ERROR
d. $f(1 + \sqrt{2}) = \ln(1 + \sqrt{2})$	$\boxed{\text{LN}} \ \boxed{(} \ 1 \ \boxed{+} \ \boxed{\sqrt{\quad}} \ 2 \ \boxed{)} \ \boxed{\text{ENTER}}$	0.8813736

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Use a calculator to evaluate the function $f(x) = \ln x$ at each value of x .

- $x = 0.01$
- $x = 4$
- $x = \sqrt{3} + 2$
- $x = \sqrt{3} - 2$

The four properties of logarithms listed on page 193 are also valid for natural logarithms.

Properties of Natural Logarithms

- $\ln 1 = 0$ because $e^0 = 1$.
- $\ln e = 1$ because $e^1 = e$.
- $\ln e^x = x$ and $e^{\ln x} = x$. Inverse Properties
- If $\ln x = \ln y$, then $x = y$. One-to-One Property

Technology Tip

On most calculators, the natural logarithm is denoted by $\boxed{\text{LN}}$, as illustrated in Example 7.

Remark

In Example 7(c), be sure you see that $\ln(-1)$ gives an error message on most calculators. This occurs because the domain of $\ln x$ is the set of *positive* real numbers (see Figure 3.16). So, $\ln(-1)$ is undefined.

EXAMPLE 8 Using Properties of Natural Logarithms

Use the properties of natural logarithms to rewrite each expression.

- a. $\ln \frac{1}{e}$ b. $e^{\ln 5}$ c. $4 \ln 1$ d. $2 \ln e$

Solution

- a. $\ln \frac{1}{e} = \ln e^{-1} = -1$ Inverse Property
 b. $e^{\ln 5} = 5$ Inverse Property
 c. $4 \ln 1 = 4(0) = 0$ Property 1
 d. $2 \ln e = 2(1) = 2$ Property 2

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Use the properties of natural logarithms to simplify each expression.

- a. $\ln e^{1/3}$ b. $5 \ln 1$ c. $\frac{3}{4} \ln e$ d. $e^{\ln 7}$

EXAMPLE 9 Finding the Domains of Logarithmic Functions

Find the domain of each function.

- a. $f(x) = \ln(x - 2)$ b. $g(x) = \ln(2 - x)$ c. $h(x) = \ln x^2$

Algebraic Solution

- a. Because $\ln(x - 2)$ is defined only when

$$x - 2 > 0$$

it follows that the domain of f is $(2, \infty)$.

- b. Because $\ln(2 - x)$ is defined only when

$$2 - x > 0$$

it follows that the domain of g is $(-\infty, 2)$.

- c. Because $\ln x^2$ is defined only when

$$x^2 > 0$$

it follows that the domain of h is all real numbers except $x = 0$.

Graphical Solution

- a.

 b.

 c.

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Find the domain of $f(x) = \ln(x + 3)$. 

In Example 9, suppose you had been asked to analyze the function $h(x) = \ln|x - 2|$. How would the domain of this function compare with the domains of the functions given in parts (a) and (b) of the example?

Application

EXAMPLE 10 Psychology

Students participating in a psychology experiment attended several lectures on a subject and were given an exam. Every month for a year after the exam, the students were retested to see how much of the material they remembered. The average scores for the group are given by the *human memory model*

$$f(t) = 75 - 6 \ln(t + 1), \quad 0 \leq t \leq 12$$

where t is the time in months. The graph of f is shown in Figure 3.17.

- What was the average score on the original exam ($t = 0$)?
- What was the average score at the end of $t = 2$ months?
- What was the average score at the end of $t = 6$ months?

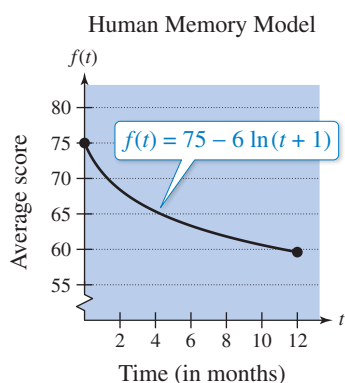


Figure 3.17



Psychologist

Algebraic Solution

- a. The original average score was

$$\begin{aligned} f(0) &= 75 - 6 \ln(0 + 1) \\ &= 75 - 6 \ln 1 \\ &= 75 - 6(0) \\ &= 75. \end{aligned}$$

- b. After 2 months, the average score was

$$\begin{aligned} f(2) &= 75 - 6 \ln(2 + 1) \\ &= 75 - 6 \ln 3 \\ &\approx 75 - 6(1.0986) \\ &\approx 68.41. \end{aligned}$$

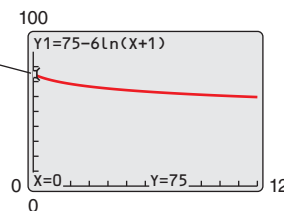
- c. After 6 months, the average score was

$$\begin{aligned} f(6) &= 75 - 6 \ln(6 + 1) \\ &= 75 - 6 \ln 7 \\ &\approx 75 - 6(1.9459) \\ &\approx 63.32. \end{aligned}$$

Graphical Solution

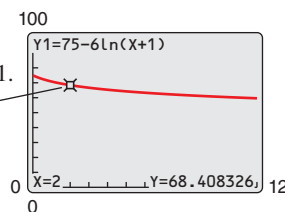
- a.

When $t = 0$, $f(0) = 75$.
So, the original average score was 75.



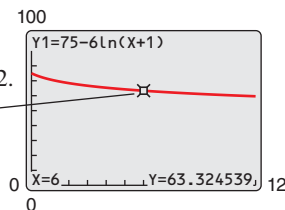
- b.

When $t = 2$, $f(2) \approx 68.41$.
So, the average score after 2 months was about 68.41.



- c.

When $t = 6$, $f(6) \approx 63.32$.
So, the average score after 6 months was about 63.32.



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In Example 10, find the average score at the end of (a) $t = 1$ month, (b) $t = 9$ months, and (c) $t = 12$ months.

3.2 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

- The inverse function of the exponential function $f(x) = a^x$ is called the _____ with base a .
- The base of the _____ logarithmic function is 10, and the base of the _____ logarithmic function is e .
- The inverse properties of logarithms are $\log_a a^x = x$ and _____.
- If $x = e^y$, then $y = \underline{\hspace{2cm}}$.
- What exponential equation is equivalent to the logarithmic equation $\log_a b = c$?
- For what value(s) of x is $\ln x = \ln 7$?

Procedures and Problem Solving

Rewriting Logarithmic Equations In Exercises 7–14, write the logarithmic equation in exponential form. For example, the exponential form of $\log_5 25 = 2$ is $5^2 = 25$.

- $\log_4 64 = 3$
- $\log_3 81 = 4$
- $\log_7 \frac{1}{49} = -2$
- $\log_{10} \frac{1}{1000} = -3$
- $\log_{32} 4 = \frac{2}{5}$
- $\log_{16} 8 = \frac{3}{4}$
- $\log_2 \sqrt{2} = \frac{1}{2}$
- $\log_5 \sqrt[3]{25} = \frac{2}{3}$

Rewriting Exponential Equations In Exercises 15–22, write the exponential equation in logarithmic form. For example, the logarithmic form of $2^3 = 8$ is $\log_2 8 = 3$.

- $5^3 = 125$
- $8^2 = 64$
- $81^{1/4} = 3$
- $9^{3/2} = 27$
- $6^{-2} = \frac{1}{36}$
- $10^{-3} = 0.001$
- $g^a = 4$
- $n^t = 10$

Evaluating Logarithms In Exercises 23–26, use the definition of logarithmic function to evaluate the function at the indicated value of x without using a calculator.

Function	Value
23. $f(x) = \log_2 x$	$x = 16$
24. $f(x) = \log_{16} x$	$x = \frac{1}{4}$
25. $g(x) = \log_{10} x$	$x = \frac{1}{1000}$
26. $g(x) = \log_{10} x$	$x = 100,000$

Evaluating Common Logarithms on a Calculator In Exercises 27–30, use a calculator to evaluate the function at the indicated value of x . Round your result to three decimal places.

Function	Value
27. $f(x) = \log_{10} x$	$x = 345$
28. $f(x) = \log_{10} x$	$x = \frac{4}{5}$

Function	Value
29. $h(x) = 6 \log_{10} x$	$x = 14.8$
30. $h(x) = 1.9 \log_{10} x$	$x = 4.3$

Using Properties of Logarithms In Exercises 31–36, solve the equation for x .

- $\log_7 x = \log_7 9$
- $\log_5 5 = \log_5 x$
- $\log_4 4^2 = x$
- $\log_3 3^{-5} = x$
- $\log_8 x = \log_8 10^{-1}$
- $\log_4 4^3 = \log_4 x$

Using Properties of Logarithms In Exercises 37–40, use the properties of logarithms to simplify the expression.

- $\log_4 4^{3x}$
- $6^{\log_6 36x}$
- $3 \log_2 \frac{1}{2}$
- $\frac{1}{4} \log_4 16$

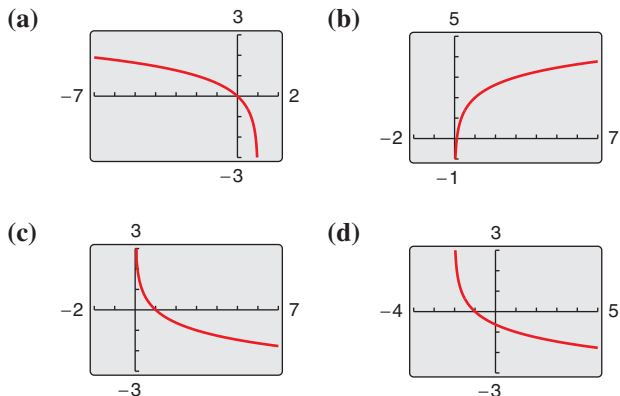
Graphs of Exponential and Logarithmic Functions In Exercises 41–44, sketch the graph of f . Then use the graph of f to sketch the graph of g .

- $f(x) = 6^x$
 $g(x) = \log_6 x$
- $f(x) = 5^x$
 $g(x) = \log_5 x$
- $f(x) = 15^x$
 $g(x) = \log_{15} x$
- $f(x) = 4^x$
 $g(x) = \log_4 x$

Sketching the Graph of a Logarithmic Function In Exercises 45–50, find the domain, vertical asymptote, and x -intercept of the logarithmic function, and sketch its graph by hand.

- $y = \log_{10}(x + 2)$
- $y = \log_{10}(x - 1)$
- $y = 1 + \log_{10} x$
- $y = 2 - \log_{10} x$
- $y = 1 + \log_{10}(x - 2)$
- $y = 2 + \log_{10}(x + 1)$

Library of Parent Functions In Exercises 51–54, use the graph of $y = \log_3 x$ to match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]



- 51. $f(x) = \log_3 x + 2$
- 52. $f(x) = -\log_3 x$
- 53. $f(x) = -\log_3(x + 2)$
- 54. $f(x) = \log_3(1 - x)$

Library of Parent Functions In Exercises 55–60, describe the transformation of the graph of f that yields the graph of g .

- 55. $f(x) = \log_{10} x, g(x) = \log_{10}(-x)$
- 56. $f(x) = \log_{10} x, g(x) = \log_{10}(x + 7)$
- 57. $f(x) = \log_2 x, g(x) = 4 - \log_2 x$
- 58. $f(x) = \log_2 x, g(x) = -3 + \log_2 x$
- 59. $f(x) = \log_8 x, g(x) = -2 + \log_8(x - 3)$
- 60. $f(x) = \log_8 x, g(x) = 4 + \log_8(x - 1)$

Rewriting Logarithmic Equations In Exercises 61–68, write the logarithmic equation in exponential form. For example, the exponential form of $\ln 5 = 1.6094 \dots$ is $e^{1.6094 \dots} = 5$.

