2.4 Complex Numbers

What you should learn

- Use the imaginary unit *i* to write complex numbers.
- Add, subtract, and multiply complex numbers.
- Use complex conjugates to write the quotient of two complex numbers in standard form.
- Find complex solutions of quadratic equations.

Why you should learn it

Complex numbers are used to model numerous aspects of the natural world, such as the impedance of an electrical circuit, as shown in Exercise 89 on page 134.

The Imaginary Unit *i*

Some quadratic equations have no real solutions. For instance, the quadratic equation $x^2 + 1 = 0$ has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit *i*,** defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, you obtain the set of **complex numbers.** Each complex number can be written in the **standard form** $a + bi$. For instance, the standard form of the complex number $\sqrt{-9} - 5$ is $-5 + 3i$ because

$$\begin{aligned}\sqrt{-9} - 5 &= \sqrt{3^2(-1)} - 5\\ &= 3\sqrt{-1} - 5\\ &= 3i - 5\\ &= -5 + 3i.\end{aligned}$$

In the standard form $a + bi$, the real number a is called the **real part** of the **complex number** $a + bi$, and the number bi (where b is a real number) is called the **imaginary part** of the complex number.

Definition of a Complex Number

If a and b are real numbers, then the number $a + bi$ is a **complex number,** and it is said to be written in **standard form.** If $b = 0$, then the number $a + bi = a$ is a real number. If $b \neq 0$, then the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 2.19. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, you can write $a = a + 0i$.

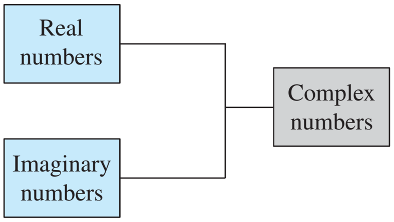

Figure 2.19

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di \qquad \text{Equality of two complex numbers}$$

if and only if $a = c$ and $b = d$.

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Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, then their sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$$-(a + bi) = -a - bi. \qquad \text{Additive inverse}$$

So, you have

$$\begin{aligned}(a + bi) + (-a - bi) &= 0 + 0i\\ &= 0.\end{aligned}$$

EXAMPLE 1 Adding and Subtracting Complex Numbers

a. $$\begin{aligned}(3 - i) + (2 + 3i) &= 3 - i + 2 + 3i && \text{Remove parentheses.}\\ &= (3 + 2) + (-i + 3i) && \text{Group like terms.}\\ &= 5 + 2i && \text{Write in standard form.}\end{aligned}$$

b. $$\begin{aligned}(1 + 2i) - (4 + 2i) &= 1 + 2i - 4 - 2i && \text{Remove parentheses.}\\ &= (1 - 4) + (2i - 2i) && \text{Group like terms.}\\ &= -3 + 0i && \text{Simplify.}\\ &= -3 && \text{Write in standard form.}\end{aligned}$$

c. $$\begin{aligned}3 - (-2 + 3i) + (-5 + i) &= 3 + 2 - 3i - 5 + i\\ &= (3 + 2 - 5) + (-3i + i)\\ &= 0 - 2i\\ &= -2i\end{aligned}$$

d. $$\begin{aligned}(3 + 2i) + (4 - i) - (7 + i) &= 3 + 2i + 4 - i - 7 - i\\ &= (3 + 4 - 7) + (2i - i - i)\\ &= 0 + 0i\\ &= 0\end{aligned}$$

For each operation on complex numbers, you can show the parallel operations on polynomials.

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Perform each operation and write the result in standard form.

a. $(7 + 3i) + (5 - 4i)$

b. $(3 + 4i) - (5 - 3i)$

c. $2i + (-3 - 4i) - (-3 - 3i)$

d. $(5 - 3i) + (3 + 5i) - (8 + 2i)$

In Examples 1(b) and 1(d), note that the sum of complex numbers can be a real number.

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication
Commutative Properties of Addition and Multiplication
Distributive Property of Multiplication over Addition

Notice how these properties are used when two complex numbers are multiplied.

$$\begin{aligned}
(a + bi)(c + di) &= a(c + di) + bi(c + di) && \text{Distributive Property}\\
&= ac + (ad)i + (bc)i + (bd)i^2 && \text{Distributive Property}\\
&= ac + (ad)i + (bc)i + (bd)(-1) && i^2 = -1\\
&= ac - bd + (ad)i + (bc)i && \text{Commutative Property}\\
&= (ac - bd) + (ad + bc)i && \text{Associative Property}
\end{aligned}$$

The procedure above is similar to multiplying two polynomials and combining like terms, as in the FOIL Method.

Explore the Concept

Complete the following:

$i^1 = i$
$i^2 = -1$
$i^3 = -i$
$i^4 = 1$
$i^5 =$
$i^6 =$

$i^7 =$
$i^8 =$
$i^9 =$
$i^{10} =$
$i^{11} =$
$i^{12} =$

What pattern do you see? Write a brief description of how you would find i raised to any positive integer power.