- 61. $\ln 6 = 1.7917 \dots$
- 62. $\ln 4 = 1.3862 \dots$
- 63. $\ln e = 1$
- 64. $\ln e^3 = 3$
- 65. $\ln \sqrt{e} = \frac{1}{2}$
- 66. $\ln \frac{1}{e^2} = -2$
- 67. $\ln 9 = 2.1972 \dots$
- 68. $\ln \sqrt[3]{e} = \frac{1}{3}$

Rewriting Exponential Equations In Exercises 69–76, write the exponential equation in logarithmic form. For example, the logarithmic form of $e^2 = 7.3890 \dots$ is $\ln 7.3890 \dots = 2$.

- 69. $e^3 = 20.0855 \dots$
- 70. $e^0 = 1$
- 71. $e^{1.3} = 3.6692 \dots$
- 72. $e^{2.5} = 12.1824 \dots$
- 73. $\sqrt[3]{e} = 1.3956 \dots$
- 74. $\frac{1}{e^4} = 0.0183 \dots$
- 75. $\sqrt{e^3} = 4.4816 \dots$
- 76. $e^{3/4} = 2.1170 \dots$

Evaluating the Natural Logarithmic Function In Exercises 77–80, use a calculator to evaluate the function at the indicated value of x . Round your result to three decimal places.

Function	Value
77. $f(x) = \ln x$	$x = 11$
78. $f(x) = \ln x$	$x = 18.31$
79. $f(x) = -\ln x$	$x = \frac{1}{2}$
80. $f(x) = 3 \ln x$	$x = \sqrt{0.65}$

Using Properties of Natural Logarithms In Exercises 81–88, use the properties of natural logarithms to rewrite the expression.

- 81. $\ln e^2$
- 82. $-\ln e$
- 83. $e^{\ln 1.8}$
- 84. $7 \ln e^0$
- 85. $e \ln 1$
- 86. $e^{\ln 22}$
- 87. $\ln e^{\ln e}$
- 88. $\ln \frac{1}{e^4}$

Library of Parent Functions In Exercises 89–92, find the domain, vertical asymptote, and x -intercept of the logarithmic function, and sketch its graph by hand. Verify using a graphing utility.

- 89. $f(x) = \ln(x - 1)$
- 90. $h(x) = \ln(x + 1)$
- 91. $g(x) = \ln(-x)$
- 92. $f(x) = \ln(3 - x)$

Library of Parent Functions In Exercises 93–98, use the graph of $f(x) = \ln x$ to describe the transformation that yields the graph of g .

- 93. $g(x) = \ln(x + 8)$
- 94. $g(x) = \ln(x - 4)$
- 95. $g(x) = \ln x - 5$
- 96. $g(x) = \ln x + 4$
- 97. $g(x) = \ln(x - 1) + 2$
- 98. $g(x) = \ln(x + 2) - 5$

Analyzing Graphs of Functions In Exercises 99–108, (a) use a graphing utility to graph the function, (b) find the domain, (c) use the graph to find the open intervals on which the function is increasing and decreasing, and (d) approximate any relative maximum or minimum values of the function. Round your results to three decimal places.

- 99. $f(x) = \frac{x}{2} - \ln \frac{x}{4}$
- 100. $g(x) = 6x \ln x$
- 101. $h(x) = \frac{14 \ln x}{x}$
- 102. $f(x) = \frac{x}{\ln x}$
- 103. $f(x) = \ln \frac{x+2}{x-1}$
- 104. $f(x) = \ln \frac{2x}{x+2}$
- 105. $f(x) = \ln \frac{x^2}{10}$
- 106. $f(x) = \ln \frac{x}{x^2+1}$
- 107. $f(x) = \sqrt{\ln x}$
- 108. $f(x) = (\ln x)^2$

- 109. Psychology** Students in a mathematics class were given an exam and then tested monthly with an equivalent exam. The average scores for the class are given by the human memory model

$$f(t) = 80 - 17 \log_{10}(t + 1), \quad 0 \leq t \leq 12$$

where t is the time in months.

- What was the average score on the original exam ($t = 0$)?
- What was the average score after 2 months?
- What was the average score after 11 months?

Verify your answers in parts (a), (b), and (c) using a graphing utility.

110. MODELING DATA

The table shows the temperatures T (in degrees Fahrenheit) at which water boils at selected pressures p (in pounds per square inch). (Source: [Standard Handbook of Mechanical Engineers](#))

DATA	Pressure, p	Temperature, T
	5	162.24°
	10	193.21°
	14.696 (1 atm)	212.00°
	20	227.96°
	30	250.33°
	40	267.25°
	60	292.71°
	80	312.03°
	100	327.81°

Spreadsheet at
LarsonPrecalculus.com



A model that approximates the data is

$$T = 87.97 + 34.96 \ln p + 7.91 \sqrt{p}.$$

- Use a graphing utility to plot the data and graph the model in the same viewing window. How well does the model fit the data?
- Use the graph to estimate the pressure at which the boiling point of water is 300°F.
- Calculate T when the pressure is 74 pounds per square inch. Verify your answer graphically.

- 111. Finance** A principal P , invested at $3\frac{1}{2}\%$ and compounded continuously, increases to an amount K times the original principal after t years, where $T = (\ln K)/0.035$.

- Complete the table and interpret your results.

K	1	2	4	6	8	10	12
t							

- Use a graphing utility to graph the function.

- 112. Audiology** The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is given by

$$\beta = 10 \log_{10} \left(\frac{I}{10^{-12}} \right).$$

- Determine the number of decibels of a sound with an intensity of 1 watt per square meter.
- Determine the number of decibels of a sound with an intensity of 10^{-2} watt per square meter.
- The intensity of the sound in part (a) is 100 times as great as that in part (b). Is the number of decibels 100 times as great? Explain.

- 113. Real Estate** The model

$$t = 16.625 \ln \frac{x}{x - 750}, \quad x > 750$$

approximates the length of a home mortgage of \$150,000 at 6% in terms of the monthly payment. In the model, t is the length of the mortgage in years and x is the monthly payment in dollars.

- Use the model to approximate the lengths of a \$150,000 mortgage at 6% when the monthly payment is \$897.72 and when the monthly payment is \$1659.24.
- Approximate the total amounts paid over the term of the mortgage with a monthly payment of \$897.72 and with a monthly payment of \$1659.24. What amount of the total is interest costs for each payment?

- 114. Why you should learn it** (p. 192) The rate of ventilation required in a public school classroom depends on the volume of air space per child. The model



$$y = 80.4 - 11 \ln x, \quad 100 \leq x \leq 1500$$

approximates the minimum required rate of ventilation y (in cubic feet per minute per child) in a classroom with x cubic feet of air space per child.

- Use a graphing utility to graph the function and approximate the required rate of ventilation in a room with 300 cubic feet of air space per child.
- A classroom of 30 students has an air conditioning system that moves 450 cubic feet of air per minute. Determine the rate of ventilation per child.
- Use the graph in part (a) to estimate the minimum required air space per child for the classroom in part (b).
- The classroom in part (b) has 960 square feet of floor space and a ceiling that is 12 feet high. Is the rate of ventilation for this classroom adequate? Explain.

Conclusions

True or False? In Exercises 115 and 116, determine whether the statement is true or false. Justify your answer.

115. You can determine the graph of $f(x) = \log_6 x$ by graphing $g(x) = 6^x$ and reflecting it about the x -axis.

116. The graph of $f(x) = \log_3 x$ contains the point $(27, 3)$.

Think About It In Exercises 117–120, find the value of the base b so that the graph of $f(x) = \log_b x$ contains the given point.

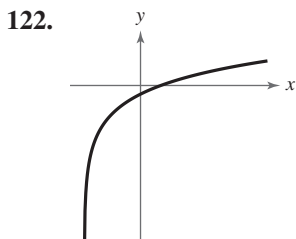
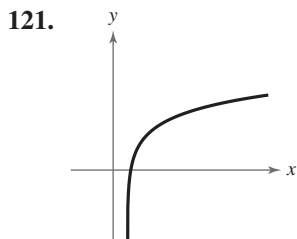
117. $(32, 5)$

118. $(81, 4)$

119. $(\frac{1}{81}, 2)$

120. $(\frac{1}{64}, 3)$

Library of Parent Functions In Exercises 121 and 122, determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)



(a) $y = \log_2(x + 1) + 2$

(a) $y = \ln(x - 1) + 2$

(b) $y = \log_2(x - 1) + 2$

(b) $y = \ln(x + 2) - 1$

(c) $y = 2 - \log_2(x - 1)$

(c) $y = 2 - \ln(x - 1)$

(d) $y = \log_2(x + 2) + 1$

(d) $y = \ln(x - 2) + 1$

123. **Writing** Explain why $\log_a x$ is defined only for $0 < a < 1$ and $a > 1$.

124. **Exploration** Let $f(x) = \ln x$ and $g(x) = x^{1/n}$.

(a) Use a graphing utility to graph g (for $n = 2$) and f in the same viewing window.

(b) Determine which function is increasing at a greater rate as x approaches infinity.

(c) Repeat parts (a) and (b) for $n = 3, 4$, and 5 . What do you notice?

125. **Exploration**

(a) Use a graphing utility to compare the graph of the function $y = \ln x$ with the graph of each function.

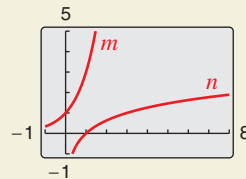
$y_1 = x - 1, y_2 = (x - 1) - \frac{1}{2}(x - 1)^2,$

$y_3 = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3$

(b) Identify the pattern of successive polynomials given in part (a). Extend the pattern one more term and compare the graph of the resulting polynomial function with the graph of $y = \ln x$. What do you think the pattern implies?



126. HOW DO YOU SEE IT? The figure shows the graphs of $f(x) = 3^x$ and $g(x) = \log_3 x$. [The graphs are labeled m and n .]



- (a) Match each function with its graph.
- (b) Given that $f(a) = b$, what is $g(b)$? Explain.

127. **Exploration**

(a) Use a graphing utility to complete the table for the function $f(x) = (\ln x)/x$.

x	1	5	10	10^2	10^4	10^6
$f(x)$						

(b) Use the table in part (a) to determine what value $f(x)$ approaches as x increases without bound. Use the graphing utility to confirm your result.

128. **Writing** Use a graphing utility to determine how many months it would take for the average score in Example 10 to decrease to 60. Explain your method of solving the problem. Describe another way that you can use the graphing utility to determine the answer. Also, based on the shape of the graph, does the rate at which a student forgets information *increase* or *decrease* with time? Explain.

Cumulative Mixed Review

Factoring a Polynomial In Exercises 129–136, factor the polynomial.

129. $x^2 + 2x - 3$

130. $x^2 + 4x - 5$

131. $12x^2 + 5x - 3$

132. $16x^2 - 54x - 7$

133. $16x^2 - 25$

134. $36x^2 - 49$

135. $2x^3 + x^2 - 45x$

136. $3x^3 - 5x^2 - 12x$

Evaluating an Arithmetic Combination of Functions In Exercises 137 and 138, evaluate the function for $f(x) = 3x + 2$ and $g(x) = x^3 - 1$.

137. $(f + g)(2)$

138. $(f - g)(-1)$

Using Graphs In Exercises 139–142, solve the equation graphically.

139. $5x - 7 = 7 + 5x$

140. $-2x + 3 = 8x$

141. $\sqrt{3x - 2} = 9$

142. $\sqrt{x - 11} = x + 2$

3.3 Properties of Logarithms

Change of Base

Most calculators have only two types of log keys, one for common logarithms (base 10) and one for natural logarithms (base e). Although common logs and natural logs are the most frequently used, you may occasionally need to evaluate logarithms to other bases. To do this, you can use the following **change-of-base formula**.

Change-of-Base Formula

Let a , b , and x be positive real numbers such that $a \neq 1$ and $b \neq 1$. Then $\log_a x$ can be converted to a different base using any of the following formulas.

<i>Base b</i>	<i>Base 10</i>	<i>Base e</i>
$\log_a x = \frac{\log_b x}{\log_b a}$	$\log_a x = \frac{\log_{10} x}{\log_{10} a}$	$\log_a x = \frac{\ln x}{\ln a}$

One way to look at the change-of-base formula is that logarithms to base a are simply *constant multiples* of logarithms to base b . The constant multiplier is $1/(\log_b a)$.

EXAMPLE 1 Changing Bases Using Common Logarithms

a. $\log_4 25 = \frac{\log_{10} 25}{\log_{10} 4}$ $\log_a x = \frac{\log_{10} x}{\log_{10} a}$

$$\approx \frac{1.39794}{0.60206}$$

Use a calculator.

$$\approx 2.32$$

Simplify.

b. $\log_3 17 = \frac{\log_{10} 17}{\log_{10} 3} \approx \frac{1.23045}{0.47712} \approx 2.58$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Evaluate $\log_2 12$ using the change-of-base formula and common logarithms.

EXAMPLE 2 Changing Bases Using Natural Logarithms

a. $\log_4 25 = \frac{\ln 25}{\ln 4}$ $\log_a x = \frac{\ln x}{\ln a}$

$$\approx \frac{3.21888}{1.38629}$$

Use a calculator.

$$\approx 2.32$$

Simplify.

b. $\log_3 17 = \frac{\ln 17}{\ln 3} \approx \frac{2.83321}{1.09861} \approx 2.58$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Evaluate $\log_2 12$ using the change-of-base formula and natural logarithms.

Notice in Examples 1 and 2 that the result is the same whether common logarithms or natural logarithms are used in the change-of-base formula.

What you should learn

- ▶ Rewrite logarithms with different bases.
- ▶ Use properties of logarithms to evaluate or rewrite logarithmic expressions.
- ▶ Use properties of logarithms to expand or condense logarithmic expressions.
- ▶ Use logarithmic functions to model and solve real-life problems.

Why you should learn it

Logarithmic functions can be used to model and solve real-life problems, such as the model for the number of decibels of a sound in Exercise 107 on page 208.



Properties of Logarithms

You know from the previous section that the logarithmic function with base a is the *inverse function* of the exponential function with base a . So, it makes sense that the properties of exponents (see Section 3.1) should have corresponding properties involving logarithms. For instance, the exponential property

$$a^0 = 1$$

has the corresponding logarithmic property

$$\log_a 1 = 0.$$

Encourage your students to know these properties well. They will be used for solving logarithmic and exponential equations, as well as in calculus.

Properties of Logarithms (See the proof on page 251.)

Let a be a positive real number such that $a \neq 1$, and let n be a real number. If u and v are positive real numbers, then the following properties are true.

	Logarithm with Base a	Natural Logarithm
1. Product Property:	$\log_a(uv) = \log_a u + \log_a v$	$\ln(uv) = \ln u + \ln v$
2. Quotient Property:	$\log_a \frac{u}{v} = \log_a u - \log_a v$	$\ln \frac{u}{v} = \ln u - \ln v$
3. Power Property:	$\log_a u^n = n \log_a u$	$\ln u^n = n \ln u$

Remark

There is no general property that can be used to rewrite $\log_a(u \pm v)$. Specifically, $\log_a(x + y)$ is *not* equal to $\log_a x + \log_a y$.

EXAMPLE 3 Using Properties of Logarithms

Write each logarithm in terms of $\ln 2$ and $\ln 3$.

a. $\ln 6$ b. $\ln \frac{2}{27}$

Solution

a. $\ln 6 = \ln(2 \cdot 3)$ Rewrite 6 as $2 \cdot 3$.
 $= \ln 2 + \ln 3$ Product Property

b. $\ln \frac{2}{27} = \ln 2 - \ln 27$ Quotient Property
 $= \ln 2 - \ln 3^3$ Rewrite 27 as 3^3 .
 $= \ln 2 - 3 \ln 3$ Power Property

 **Checkpoint**  Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Write each logarithm in terms of $\log_{10} 3$ and $\log_{10} 5$.

a. $\log_{10} 75$ b. $\log_{10} \frac{9}{125}$

EXAMPLE 4 Using Properties of Logarithms

Use the properties of logarithms to verify that $-\log_{10} \frac{1}{100} = \log_{10} 100$.

Solution

$$\begin{aligned} -\log_{10} \frac{1}{100} &= -\log_{10}(100^{-1}) && \text{Rewrite } \frac{1}{100} \text{ as } 100^{-1}. \\ &= -(-1) \log_{10} 100 && \text{Power Property} \\ &= \log_{10} 100 && \text{Simplify.} \end{aligned}$$

Remind your students to note the domain when applying the properties of logarithms to a logarithmic function. For example, the domain of $f(x) = \ln x^2$ is all real $x \neq 0$, whereas the domain of $g(x) = 2 \ln x$ is all real $x > 0$.

 **Checkpoint**  Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Use the properties of logarithms to verify that $\ln e^5 = 5$. 

Rewriting Logarithmic Expressions

The properties of logarithms are useful for rewriting logarithmic expressions in forms that simplify the operations of algebra. This is true because they convert complicated products, quotients, and exponential forms into simpler sums, differences, and products, respectively.

A common error that students make when expanding logarithmic expressions is to rewrite $\log ax^n$ as $n \log ax$ instead of $\log a + n \log x$.

EXAMPLE 5 Expanding Logarithmic Expressions

Use the properties of logarithms to expand each expression.

a. $\log_4 5x^3y$ b. $\ln \frac{\sqrt{3x-5}}{7}$

Solution

a. $\log_4 5x^3y = \log_4 5 + \log_4 x^3 + \log_4 y$
 $= \log_4 5 + 3 \log_4 x + \log_4 y$

Product Property

Power Property

b. $\ln \frac{\sqrt{3x-5}}{7} = \ln \frac{(3x-5)^{1/2}}{7}$
 $= \ln(3x-5)^{1/2} - \ln 7$
 $= \frac{1}{2} \ln(3x-5) - \ln 7$

Rewrite radical using rational exponent.

Quotient Property

Power Property

 **Checkpoint**  *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the properties of logarithms to expand the expression $\log_3 \frac{4x^2}{\sqrt{y}}$.

In Example 5, the properties of logarithms were used to *expand* logarithmic expressions. In Example 6, this procedure is reversed and the properties of logarithms are used to *condense* logarithmic expressions.

EXAMPLE 6 Condensing Logarithmic Expressions

See *LarsonPrecalculus.com* for an interactive version of this type of example.

Use the properties of logarithms to condense each expression.

a. $\frac{1}{2} \log_{10} x + 3 \log_{10}(x+1)$
 b. $2 \ln(x+2) - \ln x$
 c. $\frac{1}{3} [\log_2 x + \log_2(x-4)]$

Solution

a. $\frac{1}{2} \log_{10} x + 3 \log_{10}(x+1) = \log_{10} x^{1/2} + \log_{10}(x+1)^3$
 $= \log_{10} [\sqrt{x}(x+1)^3]$

Power Property

Product Property

b. $2 \ln(x+2) - \ln x = \ln(x+2)^2 - \ln x$
 $= \ln \frac{(x+2)^2}{x}$

Power Property

Quotient Property

c. $\frac{1}{3} [\log_2 x + \log_2(x-4)] = \frac{1}{3} \log_2 [x(x-4)]$
 $= \log_2 [x(x-4)]^{1/3}$
 $= \log_2 \sqrt[3]{x(x-4)}$

Product Property

Power Property

Rewrite with a radical.

 **Checkpoint**  *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the properties of logarithms to condense the expression

$2[\log_{10}(x+3) - 2 \log_{10}(x-2)].$

Explore the Concept

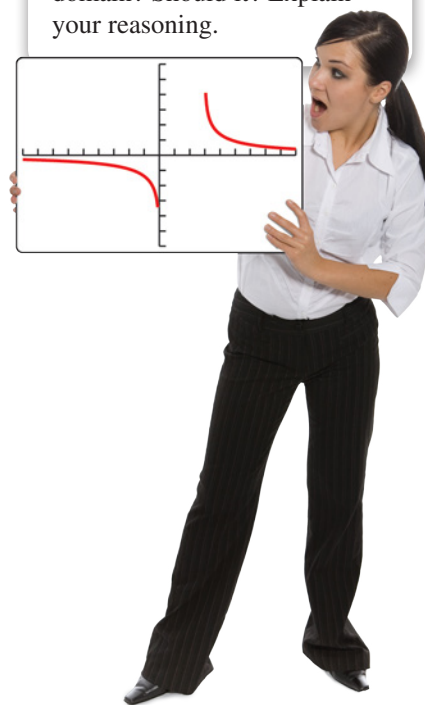
Use a graphing utility to graph the functions

$$y = \ln x - \ln(x-3)$$

and

$$y = \ln \frac{x}{x-3}$$

in the same viewing window. Does the graphing utility show the functions with the same domain? Should it? Explain your reasoning.



A common error that students make when condensing logarithmic expressions is to rewrite $\log x - \log y$ as

$$\frac{\log x}{\log y}$$

instead of

$$\log \frac{x}{y}$$

Application

EXAMPLE 7 Finding a Mathematical Model

The table shows the mean distance x from the sun and the period y (the time it takes a planet to orbit the sun) for each of the six planets that are closest to the sun. In the table, the mean distance is given in astronomical units (where the Earth's mean distance is defined as 1.0), and the period is given in years. The points in the table are plotted in Figure 3.18. Find an equation that relates y and x .

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
Mean distance, x	0.387	0.723	1.000	1.524	5.203	9.537
Period, y	0.241	0.615	1.000	1.881	11.863	29.447

Spreadsheet at LarsonPrecalculus.com

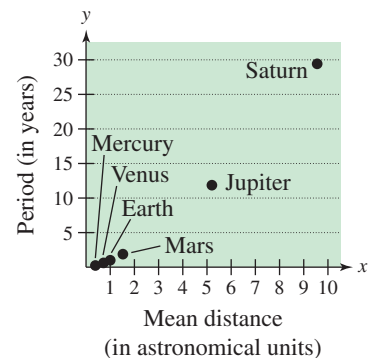


Figure 3.18

Solution

From Figure 3.18, it is not clear how to find an equation that relates y and x . To solve this problem, take the natural log of each of the x - and y -values in the table. This produces the following results.

Planet	Mercury	Venus	Earth	Mars	Jupiter	Saturn
$\ln x = X$	-0.949	-0.324	0.000	0.421	1.649	2.255
$\ln y = Y$	-1.423	-0.486	0.000	0.632	2.473	3.383

Now, by plotting the points in the table, you can see that all six of the points appear to lie in a line, as shown in Figure 3.19. To find an equation of the line through these points, you can use one of the following methods.