EXAMPLE 2 Multiplying Complex Numbers

a.
$$\begin{aligned}
5(-2 + 3i) &= 5(-2) + 5(3i) && \text{Distributive Property}\\
&= -10 + 15i && \text{Simplify.}
\end{aligned}$$

b.
$$\begin{aligned}
(2 - i)(4 + 3i) &= 2(4 + 3i) - i(4 + 3i) && \text{Distributive Property}\\
&= 8 + 6i - 4i - 3i^2 && \text{Distributive Property}\\
&= 8 + 6i - 4i - 3(-1) && i^2 = -1\\
&= 8 + 3 + 6i - 4i && \text{Group like terms.}\\
&= 11 + 2i && \text{Write in standard form.}
\end{aligned}$$

c.
$$\begin{aligned}
(3 + 2i)(3 - 2i) &= 3(3 - 2i) + 2i(3 - 2i) && \text{Distributive Property}\\
&= 9 - 6i + 6i - 4i^2 && \text{Distributive Property}\\
&= 9 - 4(-1) && i^2 = -1\\
&= 9 + 4 && \text{Simplify.}\\
&= 13 && \text{Write in standard form.}
\end{aligned}$$

d.
$$\begin{aligned}
4i(-1 + 5i) &= 4i(-1) + 4i(5i) && \text{Distributive Property}\\
&= -4i + 20i^2 && \text{Simplify.}\\
&= -4i + 20(-1) && i^2 = -1\\
&= -20 - 4i && \text{Write in standard form.}
\end{aligned}$$

e.
$$\begin{aligned}
(3 + 2i)^2 &= (3 + 2i)(3 + 2i) && \text{Property of exponents}\\
&= 3(3 + 2i) + 2i(3 + 2i) && \text{Distributive Property}\\
&= 9 + 6i + 6i + 4i^2 && \text{Distributive Property}\\
&= 9 + 6i + 6i + 4(-1) && i^2 = -1\\
&= 9 + 12i - 4 && \text{Simplify.}\\
&= 5 + 12i && \text{Write in standard form.}
\end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Perform each operation and write the result is standard form.

a. $(2 - 4i)(3 + 3i)$
b. $(4 + 5i)(4 - 5i)$
c. $(4 + 2i)^2$

Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the forms $a + bi$ and $a - bi$, called **complex conjugates.**

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2\\ &= a^2 - b^2(-1)\\ &= a^2 + b^2\end{aligned}$$

EXAMPLE 3 Multiplying Conjugates

Multiply each complex number by its complex conjugate.

a. $1 + i$ **b.** $4 - 3i$

Solution

a. The complex conjugate of $1 + i$ is $1 - i$.

$$(1 + i)(1 - i) = 1^2 - i^2 = 1 - (-1) = 2$$

b. The complex conjugate of $4 - 3i$ is $4 + 3i$.

$$(4 - 3i)(4 + 3i) = 4^2 - (3i)^2 = 16 - 9i^2 = 16 - 9(-1) = 25$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

A comparison with the method of rationalizing denominators may be helpful.

Multiply each complex number by its complex conjugate.

a. $3 + 6i$ **b.** $2 - 5i$

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right) \qquad \text{Multiply numerator and denominator by complex conjugate of denominator.}$$

$$= \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i. \qquad \text{Standard form}$$

EXAMPLE 4 Quotient of Complex Numbers in Standard Form

$$\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right) \qquad \text{Multiply numerator and denominator by complex conjugate of denominator.}$$

$$= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2} \qquad \text{Expand.}$$

$$= \frac{8 - 6 + 16i}{16 + 4} \qquad i^2 = -1$$

$$= \frac{2 + 16i}{20} \qquad \text{Simplify.}$$

$$= \frac{1}{10} + \frac{4}{5}i \qquad \text{Write in standard form.}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

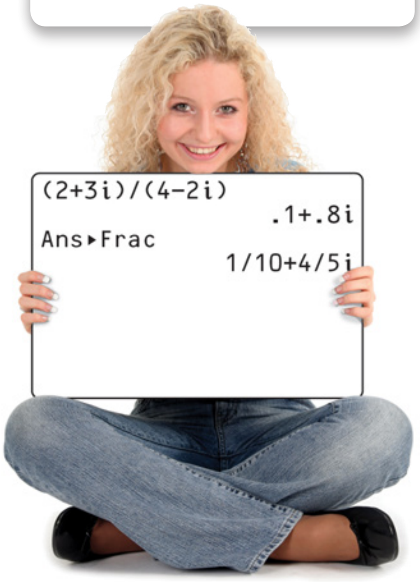
Write $\dfrac{2 + i}{2 - i}$ in standard form.

Technology Tip

Some graphing utilities can perform operations with complex numbers. For instance, on some graphing utilities, to divide $2 + 3i$ by $4 - 2i$, use the following keystrokes.

(2 + 3 i) ÷

(4 − 2 i) ENTER



```
(2+3i)/(4-2i)
                .1+.8i
Ans▸Frac
             1/10+4/5i
```

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Complex Solutions of Quadratic Equations

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as $\sqrt{-3}$, which you know is not a real number. By factoring out $i = \sqrt{-1}$, you can write this number in standard form.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

The number $\sqrt{3}i$ is called the *principal square root* of -3.

Principal Square Root of a Negative Number

If a is a positive number, then the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}i.$$

Remark

The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid when *both* a and b are negative. For example,

$$\begin{aligned}\sqrt{-5}\sqrt{-5} &= \sqrt{5(-1)}\sqrt{5(-1)}\\ &= \sqrt{5}i\sqrt{5}i\\ &= \sqrt{25}i^2\\ &= 5i^2\\ &= -5\end{aligned}$$

whereas

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

To avoid problems with square roots of negative numbers, be sure to convert complex numbers to standard form *before* multiplying.

EXAMPLE 5 Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}i\sqrt{12}i = \sqrt{36}i^2 = 6(-1) = -6$

b.
$$\begin{aligned}\sqrt{-48} - \sqrt{-27} &= \sqrt{48}i - \sqrt{27}i\\ &= 4\sqrt{3}i - 3\sqrt{3}i\\ &= \sqrt{3}i\end{aligned}$$

c.
$$\begin{aligned}\left(-1 + \sqrt{-3}\right)^2 &= \left(-1 + \sqrt{3}i\right)^2\\ &= (-1)^2 - 2\sqrt{3}i + \left(\sqrt{3}\right)^2(i^2)\\ &= 1 - 2\sqrt{3}i + 3(-1)\\ &= -2 - 2\sqrt{3}i\end{aligned}$$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write $\sqrt{-14}\sqrt{-2}$ in standard form.

EXAMPLE 6 Complex Solutions of a Quadratic Equation

See LarsonPrecalculus.com for an interactive version of this type of example.

Solve (a) $x^2 + 4 = 0$ and (b) $3x^2 - 2x + 5 = 0$.

Solution

a.
$$\begin{aligned}x^2 + 4 &= 0 &&\text{Write original equation.}\\ x^2 &= -4 &&\text{Subtract 4 from each side.}\\ x &= \pm 2i &&\text{Extract square roots.}\end{aligned}$$

b.
$$\begin{aligned}3x^2 - 2x + 5 &= 0 &&\text{Write original equation.}\\ x &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)} &&\text{Quadratic Formula}\\ &= \frac{2 \pm \sqrt{-56}}{6} &&\text{Simplify.}\\ &= \frac{2 \pm 2\sqrt{14}i}{6} &&\text{Write } \sqrt{-56} \text{ in standard form.}\\ &= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i &&\text{Write in standard form.}\end{aligned}$$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $8x^2 + 14x + 9 = 0$.

2.4 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Match the type of complex number with its definition.

(a) real number	(i) $a + bi,\ a = 0,\ b \neq 0$
(b) imaginary number	(ii) $a + bi,\ b = 0$
(c) pure imaginary number	(iii) $a + bi,\ a \neq 0,\ b \neq 0$

In Exercises 2 and 3, fill in the blanks.

2. The imaginary unit i is defined as $i =$ ________ , where $i^2 =$ ________ .

3. When you add $(7 + 6i)$ and $(8 + 5i)$, the real part of the sum is ________ and the imaginary part of the sum is ________ .

4. What method for multiplying two polynomials can you use when multiplying two complex numbers?

5. What is the additive inverse of the complex number $2 - 4i$?

6. What is the complex conjugate of the complex number $2 - 4i$?

Procedures and Problem Solving

Equality of Complex Numbers **In Exercises 7–10, find real numbers a and b such that the equation is true.**

7. $a + bi = -9 + 4i$ **8.** $a + bi = 12 + 5i$

9. $3a + (b + 3)i = 9 + 8i$ **10.** $(a + 6) + 2bi = 6 - i$

Writing a Complex Number in Standard Form **In Exercises 11–20, write the complex number in standard form.**

11. $4 + \sqrt{-9}$ **12.** $7 - \sqrt{-25}$

13. 12 **14.** -3

15. $-8i - i^2$ **16.** $2i^2 - 6i$

17. $\left(\sqrt{-16}\right)^2 + 5$ **18.** $-i - \left(\sqrt{-23}\right)^2$

19. $\sqrt{-0.09}$ **20.** $\sqrt{-0.0004}$

Adding and Subtracting Complex Numbers **In Exercises 21–30, perform the addition or subtraction and write the result in standard form.**