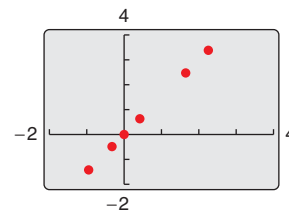


Figure 3.19

Method 1: Algebraic

Choose any two points to determine the slope of the line. Using the two points $(0.421, 0.632)$ and $(0, 0)$, you can determine that the slope of the line is

$$m = \frac{0.632 - 0}{0.421 - 0} \approx 1.5 = \frac{3}{2}$$

By the point-slope form, the equation of the line is

$$Y = \frac{3}{2}X$$

where $Y = \ln y$ and $X = \ln x$. You can therefore conclude that

$$\ln y = \frac{3}{2} \ln x.$$

Method 2: Graphical

Using the *linear regression* feature of a graphing utility, you can find a linear model for the data, as shown in Figure 3.20. You can approximate this model to be $Y = 1.5X$, where $Y = \ln y$ and $X = \ln x$. From the model, you can see that the slope of the line is $\frac{3}{2}$. So, you can conclude that $\ln y = \frac{3}{2} \ln x$.

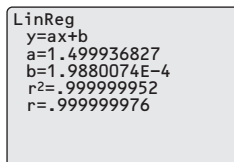


Figure 3.20

✓ Checkpoint  *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find a logarithmic equation that relates y and x for the following ordered pairs.

$$(0.37, 0.51), (1.00, 1.00), (2.72, 1.95), (7.39, 3.79), (20.09, 7.39)$$

In Example 7, try to convert the final equation to $y = f(x)$ form. You will get a function of the form $y = ax^b$, which is called a *power model*.

3.3 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank(s).

- You can evaluate logarithms to any base using the _____ formula.
- Two properties of logarithms are _____ = $n \log_a u$ and $\ln(uv) = \text{_____}$.
- Is $\log_3 24 = \frac{\ln 3}{\ln 24}$ or $\log_3 24 = \frac{\ln 24}{\ln 3}$ correct?
- Which property of logarithms can you use to condense the expression $\ln x - \ln 2$?

Procedures and Problem Solving

Changing the Base In Exercises 5–12, rewrite the logarithm as a ratio of (a) common logarithms and (b) natural logarithms.

- $\log_5 x$
- $\log_3 x$
- $\log_{1/6} x$
- $\log_{1/4} x$
- $\log_a \frac{3}{10}$
- $\log_a \frac{4}{5}$
- $\log_{2.6} x$
- $\log_{7.1} x$

Changing the Base In Exercises 13–20, evaluate the logarithm using the change-of-base formula. Round your result to three decimal places.

- $\log_3 7$
- $\log_9 4$
- $\log_{1/2} 16$
- $\log_{1/8} 64$
- $\log_6 0.9$
- $\log_4 0.045$
- $\log_{15} 1460$
- $\log_{20} 175$

Using Properties of Logarithms In Exercises 21–24, rewrite the expression in terms of $\ln 4$ and $\ln 5$.

- $\ln 20$
- $\ln 500$
- $\ln \frac{25}{4}$
- $\ln \frac{5}{2}$

Using Properties to Evaluate Logarithms In Exercises 25–28, approximate the logarithm using the properties of logarithms, given the values $\log_b 2 \approx 0.3562$, $\log_b 3 \approx 0.5646$, and $\log_b 5 \approx 0.8271$. Round your result to four decimal places.

- $\log_b 8$
- $\log_b 30$
- $\log_b \frac{16}{25}$
- $\log_b \sqrt{3}$

Graphing a Logarithm with Any Base In Exercises 29–36, use the change-of-base formula $\log_a x = (\ln x)/(\ln a)$ and a graphing utility to graph the function.

- $f(x) = \log_3(x + 1)$
- $f(x) = \log_2(x - 1)$
- $f(x) = \log_{1/2}(x - 2)$
- $f(x) = \log_{1/3}(x + 2)$
- $f(x) = \log_{1/4} x^2$
- $f(x) = \log_3 \sqrt{x}$

$$35. f(x) = \log_5 \left(\frac{x}{2} \right) \qquad 36. f(x) = \log_{1/3} \left(\frac{x}{3} \right)$$

Simplifying a Logarithm In Exercises 37–44, use the properties of logarithms to rewrite and simplify the logarithmic expression.

- $\log_4 8$
- $\log_9 243$
- $\log_2 4^2 \cdot 3^4$
- $\log_3 9^2 \cdot 2^4$
- $\ln 5e^6$
- $\ln 8e^3$
- $\ln \frac{6}{e^2}$
- $\ln \frac{e^5}{7}$

Using Properties of Logarithms In Exercises 45 and 46, use the properties of logarithms to verify the equation.

- $\log_5 \frac{1}{250} = -3 - \log_5 2$
- $-\ln 24 = -(3 \ln 2 + \ln 3)$

Expanding Logarithmic Expressions In Exercises 47–64, use the properties of logarithms to expand the expression as a sum, difference, and/or constant multiple of logarithms. (Assume all variables are positive.)

- $\log_{10} 10x$
- $\log_{10} 100x$
- $\log_{10} \frac{t}{8}$
- $\log_{10} \frac{7}{z}$
- $\log_8 x^4$
- $\log_6 z^{-3}$
- $\ln \sqrt{z}$
- $\ln \sqrt[3]{t}$
- $\ln xyz$
- $\ln \frac{xy}{z}$
- $\log_6 ab^3c^{-2}$
- $\log_4 xy^6z^4$
- $\ln \sqrt[3]{\frac{x^4}{y^3}}$
- $\ln \sqrt{\frac{x^2}{y^3}}$
- $\ln \frac{x^2 - 1}{x^3}, x > 1$
- $\ln \frac{x}{\sqrt{x^2 + 1}}$
- $\log_b \frac{x^4 \sqrt{y}}{z^5}$
- $\log_b \frac{\sqrt{xy^4}}{z^4}$

Algebraic-Graphical-Numerical In Exercises 65–68, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) What do the graphs and tables suggest? Verify your conclusion algebraically.

65. $y_1 = \ln[x^2(x - 4)]$
 $y_2 = 2 \ln x + \ln(x - 4)$

66. $y_1 = \ln 9x^3$
 $y_2 = \ln 9 + 3 \ln x$

67. $y_1 = \ln\left(\frac{x^4}{x - 2}\right)$
 $y_2 = 4 \ln x - \ln(x - 2)$

68. $y_1 = \ln\left(\frac{\sqrt{x}}{x + 3}\right)$
 $y_2 = \frac{1}{2} \ln x - \ln(x + 3)$

Condensing Logarithmic Expressions In Exercises 69–84, use the properties of logarithms to condense the expression.

69. $\ln x + \ln 4$ 70. $\ln y + \ln z$
 71. $\log_4 z - \log_4 y$ 72. $\log_5 8 - \log_5 t$

73. $4 \log_3(x + 2)$
 74. $\frac{5}{2} \log_7(z - 4)$
 75. $\frac{1}{2} \ln(x^2 + 4) + \ln x$
 76. $2 \ln x + \ln(x + 1)$
 77. $\ln x - 3 \ln(x + 1)$
 78. $\ln x - 2 \ln(x + 2)$

79. $\ln(x - 2) + \ln 2 - 3 \ln y$
 80. $3 \ln x + 2 \ln y - 4 \ln z$
 81. $\ln x - 2[\ln(x + 2) + \ln(x - 2)]$
 82. $4[\ln z + \ln(z + 5)] - 2 \ln(z - 5)$
 83. $\frac{1}{3}[2 \ln(x + 3) + \ln x - \ln(x^2 - 1)]$
 84. $2[\ln x - \ln(x + 1) - \ln(x - 1)]$

Algebraic-Graphical-Numerical In Exercises 85–88, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) What do the graphs and tables suggest? Verify your conclusion algebraically.

85. $y_1 = 2[\ln 8 - \ln(x^2 + 1)]$, $y_2 = \ln\left[\frac{64}{(x^2 + 1)^2}\right]$

86. $y_1 = 2[\ln 6 + \ln(x^2 + 1)]$, $y_2 = \ln[36(x^2 + 1)^2]$

87. $y_1 = \ln x + \frac{1}{2} \ln(x + 1)$, $y_2 = \ln(x\sqrt{x + 1})$

88. $y_1 = \frac{1}{2} \ln x - \ln(x + 2)$, $y_2 = \ln\left(\frac{\sqrt{x}}{x + 2}\right)$

Using Properties to Evaluate Logarithms In Exercises 89–102, find the exact value of the logarithm without using a calculator. If this is not possible, state the reason.

89. $\log_3 9$ 90. $\log_6 6$
 91. $\log_4 16^{3.4}$ 92. $\log_5\left(\frac{1}{125}\right)$
 93. $\log_2(-4)$ 94. $\log_4(-16)$
 95. $\log_5 375 - \log_5 3$ 96. $\log_4 2 + \log_4 32$
 97. $\ln e^3 - \ln e^7$ 98. $\ln e^6 - 2 \ln e^7$
 99. $2 \ln e^4$ 100. $3 \ln e^5$
 101. $\ln \frac{1}{\sqrt{e}}$ 102. $\ln \sqrt[5]{e^3}$

Algebraic-Graphical-Numerical In Exercises 103–106, (a) use a graphing utility to graph the two equations in the same viewing window and (b) use the *table* feature of the graphing utility to create a table of values for each equation. (c) Are the expressions equivalent? Explain. Verify your conclusion algebraically.

103. $y_1 = \ln x^2$, $y_2 = 2 \ln x$
 104. $y_1 = 2(\ln 2 + \ln x)$, $y_2 = \ln 4x^2$
 105. $y_1 = \ln(x - 2) + \ln(x + 2)$, $y_2 = \ln(x^2 - 4)$
 106. $y_1 = \frac{1}{4} \ln[x^4(x^2 + 1)]$, $y_2 = \ln x + \frac{1}{4} \ln(x^2 + 1)$

107. **Why you should learn it** (p. 203) The relationship between the number of decibels β and the intensity of a sound I in watts per square meter is given by



$$\beta = 10 \log_{10}\left(\frac{I}{10^{-12}}\right).$$

- (a) Use the properties of logarithms to write the formula in a simpler form.
 (b) Use a graphing utility to complete the table. Verify your answers algebraically.


I	10^{-4}	10^{-6}	10^{-8}	10^{-10}	10^{-12}	10^{-14}
β						

108. **Nail Length** The table shows the approximate lengths and diameters (in inches) of common nails. Find a logarithmic equation that relates the diameter y of a common nail to its length x .

	Length, x	Diameter, y
Spreadsheet at LarsonPrecalculus.com	1	0.072
	2	0.120
	3	0.148
	4	0.203
	5	0.238
	6	0.284

109. MODELING DATA

A beaker of liquid at an initial temperature of 78°C is placed in a room at a constant temperature of 21°C. The temperature of the liquid is measured every 5 minutes during a half-hour period. The results are recorded as ordered pairs of the form (t, T) , where t is the time (in minutes) and T is the temperature (in degrees Celsius). (*Spreadsheet at LarsonPrecalculus.com*)

 $(0, 78.0^\circ), (5, 66.0^\circ), (10, 57.5^\circ), (15, 51.2^\circ), (20, 46.3^\circ), (25, 42.5^\circ), (30, 39.6^\circ)$

(a) The graph of the temperature of the room should be an asymptote of the graph of the model for the data. Subtract the room temperature from each of the temperatures in the ordered pairs. Use a graphing utility to plot the data points (t, T) and $(t, T - 21)$.