21. $(4 + i) - (7 - 2i)$ **22.** $(11 - 2i) - (-3 + 6i)$

23. $(-1 + 8i) + (8 - 5i)$ **24.** $(7 + 6i) + (3 + 12i)$

25. $13i - (14 - 7i)$ **26.** $22 + (-5 + 8i) - 9i$

27. $\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$ **28.** $\left(\frac{3}{4} + \frac{7}{5}i\right) - \left(\frac{5}{6} - \frac{1}{6}i\right)$

29. $(1.6 + 3.2i) + (-5.8 + 4.3i)$

30. $-(-3.7 - 12.8i) - (6.1 - 16.3i)$

Multiplying Complex Numbers **In Exercises 31–42, perform the operation(s) and write the result in standard form.**

31. $4(3 + 5i)$ **32.** $-6(5 - 3i)$

33. $(1 + i)(3 - 2i)$ **34.** $(6 - 2i)(2 - 3i)$

35. $4i(8 + 5i)$ **36.** $-3i(6 - i)$

37. $\left(\sqrt{14} + \sqrt{10}i\right)\left(\sqrt{14} - \sqrt{10}i\right)$

38. $\left(\sqrt{3} + \sqrt{15}i\right)\left(\sqrt{3} - \sqrt{15}i\right)$

39. $(6 + 7i)^2$ **40.** $(5 - 4i)^2$

41. $(4 + 5i)^2 - (4 - 5i)^2$ **42.** $(1 - 2i)^2 - (1 + 2i)^2$

Multiplying Conjugates **In Exercises 43–50, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.**

43. $6 - 2i$ **44.** $3 + 5i$

45. $-1 + \sqrt{7}i$ **46.** $-4 - \sqrt{3}i$

47. $\sqrt{-29}$ **48.** $\sqrt{-10}$

49. $9 - \sqrt{6}i$ **50.** $-8 + \sqrt{15}i$

Writing a Quotient of Complex Numbers in Standard Form **In Exercises 51–58, write the quotient in standard form.**

51. $\dfrac{6}{i}$ **52.** $-\dfrac{5}{2i}$

53. $\dfrac{2}{4 - 5i}$ **54.** $\dfrac{3}{1 - i}$

55. $\dfrac{2 + i}{2 - i}$ **56.** $\dfrac{8 - 7i}{1 - 2i}$

57. $i/(4 - 5i)^2$ **58.** $5i/(2 + 3i)^2$

Adding or Subtracting Quotients of Complex Numbers **In Exercises 59 and 60, perform the operation and write the result in standard form.**

59. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i}$ **60.** $\dfrac{2i}{2 + i} + \dfrac{5}{2 - i}$

Writing Complex Numbers in Standard Form **In Exercises 61–70, perform the operation and write the result in standard form.**

61. $\sqrt{-18} - \sqrt{-54}$
62. $\sqrt{-50} + \sqrt{-275}$
63. $(-3 + \sqrt{-24}) + (7 - \sqrt{-44})$
64. $(-12 - \sqrt{-72}) + (9 + \sqrt{-108})$
65. $\sqrt{-6}\sqrt{-2}$
66. $\sqrt{-5}\sqrt{-10}$
67. $(\sqrt{-10})^2$
68. $(\sqrt{-75})^2$
69. $(2 - \sqrt{-6})^2$
70. $(3 + \sqrt{-5})(7 - \sqrt{-10})$

Complex Solutions of a Quadratic Equation **In Exercises 71–82, solve the quadratic equation.**

71. $x^2 + 25 = 0$
72. $x^2 + 32 = 0$
73. $x^2 - 2x + 2 = 0$
74. $x^2 + 6x + 10 = 0$
75. $4x^2 + 16x + 17 = 0$
76. $9x^2 - 6x + 37 = 0$
77. $16t^2 - 4t + 3 = 0$
78. $4x^2 + 16x + 15 = 0$
79. $\frac{3}{2}x^2 - 6x + 9 = 0$
80. $\frac{7}{8}x^2 - \frac{3}{4}x + \frac{5}{16} = 0$
81. $1.4x^2 - 2x - 10 = 0$
82. $4.5x^2 - 3x + 12 = 0$

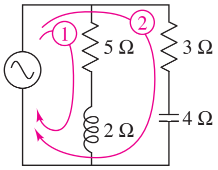
Expressions Involving Powers of *i* **In Exercises 83–88, simplify the complex number and write the result in standard form.**

83. $-6i^3 + i^2$
84. $4i^2 - 2i^3$
85. $(\sqrt{-75})^3$
86. $(\sqrt{-2})^6$
87. $\dfrac{1}{i^3}$
88. $\dfrac{1}{(2i)^3}$

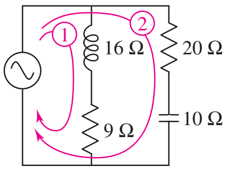
89. ***Why you should learn it*** *(p. 128)* The opposition to current in an electrical circuit is called its impedance. The impedance z in a parallel circuit with two pathways satisfies the equation $1/z = 1/z_1 + 1/z_2$, where z_1 is the impedance (in ohms) of pathway 1, and z_2 is the impedance (in ohms) of pathway 2. Use the table to determine the impedance of each parallel circuit. (*Hint:* You can find the impedance of each pathway in a parallel circuit by adding the impedances of all components in the pathway.)

	Resistor	Inductor	Capacitor
Symbol	$a\ \Omega$	$b\ \Omega$	$c\ \Omega$
Impedance	a	bi	$-ci$

(a)



(b)



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90. **Exploration** Consider the functions

$$f(x) = 2(x - 3)^2 - 4 \quad \text{and} \quad g(x) = -2(x - 3)^2 - 4.$$

(a) Without graphing either function, determine whether the graph of f and the graph of g have x-intercepts. Explain your reasoning.

(b) Solve $f(x) = 0$ and $g(x) = 0$.

(c) Explain how the zeros of f and g are related to whether their graphs have x-intercepts.

(d) For the function $f(x) = a(x - h)^2 + k$, make a general statement about how a, h, and k affect whether the graph of f has x-intercepts, and whether the zeros of f are real or complex.

Conclusions

True or False? **In Exercises 91–94, determine whether the statement is true or false. Justify your answer.**

91. No complex number is equal to its complex conjugate.

92. $i^{44} + i^{150} - i^{74} - i^{109} + i^{61} = -1$

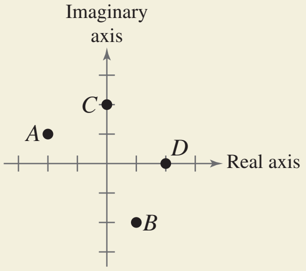
93. The conjugate of the product of two complex numbers is equal to the product of the conjugates of the two complex numbers.

94. The conjugate of the sum of two complex numbers is equal to the sum of the conjugates of the two complex numbers.

95. **Error Analysis** Describe the error.

~~$\sqrt{-6}\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6$~~

96. **HOW DO YOU SEE IT?** The coordinate system shown below is called the *complex plane*. In the complex plane, the point that corresponds to the complex number $a + bi$ is (a, b).



Match each complex number with its corresponding point.

(i) 2 (ii) $2i$ (iii) $-2 + i$ (iv) $1 - 2i$

Cumulative Mixed Review

Multiplying Polynomials **In Exercises 97–100, perform the operation and write the result in standard form.**

97. $(4x - 5)(4x + 5)$
98. $(x + 2)^3$
99. $(3x - \frac{1}{2})(x + 4)$
100. $(2x - 5)^2$

2.5 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra

You know that an nth-degree polynomial can have at most n real zeros. In the complex number system, this statement can be improved. That is, in the complex number system, every nth-degree polynomial function has *precisely n zeros*. This important result is derived from the **Fundamental Theorem of Algebra,** first proved by the German mathematician Carl Friedrich Gauss (1777–1855).

The Fundamental Theorem of Algebra

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.

Using the Fundamental Theorem of Algebra and the equivalence of zeros and factors, you obtain the **Linear Factorization Theorem.**

Linear Factorization Theorem (See the proof on page 177.)

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, . . . , c_n$ are complex numbers.

Note that the Fundamental Theorem of Algebra and the Linear Factorization Theorem tell you only that the zeros or factors of a polynomial exist, not how to find them. Such theorems are called *existence theorems.* To find the zeros of a polynomial function, you still must rely on other techniques.

EXAMPLE 1 Zeros of Polynomial Functions

See LarsonPrecalculus.com for an interactive version of this type of example.

a. The first-degree polynomial $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function

$$\begin{aligned} f(x) &= x^2 - 6x + 9 \\ &= (x - 3)(x - 3) \end{aligned}$$

has exactly *two* zeros: $x = 3$ and $x = 3$. (This is called a *repeated zero*.)

c. The third-degree polynomial function

$$f(x) = x^3 + 4x = x(x^2 + 4) = x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$.

d. The fifth-degree polynomial function

$$f(x) = x^5 + 9x^3 = x^3(x^2 + 9) = x^3(x - 3i)(x + 3i)$$

has exactly *five* zeros: $x = 0$, $x = 0$, $x = 0$, $x = 3i$, and $x = -3i$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