(b) An exponential model for the data $(t, T - 21)$ is given by

$$T - 21 = 54.4(0.964)^t.$$

Solve for T and graph the model. Compare the result with the plot of the original data.

(c) Take the natural logarithms of the revised temperatures. Use the graphing utility to plot the points $(t, \ln(T - 21))$ and observe that the points appear linear. Use the *regression* feature of the graphing utility to fit a line to the data. The resulting line has the form

$$\ln(T - 21) = at + b.$$

Use the properties of logarithms to solve for T . Verify that the result is equivalent to the model in part (b).

(d) Fit a rational model to the data. Take the reciprocals of the y -coordinates of the revised data points to generate the points

$$\left(t, \frac{1}{T - 21}\right).$$

Use the graphing utility to plot these points and observe that they appear linear. Use the *regression* feature of the graphing utility to fit a line to the data. The resulting line has the form

$$\frac{1}{T - 21} = at + b.$$

Solve for T and use the graphing utility to graph the rational function and the original data points.

110. Writing Write a short paragraph explaining why the transformations of the data in Exercise 109 were necessary to obtain the models. Why did taking the logarithms of the temperatures lead to a linear scatter plot? Why did taking the reciprocals of the temperatures lead to a linear scatter plot?

Conclusions

True or False? In Exercises 111–116, determine whether the statement is true or false given that $f(x) = \ln x$, where $x > 0$. Justify your answer.

111. $f(ax) = f(a) + f(x), a > 0$

112. $f(x - a) = f(x) - f(a), x > a$

113. $\sqrt{f(x)} = \frac{1}{2}f(x)$

114. $[f(x)]^n = nf(x)$

115. If $f(x) < 0$, then $0 < x < 1$.

116. If $f(x) > 0$, then $x > e$.

117. **Error Analysis** Describe the error.

~~$$\ln\left(\frac{x^2}{\sqrt{x^2 + 4}}\right) = \frac{\ln x^2}{\ln \sqrt{x^2 + 4}}$$~~

118. **Think About It** Consider the functions below.

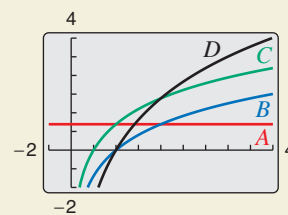
$$f(x) = \ln \frac{x}{2}, \quad g(x) = \frac{\ln x}{\ln 2}, \quad h(x) = \ln x - \ln 2$$

Which two functions have identical graphs? Verify your answer by using a graphing utility to graph all three functions in the same viewing window.

119. **Exploration** For how many integers between 1 and 20 can the natural logarithms be approximated given that $\ln 2 \approx 0.6931, \ln 3 \approx 1.0986$, and $\ln 5 \approx 1.6094$? Approximate these logarithms without a calculator.



120. HOW DO YOU SEE IT? The figure shows the graphs of $y = \ln x, y = \ln x^2, y = \ln 2x$, and $y = \ln 2$. Match each function with its graph. (The graphs are labeled A through D.) Explain your reasoning.



121. **Think About It** Does $y_1 = \ln[x(x - 2)]$ have the same domain as $y_2 = \ln x + \ln(x - 2)$? Explain.

122. **Proof** Prove that $\frac{\log_a x}{\log_a/b x} = 1 + \log_a \frac{1}{b}$.

Cumulative Mixed Review

Using Properties of Exponents In Exercises 123–126, simplify the expression.

123. $(64x^3y^4)^{-3}(8x^3y^2)^4$ 124. $xy(x^{-1} + y^{-1})^{-1}$

125. $\frac{24xy^{-2}}{16x^{-3}y}$ 126. $\left(\frac{2x^3}{3y}\right)^{-3}$

3.4 Solving Exponential and Logarithmic Equations

Introduction

So far in this chapter, you have studied the definitions, graphs, and properties of exponential and logarithmic functions. In this section, you will study procedures for solving equations involving exponential and logarithmic functions.

There are two basic strategies for solving exponential or logarithmic equations. The first is based on the One-to-One Properties and the second is based on the Inverse Properties. For $a > 0$ and $a \neq 1$, the following properties are true for all x and y for which $\log_a x$ and $\log_a y$ are defined.

One-to-One Properties

$$a^x = a^y \quad \text{if and only if} \quad x = y.$$

$$\log_a x = \log_a y \quad \text{if and only if} \quad x = y.$$

Inverse Properties

$$a^{\log_a x} = x$$

$$\log_a a^x = x$$

What you should learn

- ▶ Solve simple exponential and logarithmic equations.
- ▶ Solve more complicated exponential equations.
- ▶ Solve more complicated logarithmic equations.
- ▶ Use exponential and logarithmic equations to model and solve real-life problems.

Why you should learn it

Exponential and logarithmic equations can be used to model and solve real-life problems. For instance, Exercise 148 on page 219 shows how to use an exponential function to model the average heights of men and women.



EXAMPLE 1 Solving Simple Equations

Original Equation	Rewritten Equation	Solution	Property
a. $2^x = 32$	$2^x = 2^5$	$x = 5$	One-to-One
b. $\log_4 x - \log_4 8 = 0$	$\log_4 x = \log_4 8$	$x = 8$	One-to-One
c. $\ln x - \ln 3 = 0$	$\ln x = \ln 3$	$x = 3$	One-to-One
d. $(\frac{1}{3})^x = 9$	$3^{-x} = 3^2$	$x = -2$	One-to-One
e. $e^x = 7$	$\ln e^x = \ln 7$	$x = \ln 7$	Inverse
f. $\ln x = -3$	$e^{\ln x} = e^{-3}$	$x = e^{-3}$	Inverse
g. $\log_{10} x = -1$	$10^{\log_{10} x} = 10^{-1}$	$x = 10^{-1} = \frac{1}{10}$	Inverse
h. $\log_3 x = 4$	$3^{\log_3 x} = 3^4$	$x = 81$	Inverse

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Solve each equation for x .

- a. $2^x = 512$ b. $\log_6 x = 3$ c. $5 - e^x = 0$ d. $9^x = \frac{1}{3}$

The strategies used in Example 1 are summarized as follows.

Strategies for Solving Exponential and Logarithmic Equations

1. Rewrite the original equation in a form that allows the use of the One-to-One Properties of exponential or logarithmic functions.
2. Rewrite an exponential equation in logarithmic form and apply the Inverse Property of logarithmic functions.
3. Rewrite a logarithmic equation in exponential form and apply the Inverse Property of exponential functions.

Solving Exponential Equations

EXAMPLE 2 Solving Exponential Equations

Solve each equation.

a. $e^x = 72$

b. $3(2^x) = 42$

Algebraic Solution

a. $e^x = 72$

Write original equation.

$$\ln e^x = \ln 72$$

Take natural log of each side.

$$x = \ln 72$$

Inverse Property

$$x \approx 4.28$$

Use a calculator.

The solution is $x = \ln 72 \approx 4.28$. Check this in the original equation.

b. $3(2^x) = 42$

Write original equation.

$$2^x = 14$$

Divide each side by 3.

$$\log_2 2^x = \log_2 14$$

Take log (base 2) of each side.

$$x = \log_2 14$$

Inverse Property

$$x = \frac{\ln 14}{\ln 2}$$

Change-of-base formula

$$x \approx 3.81$$

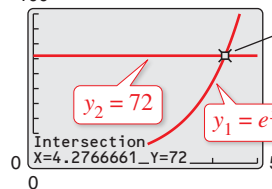
Use a calculator.

The solution is $x = \log_2 14 \approx 3.81$. Check this in the original equation.

Graphical Solution

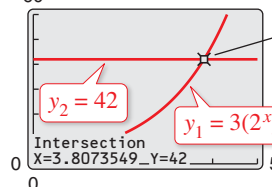
To solve an equation using a graphing utility, you can graph the left- and right-hand sides of the equation and use the *intersect* feature.

a. 100



The intersection point is about $(4.28, 72)$. So, the solution is $x \approx 4.28$.

b. 60



The intersection point is about $(3.81, 42)$. So, the solution is $x \approx 3.81$.

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Solve (a) $e^x = 10$ and (b) $2(5^x) = 32$.

EXAMPLE 3 Solving an Exponential Equation

Solve $4e^{2x} - 3 = 2$.

Algebraic Solution

$$4e^{2x} - 3 = 2$$

Write original equation.

$$4e^{2x} = 5$$

Add 3 to each side.

$$e^{2x} = \frac{5}{4}$$

Divide each side by 4.

$$\ln e^{2x} = \ln \frac{5}{4}$$

Take natural log of each side.

$$2x = \ln \frac{5}{4}$$

Inverse Property

$$x = \frac{1}{2} \ln \frac{5}{4}$$

Divide each side by 2.

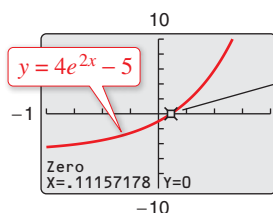
$$x \approx 0.11$$

Use a calculator.

The solution is $x = \frac{1}{2} \ln \frac{5}{4} \approx 0.11$. Check this in the original equation.

Graphical Solution

Rather than using the procedure in Example 2, another way to solve the equation graphically is first to rewrite the equation as $4e^{2x} - 5 = 0$, and then use a graphing utility to graph $y = 4e^{2x} - 5$. Use the *zero* or *root* feature of the graphing utility to approximate the value of x for which $y = 0$, as shown in the figure.



The zero occurs at $x \approx 0.11$. So, the solution of the original equation is $x \approx 0.11$.

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Solve $2e^{3x} + 5 = 29$.

EXAMPLE 4 Solving an Exponential Equation

Solve $2(3^{2t-5}) - 4 = 11$.

Solution

$2(3^{2t-5}) - 4 = 11$	Write original equation.
$2(3^{2t-5}) = 15$	Add 4 to each side.
$3^{2t-5} = \frac{15}{2}$	Divide each side by 2.
$\log_3 3^{2t-5} = \log_3 \frac{15}{2}$	Take log (base 3) of each side.
$2t - 5 = \log_3 \frac{15}{2}$	Inverse Property
$2t = 5 + \log_3 7.5$	Add 5 to each side.
$t = \frac{5}{2} + \frac{1}{2} \log_3 7.5$	Divide each side by 2.
$t \approx 3.42$	Use a calculator.

The solution is $t = \frac{5}{2} + \frac{1}{2} \log_3 7.5 \approx 3.42$. Check this in the original equation.

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Solve $6(2^{t+5}) + 4 = 11$.

When an equation involves two or more exponential expressions, you can still use a procedure similar to that demonstrated in the previous three examples. However, the algebra is a bit more complicated.

Remark

Remember that to evaluate a logarithm such as $\log_3 7.5$, you need to use the change-of-base formula.

$$\log_3 7.5 = \frac{\ln 7.5}{\ln 3} \approx 1.834$$

EXAMPLE 5 Solving an Exponential Equation in Quadratic Form

Solve $e^{2x} - 3e^x + 2 = 0$.

Algebraic Solution

$e^{2x} - 3e^x + 2 = 0$	Write original equation.
$(e^x)^2 - 3e^x + 2 = 0$	Write in quadratic form.
$(e^x - 2)(e^x - 1) = 0$	Factor.
$e^x - 2 = 0$	Set 1st factor equal to 0.
$e^x = 2$	Add 2 to each side.
$x = \ln 2$	Solution
$e^x - 1 = 0$	Set 2nd factor equal to 0.
$e^x = 1$	Add 1 to each side.
$x = \ln 1$	Inverse Property
$x = 0$	Solution

The solutions are

$x = \ln 2 \approx 0.69$ and $x = 0$.