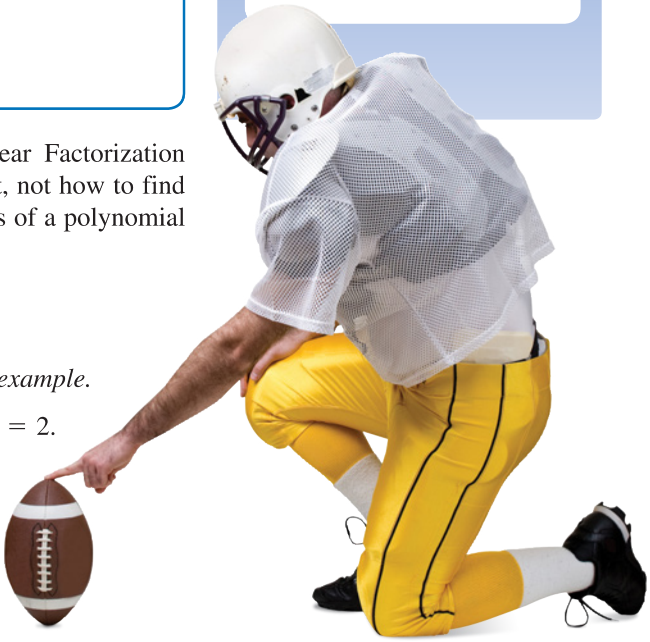
Determine the number of zeros of the polynomial function $f(x) = x^4 - 1$.

What you should learn

- Use the Fundamental Theorem of Algebra to determine the number of zeros of a polynomial function.
- Find all zeros of polynomial functions.
- Find conjugate pairs of complex zeros.
- Find zeros of polynomials by factoring.

Why you should learn it

Being able to find zeros of polynomial functions is an important part of modeling real-life problems. For instance, Exercise 69 on page 141 shows how to determine whether a football kicked with a given velocity can reach a certain height.



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Finding Zeros of a Polynomial Function

Remember that the n zeros of a polynomial function can be real or imaginary, and they may be repeated. Examples 2 and 3 illustrate several cases.

EXAMPLE 2 Complex Zeros of a Polynomial Function

Confirm that the third-degree polynomial function $f(x) = x^3 + 4x$ has exactly three zeros: $x = 0$, $x = 2i$, and $x = -2i$.

Solution

Factor the polynomial completely as $x(x - 2i)(x + 2i)$. So, the zeros are

$$x(x - 2i)(x + 2i) = 0$$

$$x = 0 \qquad \text{Set 1st factor equal to 0.}$$

$$x - 2i = 0 \Rightarrow x = 2i \qquad \text{Set 2nd factor equal to 0.}$$

$$x + 2i = 0 \Rightarrow x = -2i. \qquad \text{Set 3rd factor equal to 0.}$$

In Figure 2.20, only the real zero $x = 0$ appears as an x-intercept.

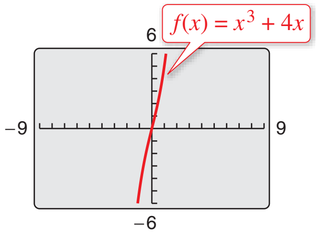


Figure 2.20

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Confirm that the fourth-degree polynomial function $f(x) = x^4 - 1$ has exactly four zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$. ■

The next example shows how to use the methods described in Sections 2.2 and 2.3 (the Rational Zero Test, synthetic division, and factoring) to find all the zeros of a polynomial function, including imaginary zeros.

EXAMPLE 3 Finding the Zeros of a Polynomial Function

You may want to remind students that a graphing utility is helpful for determining real zeros, which in turn are useful in finding complex zeros.

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all the zeros of f.

Solution

The possible rational zeros are ±1, ±2, ±4, and ±8. The graph shown in Figure 2.21 indicates that 1 and -2 are likely zeros, and that 1 is possibly a repeated zero because it appears that the graph touches (but does not cross) the x-axis at this point. Using synthetic division, you can determine that -2 is a zero and 1 is a repeated zero of f. So, you have

$$f(x) = x^5 + x^3 + 2x^2 - 12x + 8 = (x - 1)(x - 1)(x + 2)(x^2 + 4).$$

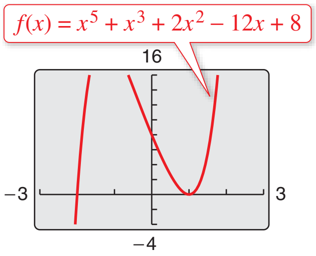


Figure 2.21

By factoring $x^2 + 4$ as

$$x^2 - (-4) = \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) = (x - 2i)(x + 2i)$$

you obtain

$$f(x) = (x - 1)(x - 1)(x + 2)(x - 2i)(x + 2i)$$

which gives the following five zeros of f.

$$x = 1, x = 1, x = -2, x = 2i, \quad \text{and} \quad x = -2i$$

Note from the graph of f shown in Figure 2.21 that the *real* zeros are the only ones that appear as x-intercepts.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write $f(x) = x^4 - 256$ as the product of linear factors, and list all the zeros of f. ■

Conjugate Pairs

In Example 3, note that the two complex zeros $2i$ and $-2i$ are **conjugates.** That is, they are of the forms $a + bi$ and $a - bi$.

Complex Zeros Occur in Conjugate Pairs

Let f be a polynomial function that has *real coefficients.* If $a + bi$, where $b \neq 0$, is a zero of the function, then the conjugate $a - bi$ is also a zero of the function.

Be sure you see that this result is true only when the polynomial function has *real coefficients.* For instance, the result applies to the function $f(x) = x^2 + 1$, but not to the function $g(x) = x - i$.

EXAMPLE 4 Finding a Polynomial with Given Zeros

Find a *fourth-degree* polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$f(x) = (x^2 + 2x + 1)(x^2 + 9) = x^4 + 2x^3 + 10x^2 + 18x + 9.$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find a *fourth-degree* polynomial function with real coefficients that has 2, -2, and $7i$ as zeros.

EXAMPLE 5 Finding a Polynomial with Given Zeros

Find a *cubic* polynomial function f with real coefficients that has 2 and $1-i$ as zeros, and $f(1) = 3$.

Solution

Because $1 - i$ is a zero of f, the conjugate $1 + i$ must also be a zero.

$$\begin{aligned} f(x) &= a(x - 2)[x - (1 - i)][x - (1 + i)] \\ &= a(x - 2)[x^2 - x(1 + i) - x(1 - i) + 1 - i^2] \\ &= a(x - 2)(x^2 - 2x + 2) \\ &= a(x^3 - 4x^2 + 6x - 4) \end{aligned}$$

To find the value of a, use the fact that $f(1) = 3$ to obtain

$$a[(1)^3 - 4(1)^2 + 6(1) - 4] = 3.$$

So, $a = -3$ and you can conclude that

$$f(x) = -3(x^3 - 4x^2 + 6x - 4) = -3x^3 + 12x^2 - 18x + 12.$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find a *quartic* polynomial function f with real coefficients that has 1, -2, and $2i$ as zeros, and $f(-1) = 10$.

Factoring a Polynomial

The Linear Factorization Theorem states that you can write any nth-degree polynomial as the product of n linear factors.

$$f(x) = a_n(x - c_1)(x - c_2)(x - c_3) \cdot \cdot \cdot (x - c_n)$$

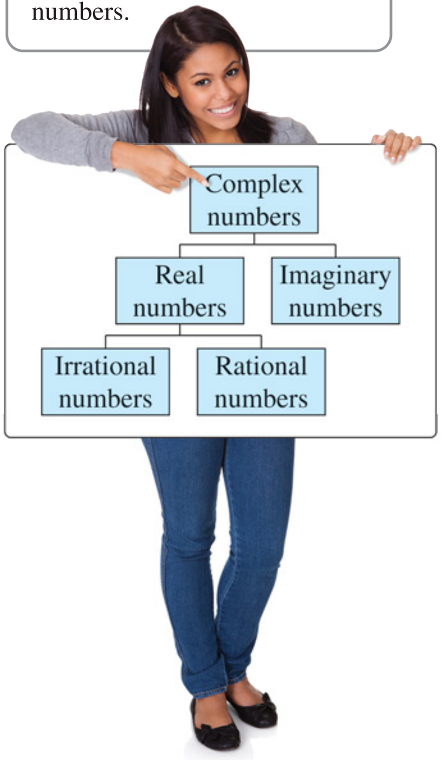
This result, however, includes the possibility that some of the values of c_i are imaginary. The next theorem states that even when you do not want to get involved with "imaginary factors," you can still write $f(x)$ as the product of linear and quadratic factors.

Factors of a Polynomial (See the proof on page 177.)

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Remark

Recall that irrational and rational numbers are subsets of the set of real numbers, and the real numbers are a subset of the set of complex numbers.



A quadratic factor with no real zeros is said to be **prime** or **irreducible over the reals.** Be sure you see that this is not the same as being *irreducible over the rationals.* For example, the quadratic

$$x^2 + 1 = (x - i)(x + i)$$

is irreducible over the reals (and therefore over the rationals). On the other hand, the quadratic

$$x^2 - 2 = \left(x - \sqrt{2}\right)\left(x + \sqrt{2}\right)$$

is irreducible over the rationals, but *reducible* over the reals.

EXAMPLE 6 Factoring a Polynomial

Write the polynomial $f(x) = x^4 - x^2 - 20$

a. as the product of factors that are irreducible over the *rationals*,

b. as the product of linear factors and quadratic factors that are irreducible over the *reals*, and

c. in completely factored form.

Solution

a. Begin by factoring the polynomial as the product of two quadratic polynomials.

$$x^4 - x^2 - 20 = (x^2 - 5)(x^2 + 4)$$

Both of these factors are irreducible over the rationals.

b. By factoring over the reals, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x^2 + 4)$$

where the quadratic factor is irreducible over the reals.

c. In completely factored form, you have

$$x^4 - x^2 - 20 = \left(x + \sqrt{5}\right)\left(x - \sqrt{5}\right)(x - 2i)(x + 2i).$$

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write the polynomial $f(x) = x^4 - 2x^2 - 3$ (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear factors and quadratic factors that are irreducible over the *reals*, and (c) in completely factored form.