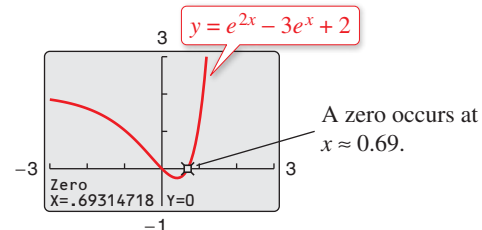
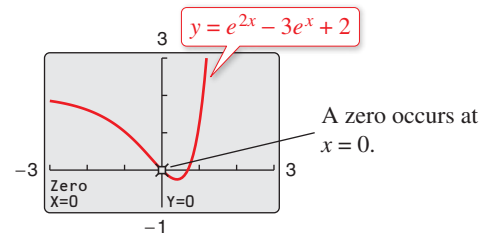
Check these in the original equation.

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Solve $e^{2x} - 7e^x + 12 = 0$.

Graphical Solution

Use a graphing utility to graph $y = e^{2x} - 3e^x + 2$ and then find the zeros.



From the figures, you can conclude that the solutions are $x = 0$ and $x \approx 0.69$.

Solving Logarithmic Equations

To solve a logarithmic equation, you can write it in exponential form.

$$\begin{array}{ll} \ln x = 3 & \text{Logarithmic form} \\ e^{\ln x} = e^3 & \text{Exponentiate each side.} \\ x = e^3 & \text{Exponential form} \end{array}$$

This procedure is called *exponentiating* each side of an equation. It is applied after the logarithmic expression has been isolated.

EXAMPLE 6 Solving Logarithmic Equations

Solve each logarithmic equation.

- a. $\ln 3x = 2$
b. $\log_3(5x - 1) = \log_3(x + 7)$

Solution

a. $\ln 3x = 2$ Write original equation.
 $e^{\ln 3x} = e^2$ Exponentiate each side.
 $3x = e^2$ Inverse Property
 $x = \frac{1}{3}e^2$ Multiply each side by $\frac{1}{3}$.
 $x \approx 2.46$ Use a calculator.

The solution is $x = \frac{1}{3}e^2 \approx 2.46$. Check this in the original equation.

b. $\log_3(5x - 1) = \log_3(x + 7)$ Write original equation.
 $5x - 1 = x + 7$ One-to-One Property
 $x = 2$ Solve for x .

The solution is $x = 2$. Check this in the original equation.

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Solve each logarithmic equation.

- a. $\ln x = \frac{2}{3}$ b. $\log_2(2x - 3) = \log_2(x + 4)$

EXAMPLE 7 Solving a Logarithmic Equation

Solve $5 + 2 \ln x = 4$.

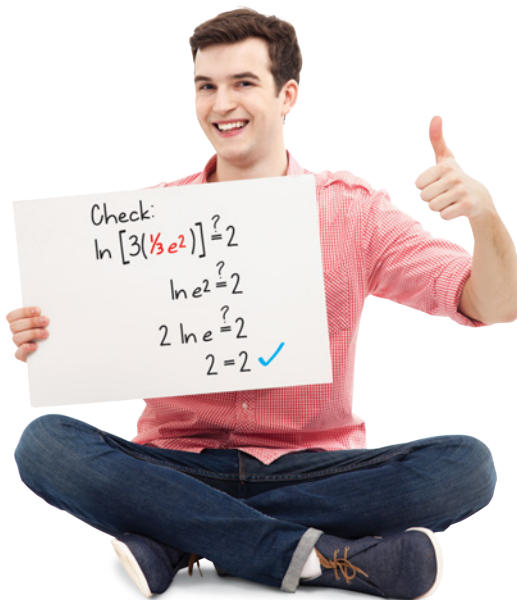
Algebraic Solution

$$\begin{array}{ll} 5 + 2 \ln x = 4 & \text{Write original equation.} \\ 2 \ln x = -1 & \text{Subtract 5 from each side.} \\ \ln x = -\frac{1}{2} & \text{Divide each side by 2.} \\ e^{\ln x} = e^{-1/2} & \text{Exponentiate each side.} \\ x = e^{-1/2} & \text{Inverse Property} \\ x \approx 0.61 & \text{Use a calculator.} \end{array}$$

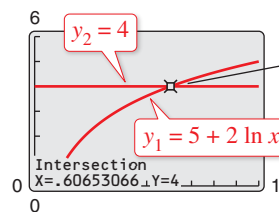
The solution is $x = e^{-1/2} \approx 0.61$. Check this in the original equation.

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Solve $7 + 3 \ln x = 5$.



Graphical Solution



The intersection point is about $(0.61, 4)$. So, the solution is $x \approx 0.61$.

EXAMPLE 8 Solving a Logarithmic Equation

Solve $2 \log_5 3x = 4$.

Solution

$$\begin{aligned}
 2 \log_5 3x &= 4 && \text{Write original equation.} \\
 \log_5 3x &= 2 && \text{Divide each side by 2.} \\
 5^{\log_5 3x} &= 5^2 && \text{Exponentiate each side (base 5).} \\
 3x &= 25 && \text{Inverse Property} \\
 x &= \frac{25}{3} && \text{Divide each side by 3.}
 \end{aligned}$$

The solution is $x = \frac{25}{3}$. Check this in the original equation. Or, perform a graphical check by graphing

$$y_1 = 2 \log_5 3x = 2 \left(\frac{\log_{10} 3x}{\log_{10} 5} \right) \quad \text{and} \quad y_2 = 4$$

in the same viewing window. The two graphs should intersect at $x = \frac{25}{3} \approx 8.33$ and $y = 4$, as shown in Figure 3.21.

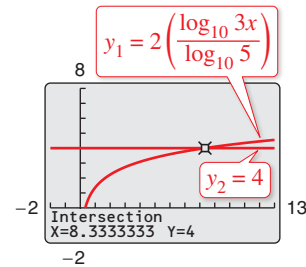


Figure 3.21

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Solve $3 \log_4 6x = 9$.

Because the domain of a logarithmic function generally does not include all real numbers, you should be sure to check for extraneous solutions of logarithmic equations, as shown in the next example.

EXAMPLE 9 Checking for Extraneous Solutions

Solve $\ln(x - 2) + \ln(2x - 3) = 2 \ln x$.

Algebraic Solution

$$\begin{aligned}
 \ln(x - 2) + \ln(2x - 3) &= 2 \ln x && \text{Write original equation.} \\
 \ln[(x - 2)(2x - 3)] &= \ln x^2 && \text{Use properties of logarithms.} \\
 \ln(2x^2 - 7x + 6) &= \ln x^2 && \text{Multiply binomials.} \\
 2x^2 - 7x + 6 &= x^2 && \text{One-to-One Property} \\
 x^2 - 7x + 6 &= 0 && \text{Write in general form.} \\
 (x - 6)(x - 1) &= 0 && \text{Factor.} \\
 x - 6 &= 0 &\rightarrow& x = 6 && \text{Set 1st factor equal to 0.} \\
 x - 1 &= 0 &\rightarrow& x = 1 && \text{Set 2nd factor equal to 0.}
 \end{aligned}$$

Finally, by checking these two “solutions” in the original equation, you can conclude that $x = 1$ is not valid. This is because when $x = 1$,

$$\ln(x - 2) + \ln(2x - 3) = \ln(-1) + \ln(-1)$$

which is invalid because -1 is not in the domain of the natural logarithmic function. So, the only solution is $x = 6$.

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Solve $\log_{10} x + \log_{10}(x - 9) = 1$.

Graphical Solution

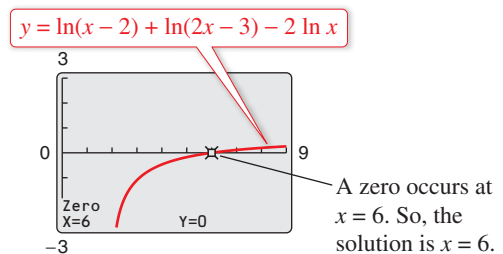
First, rewrite the original equation as

$$\ln(x - 2) + \ln(2x - 3) - 2 \ln x = 0.$$

Then, use a graphing utility to graph the equation

$$y = \ln(x - 2) + \ln(2x - 3) - 2 \ln x$$

and find the zeros, as shown in the figure.



EXAMPLE 10 The Change-of-Base Formula

Prove the change-of-base formula: $\log_a x = \frac{\log_b x}{\log_b a}$.

Solution

Begin by letting $y = \log_a x$ and writing the equivalent exponential form

$$a^y = x.$$

Now, taking the logarithms *with base b* of each side produces the following.


$$\log_b a^y = \log_b x$$

$$y \log_b a = \log_b x \quad \text{Power Property}$$

$$y = \frac{\log_b x}{\log_b a} \quad \text{Divide each side by } \log_b a.$$

$$\log_a x = \frac{\log_b x}{\log_b a} \quad \text{Replace } y \text{ with } \log_a x.$$

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Prove that $\log_a \frac{1}{x} = -\log_a x$. 

Equations that involve combinations of algebraic functions, exponential functions, and logarithmic functions can be very difficult to solve by algebraic procedures. Here again, you can take advantage of a graphing utility.

EXAMPLE 11 Approximating the Solution of an Equation

Approximate (to three decimal places) the solution of $\ln x = x^2 - 2$.

Solution

First, rewrite the equation as

$$\ln x - x^2 + 2 = 0.$$

Then, use a graphing utility to graph

$$y = -x^2 + 2 + \ln x$$

as shown in Figure 3.22. From this graph, you can see that the equation has two solutions. Next, using the *zero* or *root* feature, you can approximate the two solutions to be $x \approx 0.138$ and $x \approx 1.564$.

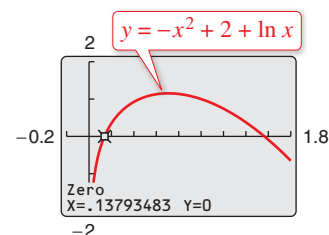


Figure 3.22

Check

$$\ln x = x^2 - 2 \quad \text{Write original equation.}$$

$$\ln(0.138) \stackrel{?}{\approx} (0.138)^2 - 2 \quad \text{Substitute 0.138 for } x.$$

$$-1.9805 \approx -1.9810 \quad \text{Solution checks. } \checkmark$$

$$\ln(1.564) \stackrel{?}{\approx} (1.564)^2 - 2 \quad \text{Substitute 1.564 for } x.$$

$$0.4472 \approx 0.4461 \quad \text{Solution checks. } \checkmark$$

So, the two solutions $x \approx 0.138$ and $x \approx 1.564$ seem reasonable.

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Approximate (to three decimal places) the solution of $5e^{0.3x} = 17$. 

Activities

1. Solve for x : $7^x = 3$

Answer: $x = \log_7 3$

$$= \frac{\ln 3}{\ln 7} \\ \approx 0.565$$

2. Solve for x :

$$\log_{10}(x + 4) + \log_{10}(x + 1) = 1$$

Answer: $x = 1$ ($x = -6$ is not in the domain.)

Applications

EXAMPLE 12 Doubling an Investment

See *LarsonPrecalculus.com* for an interactive version of this type of example.

You have deposited \$500 in an account that pays 6.75% interest, compounded continuously. How long will it take your money to double?

Solution

Using the formula for continuous compounding, you can find that the balance in the account is $A = Pe^{rt} = 500e^{0.0675t}$. To find the time required for the balance to double, let $A = 1000$ and solve the resulting equation for t .

$$\begin{aligned}
 500e^{0.0675t} &= 1000 && \text{Substitute 1000 for } A. \\
 e^{0.0675t} &= 2 && \text{Divide each side by 500.} \\
 \ln e^{0.0675t} &= \ln 2 && \text{Take natural log of each side.} \\
 0.0675t &= \ln 2 && \text{Inverse Property} \\
 t &= \frac{\ln 2}{0.0675} && \text{Divide each side by 0.0675.} \\
 t &\approx 10.27 && \text{Use a calculator.}
 \end{aligned}$$

The balance in the account will double after approximately 10.27 years. This result is demonstrated graphically in Figure 3.23.

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You invest \$500 at an annual interest rate of 5.25%, compounded continuously. How long will it take your money to double? Compare your result with that of Example 12.

EXAMPLE 13 Average Salary for Public School Teachers

From 2004 through 2013, the average salary y (in thousands of dollars) for public school teachers for the year t can be modeled by the equation $y = 33.374 + 9.162 \ln t$, where $t = 4$ represents 2004. During which year did the average salary for public school teachers reach \$50 thousand? (Source: National Center for Education Statistics)

Solution

$$\begin{aligned}
 33.374 + 9.162 \ln t &= y && \text{Write original equation.} \\
 33.374 + 9.162 \ln t &= 50 && \text{Substitute 50 for } y. \\
 9.162 \ln t &= 16.626 && \text{Subtract 33.374 from each side.} \\
 \ln t &= \frac{16.626}{9.162} && \text{Divide each side by 9.162.} \\
 e^{\ln t} &= e^{16.626/9.162} && \text{Exponentiate each side.} \\
 t &= e^{16.626/9.162} && \text{Inverse Property} \\
 t &\approx 6.14 && \text{Use a calculator.}
 \end{aligned}$$

The solution is $t \approx 6.14$ years. Because $t = 4$ represents 2004, it follows that the average salary for public school teachers reached \$50 thousand in 2006.

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In Example 13, during which year did the average salary reach \$52.5 thousand?

Additional Example

A deposit of \$5000 is placed in a savings account and compounded continuously for 2 years. After 2 years, the balance in the account is \$5416.50. What is the annual interest rate for this account?

Solution

Using the formula for continuous compounding, $A = Pe^{rt}$, you have

$$\begin{aligned}
 5416.50 &= 5000e^{2r} \\
 \frac{5416.50}{5000} &= e^{2r} \\
 1.0833 &= e^{2r} \\
 \ln 1.0833 &= 2r \\
 0.08 &\approx 2r \\
 0.04 &\approx r.
 \end{aligned}$$

The rate is approximately 4%.

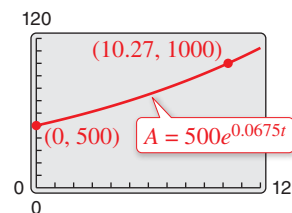
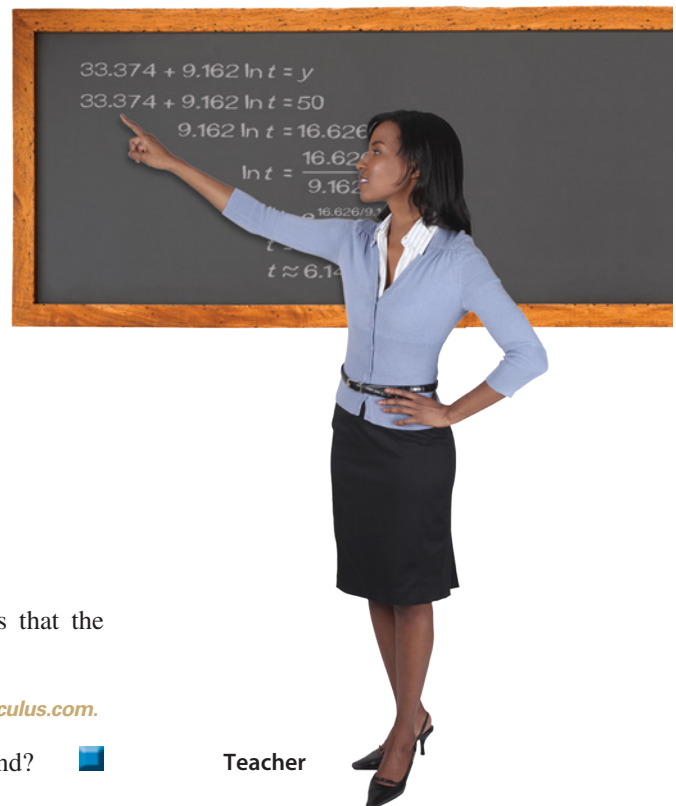


Figure 3.23



Teacher

3.4 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

- To solve exponential and logarithmic equations, you can use the following One-to-One and Inverse Properties.
 - $a^x = a^y$ if and only if _____ .
 - $\log_a x = \log_a y$ if and only if _____ .
 - $a^{\log_a x} =$ _____
 - $\log_a a^x =$ _____
- A(n) _____ solution does not satisfy the original equation.
- What is the value of $\ln e^7$?
- Can you solve $5^x = 125$ using a One-to-One Property?
- What is the first step in solving the equation $3 + \ln x = 10$?
- Do you solve $\log_4 x = 2$ by using a One-to-One Property or an Inverse Property?

Procedures and Problem Solving

Checking Solutions In Exercises 7–14, determine whether each x -value is a solution of the equation.

- | | |
|-------------------------|---|
| 7. $4^{2x-7} = 64$ | 8. $2^{3x+1} = 128$ |
| (a) $x = 5$ | (a) $x = -1$ |
| (b) $x = 2$ | (b) $x = 2$ |
| 9. $3e^{x+2} = 75$ | 10. $4e^{x-1} = 60$ |
| (a) $x = -2 + e^{25}$ | (a) $x = 1 + \ln 15$ |
| (b) $x = -2 + \ln 25$ | (b) $x \approx 3.7081$ |
| (c) $x \approx 1.2189$ | (c) $x = \ln 16$ |
| 11. $\log_4(3x) = 3$ | 12. $\log_6\left(\frac{5}{3}x\right) = 2$ |
| (a) $x \approx 21.3560$ | (a) $x \approx 20.2882$ |
| (b) $x = -4$ | (b) $x = \frac{108}{5}$ |
| (c) $x = \frac{64}{3}$ | (c) $x = 21.6$ |
| 13. $\ln(x - 1) = 3.8$ | 14. $\ln(2 + x) = 2.5$ |
| (a) $x = 1 + e^{3.8}$ | (a) $x = e^{2.5} - 2$ |
| (b) $x \approx 45.7012$ | (b) $x \approx \frac{4073}{400}$ |
| (c) $x = 1 + \ln 3.8$ | (c) $x = \frac{1}{2}$ |

Solving Equations Graphically In Exercises 15–22, use a graphing utility to graph f and g in the same viewing window. Approximate the point of intersection of the graphs of f and g . Then solve the equation $f(x) = g(x)$ algebraically.

- | | |
|---------------------------|---------------------------|
| 15. $f(x) = 64^x$ | 16. $f(x) = 27^x$ |
| $g(x) = 8$ | $g(x) = 9$ |
| 17. $f(x) = 5^{x-2} - 15$ | 18. $f(x) = 2^{-x+1} - 3$ |
| $g(x) = 10$ | $g(x) = 13$ |
| 19. $f(x) = 4 \log_3 x$ | 20. $f(x) = 3 \log_5 x$ |
| $g(x) = 20$ | $g(x) = 6$ |

$$21. \begin{aligned} f(x) &= \ln e^{x+1} \\ g(x) &= 2x + 5 \end{aligned}$$

$$22. \begin{aligned} f(x) &= \ln e^{x-2} \\ g(x) &= 3x + 2 \end{aligned}$$

Solving an Exponential Equation In Exercises 23–36, solve the exponential equation.

- | | |
|--|--|
| 23. $4^x = 16$ | 24. $3^x = 243$ |
| 25. $5^x = \frac{1}{625}$ | 26. $7^x = \frac{1}{49}$ |
| 27. $\left(\frac{1}{8}\right)^x = 64$ | 28. $\left(\frac{1}{2}\right)^x = 32$ |
| 29. $\left(\frac{2}{3}\right)^x = \frac{81}{16}$ | 30. $\left(\frac{3}{4}\right)^x = \frac{64}{27}$ |
| 31. $e^x = 14$ | 32. $e^x = 66$ |
| 33. $6(10^x) = 216$ | 34. $5(8^x) = 325$ |
| 35. $2^{x+3} = 256$ | 36. $4^{x-1} = 256$ |

Solving a Logarithmic Equation In Exercises 37–46, solve the logarithmic equation.

- | | |
|-------------------------|----------------------------------|
| 37. $\ln x - \ln 5 = 0$ | 38. $\ln x - \ln 2 = 0$ |
| 39. $\ln x = -9$ | 40. $\ln x = -14$ |
| 41. $\log_x 625 = 4$ | 42. $\log_x 81 = 2$ |
| 43. $\log_{11} x = -1$ | 44. $\log_{10} x = -\frac{1}{4}$ |
| 45. $\ln(2x - 1) = 5$ | 46. $\ln(3x + 5) = 8$ |

Using Inverse Properties In Exercises 47–54, simplify the expression.

- | | |
|--------------------------|--------------------|
| 47. $\ln e^{x^2}$ | 48. $\ln e^{2x-1}$ |
| 49. $e^{\ln(x^2-3)}$ | 50. $e^{\ln x^2}$ |
| 51. $-1 + \ln e^{2x}$ | |
| 52. $-4 + e^{\ln x^4}$ | |
| 53. $5 - e^{\ln(x^2+1)}$ | |
| 54. $3 - \ln(e^{x^2+2})$ | |

Solving an Exponential Equation In Exercises 55–80, solve the exponential equation algebraically. Round your result to three decimal places. Use a graphing utility to verify your answer.

55. $8^{3x} = 360$ 56. $6^{5x} = 3000$
 57. $5^{-t/2} = 0.20$ 58. $4^{-2t} = 0.0625$
 59. $250e^{0.02x} = 10,000$ 60. $100e^{0.005x} = 125,000$
 61. $500e^{-x} = 300$ 62. $1000e^{-4x} = 75$
 63. $7 - 2e^x = 1$ 64. $-14 + 3e^x = 11$
 65. $5(2^{3-x}) - 13 = 100$ 66. $6(8^{-2-x}) + 15 = 2601$
 67. $\left(1 + \frac{0.10}{12}\right)^{12t} = 2$ 68. $\left(16 + \frac{0.878}{26}\right)^{3t} = 30$
 69. $5000\left[\frac{(1 + 0.005)^x}{0.005}\right] = 250,000$
 70. $250\left[\frac{(1 + 0.01)^x}{0.01}\right] = 150,000$
 71. $e^{2x} - 4e^x - 5 = 0$ 72. $e^{2x} - e^x - 6 = 0$
 73. $e^x = e^{x^2-2}$ 74. $e^{2x} = e^{x^2-8}$
 75. $e^{x^2-3x} = e^{x-2}$ 76. $e^{-x^2} = e^{x^2-2x}$
 77. $\frac{400}{1 + e^{-x}} = 350$ 78. $\frac{525}{1 + e^{-x}} = 275$
 79. $\frac{40}{1 - 5e^{-0.01x}} = 200$ 80. $\frac{50}{1 - 2e^{-0.001x}} = 1000$

Algebraic-Graphical-Numerical In Exercises 81–84, (a) complete the table to find an interval containing the solution of the equation, (b) use a graphing utility to graph both sides of the equation to estimate the solution, and (c) solve the equation algebraically. Round your results to three decimal places.

81. $e^{3x} = 12$

x	0.6	0.7	0.8	0.9	1.0
e^{3x}					

82. $4e^{5x} = 24$

x	0.1	0.2	0.3	0.4	0.5
$4e^{5x}$					

83. $20(100 - e^{x/2}) = 500$

x	5	6	7	8	9
$20(100 - e^{x/2})$					

84. $11(77 - e^{x-4}) = 264$

x	5	6	7	8	9
$11(77 - e^{x-4})$					

Solving an Exponential Equation Graphically In Exercises 85–88, use the *zero* or *root* feature or the *zoom* and *trace* features of a graphing utility to approximate the solution of the exponential equation accurate to three decimal places.

85. $\left(1 + \frac{0.065}{365}\right)^{365t} = 4$ 86. $\left(4 - \frac{2.471}{40}\right)^{9t} = 21$
 87. $\frac{7000}{5 + e^{3x}} = 2$ 88. $\frac{119}{e^{6x} - 14} = 7$

Finding the Zero of a Function In Exercises 89–92, use a graphing utility to graph the function and approximate its zero accurate to three decimal places.

89. $g(x) = 6e^{1-x} - 25$ 90. $f(x) = 3e^{3x/2} - 962$
 91. $g(t) = e^{0.09t} - 3$ 92. $h(t) = e^{-0.125t} - 8$

Solving a Logarithmic Equation In Exercises 93–114, solve the logarithmic equation algebraically. Round the result to three decimal places. Verify your answer(s) using a graphing utility.

93. $\ln x = -3$ 94. $\ln x = -4$
 95. $\ln 4x = 2.1$ 96. $\ln 2x = 1.5$
 97. $\log_5(3x + 2) = \log_5(-x)$ 98. $\log_9(4 + x) = \log_9 2x$
 99. $-2 + 2 \ln 3x = 17$ 100. $3 + 2 \ln x = 10$
 101. $7 \log_4(0.6x) = 12$ 102. $4 \log_{10}(x - 6) = 11$
 103. $\log_{10}(z^2 + 19) = 2$ 104. $\log_{12} x^2 = 6$
 105. $\ln \sqrt{x+2} = 1$ 106. $\ln \sqrt{x-8} = 5$
 107. $\ln(x + 1)^2 = 2$ 108. $\ln(x - 4)^2 = 5$
 109. $\log_4 x - \log_4(x - 1) = \frac{1}{2}$
 110. $\log_3 x + \log_3(x - 8) = 2$
 111. $\ln(x + 5) = \ln(x - 1) + \ln(x + 1)$
 112. $\ln(x + 1) - \ln(x - 2) = \ln x$
 113. $\log_{10} 8x - \log_{10}(1 + \sqrt{x}) = 2$
 114. $\log_{10} 4x - \log_{10}(12 + \sqrt{x}) = 2$

Algebraic-Graphical-Numerical In Exercises 115–118, (a) complete the table to find an interval containing the solution of the equation, (b) use a graphing utility to graph both sides of the equation to estimate the solution, and (c) solve the equation algebraically. Round your results to three decimal places.

115. $\ln 2x = 2.4$

x	2	3	4	5	6
$\ln 2x$					

116. $3 \ln 5x = 10$

x	4	5	6	7	8
$3 \ln 5x$					

117. $6 \log_3(0.5x) = 11$

x	12	13	14	15	16
$6 \log_3(0.5x)$					

118. $5 \log_{10}(x - 2) = 11$

x	150	155	160	165	170
$5 \log_{10}(x - 2)$					

Approximating the Solution of an Equation In Exercises 119–124, use the *zero* or *root* feature of a graphing utility to approximate the solution of the logarithmic equation.

119. $\log_{10} x = x^3 - 3$

120. $\log_{10} x = (x - 3)^2$

121. $\log_{10} x + e^{0.5x} = 6$

122. $e^x \log_{10} x = 7$

123. $\ln(x + 2) - 3^{x-2} + 10 = 5$

124. $\ln x^2 - e^x = -3 - \ln x^2$

Finding the Point of Intersection In Exercises 125–130, use a graphing utility to approximate the point of intersection of the graphs. Round your result to three decimal places.

125. $y_1 = 7$

$y_2 = 2^{x-1} - 5$

126. $y_1 = 4$

$y_2 = 3^{x+1} - 2$

127. $y_1 = 80$

$y_2 = 4e^{-0.2x}$

128. $y_1 = 500$

$y_2 = 1500e^{-x/2}$

129. $y_1 = 3.25$

$y_2 = \frac{1}{2} \ln(x + 2)$

130. $y_1 = 1.05$

$y_2 = \ln \sqrt{x - 2}$

Solving Exponential and Logarithmic Equations In Exercises 131–138, solve the equation algebraically. Round the result to three decimal places. Verify your answer using a graphing utility.

131. $2x^2e^{2x} + 2xe^{2x} = 0$

132. $-x^2e^{-x} + 2xe^{-x} = 0$

133. $-xe^{-x} + e^{-x} = 0$

134. $e^{-2x} - 2xe^{-2x} = 0$

135. $2x \ln x + x = 0$

136. $\frac{1 - \ln x}{x^2} = 0$

137. $\frac{1 + \ln x}{2} = 0$

138. $3x \ln\left(\frac{1}{x}\right) - x = 0$

Solving a Model for x In Exercises 139–142, the equation represents the given type of model, which you will use in Section 3.5. Solve the equation for x .

Model Type

Equation

139. Exponential growth

$y = ae^{bx}$

140. Exponential decay

$y = ae^{-bx}$

141. Gaussian

$y = ae^{-(x-b)^2/c}$

142. Logarithmic

$y = a + b \ln x$

Doubling and Tripling an Investment In Exercises 143–146, find the time required for a \$1000 investment to (a) double at interest rate r , compounded continuously, and (b) triple at interest rate r , compounded continuously. Round your results to two decimal places.

143. $r = 7\%$

144. $r = 6\%$

145. $r = 2.5\%$

146. $r = 3.75\%$

147. **Economics** The percent p (in decimal form) of the United States population who own a smartphone is given by

$$p = \frac{1}{1 + e^{-(t-93)/22.5}}$$

where t is the number of months after smartphones were available on the market. Find the number of months t when the percent of the population owning smartphones is (a) 50% and (b) 80%.

148. **Why you should learn it** (p. 210) The percent m of American males between the ages of 18 and 24 who are no more than x inches tall is modeled by



$$m(x) = \frac{100}{1 + e^{-0.6114(x-69.71)}}$$

and the percent f of American females between the ages of 18 and 24 who are no more than x inches tall is modeled by

$$f(x) = \frac{100}{1 + e^{-0.66607(x-64.51)}}$$

(Source: U.S. National Center for Health Statistics)

(a) Use a graphing utility to graph the two functions in the same viewing window.

(b) Use the graphs in part (a) to determine the horizontal asymptotes of the functions. Interpret their meanings in the context of the problem.

(c) What is the average height for each sex?

149. **Finance** The numbers y of commercial banks in the United States from 2007 through 2013 can be modeled by

$$y = 11,912 - 2340.1 \ln t, \quad 7 \leq t \leq 13$$

where t represents the year, with $t = 7$ corresponding to 2007. In what year were there about 6300 commercial banks? (Source: Federal Deposit Insurance Corp.)

150. **Forestry** The yield V (in millions of cubic feet per acre) for a forest at age t years is given by $V = 6.7e^{-48.1/t}$.

(a) Use a graphing utility to graph the function.

(b) Determine the horizontal asymptote of the function. Interpret its meaning in the context of the problem.

(c) Find the time necessary to obtain a yield of 1.3 million cubic feet.

- 151. Science** An object at a temperature of 160°C was removed from a furnace and placed in a room at 20°C . The temperature T (in degrees Celsius) of the object was measured after each hour h and recorded in the table. A model for the data is given by $T = 20[1 + 7(2^{-h})]$.

Hour, h	Temperature, T
0	160°
1	90°
2	56°
3	38°
4	29°
5	24°

- Use a graphing utility to plot the data and graph the model in the same viewing window.
- Identify the horizontal asymptote of the graph. Interpret its meaning in the context of the problem.
- Approximate the time when the temperature of the object is 100°C .
- How long does it take the object to cool to 80°C , one-half the initial temperature?

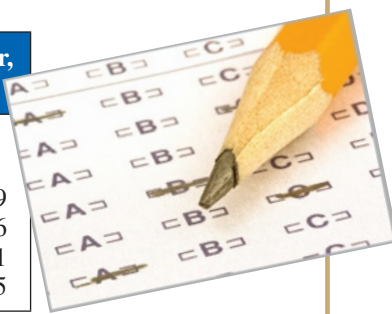
152. MODELING DATA

The table shows the numbers N of college-bound seniors intending to major in engineering who took the SAT exam from 2008 through 2013. The data can be modeled by the logarithmic function

$$N = -152,656 + 111,959.9 \ln t$$

where t represents the year, with $t = 8$ corresponding to 2008. (Source: The College Board)

Year	Number, N
2008	81,338
2009	88,719
2010	108,389
2011	116,746
2012	127,061
2013	132,275



- According to the model, in what year would 150,537 seniors intending to major in engineering take the SAT exam?
- Use a graphing utility to graph the model with the data, and use the graph to verify your answer in part (a).
- Do you think this is a good model for predicting future values? Explain.

Conclusions

True or False? In Exercises 153 and 154, determine whether the statement is true or false. Justify your answer.

- An exponential equation must have at least one solution.
- A logarithmic equation can have at most one extraneous solution.
- Error Analysis** Describe the error.

~~$$2e^x = 10$$

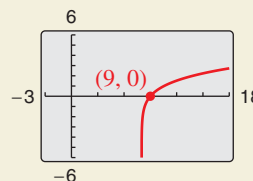
$$\ln(2e^x) = \ln 10$$

$$2x = \ln 10$$

$$x = \frac{1}{2} \ln 10$$~~



- 156. HOW DO YOU SEE IT?** Solving $\log_3 x + \log_3(x - 8) = 2$ algebraically, the solutions appear to be $x = 9$ and $x = -1$. Use the graph of $y = \log_3 x + \log_3(x - 8) - 2$ to determine whether each value is an actual solution of the equation. Explain your reasoning.



- Exploration** Let $f(x) = \log_a x$ and $g(x) = a^x$, where $a > 1$.
 - Let $a = 1.2$ and use a graphing utility to graph the two functions in the same viewing window. What do you observe? Approximate any points of intersection of the two graphs.
 - Determine the value(s) of a for which the two graphs have one point of intersection.
 - Determine the value(s) of a for which the two graphs have two points of intersection.
- ThinkAboutIt** Is the time required for a continuously compounded investment to quadruple twice as long as the time required for it to double? Give a reason for your answer and verify your answer algebraically.

Cumulative Mixed Review

Sketching Graphs In Exercises 159–162, sketch the graph of the function.

- $f(x) = 3x^3 - 4$
- $f(x) = |x - 2| - 8$
- $f(x) = \begin{cases} 2x + 1, & x < 0 \\ -x^2, & x \geq 0 \end{cases}$
- $f(x) = \begin{cases} x - 9, & x \leq 1 \\ x^2 + 1, & x > 1 \end{cases}$