In Example 6, notice from the completely factored form that the *fourth*-degree polynomial has *four* zeros.

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Throughout this chapter, the results and theorems have been stated in terms of zeros of polynomial functions. Be sure you see that the same results could have been stated in terms of solutions of polynomial equations. This is true because the zeros of the polynomial function

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

are precisely the solutions of the polynomial equation

$$a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0 = 0.$$

EXAMPLE 7 Finding the Zeros of a Polynomial Function

Find all the zeros of

$$f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$$

given that $1 + 3i$ is a zero of f.

Algebraic Solution

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$x - (1 + 3i) \quad \text{and} \quad x - (1 - 3i)$$

are factors of f. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$\begin{array}{r@{}r}
 & x^2 - x - 6 \\
x^2 - 2x + 10 \big) & x^4 - 3x^3 + 6x^2 + 2x - 60 \\
 & \underline{x^4 - 2x^3 + 10x^2} \\
 & -x^3 - 4x^2 + 2x \\
 & \underline{-x^3 + 2x^2 - 10x} \\
 & -6x^2 + 12x - 60 \\
 & \underline{-6x^2 + 12x - 60} \\
 & 0
\end{array}$$

So, you have

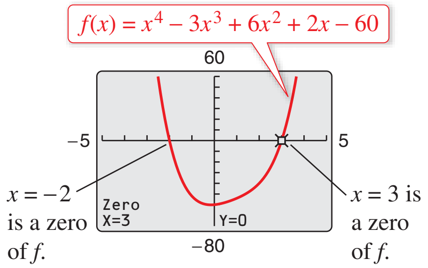
$$\begin{aligned}f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2)\end{aligned}$$

and you can conclude that the zeros of f are

$$x = 1 + 3i,\ x = 1 - 3i,\ x = 3, \text{ and } x = -2.$$

Graphical Solution

Using the *zero* feature of a graphing utility, you can conclude that $x = -2$ and $x = 3$ are zeros of f. (See figure.)



Because $1 + 3i$ is a zero of f, you know that the conjugate $1 - 3i$ must also be a zero. So, you can conclude that the zeros of f are

$$x = 1 + 3i,\ x = 1 - 3i,\ x = 3, \text{ and } x = -2.$$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all the zeros of $f(x) = 3x^3 - 2x^2 + 48x - 32$ given that $4i$ is a zero of f.

In Example 7, if you were not told that $1 + 3i$ is a zero of f, you could still find all the zeros of the function by using synthetic division to find the real zeros -2 and 3. Then, you could factor the polynomial as $(x + 2)(x - 3)(x^2 - 2x + 10)$. Finally, by using the Quadratic Formula, you could determine that the zeros are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

Activities

1. Write as a product of linear factors: $f(x) = x^4 - 16$.
   Answer: $(x - 2)(x + 2)(x - 2i)(x + 2i)$
2. Find a third-degree polynomial with integer coefficients that has 2 and $3 - i$ as zeros.
   Answer: $x^3 - 8x^2 + 22x - 20$
3. Write the polynomial $f(x) = x^4 - 4x^3 + 5x^2 - 2x - 6$ in completely factored form. (*Hint:* One factor is $x^2 - 2x - 2$.)
   Answer: $(x - 1 + \sqrt{3})(x - 1 - \sqrt{3})$
   $(x - 1 + \sqrt{2}i)(x - 1 - \sqrt{2}i)$

2.5 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blanks.

1. The ________ of ________ states that if $f(x)$ is a polynomial of degree n $(n > 0)$, then f has at least one zero in the complex number system.

2. A quadratic factor that cannot be factored as a product of linear factors containing real numbers is said to be ________ over the ________ .

3. How many linear factors does a polynomial function f of degree n have, where $n > 0$?

4. Three of the zeros of a fourth-degree polynomial function f are -1, 3, and $2i$. What is the other zero of f?

Procedures and Problem Solving

Zeros of a Polynomial Function **In Exercises 5–8, match the function with its exact number of zeros.**

5. $f(x) = -2x^4 + 32$	(a) 1 zero
6. $f(x) = x^5 - x^3$	(b) 3 zeros
7. $f(x) = x^3 + 3x^2 + 2x$	(c) 4 zeros
8. $f(x) = x - 14$	(d) 5 zeros

Complex Zeros of a Polynomial Function **In Exercises 9–12, confirm that the function has the indicated zeros.**

9. $f(x) = x^2 + 5;\ -\sqrt{5}i,\ \sqrt{5}i$

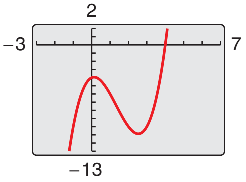
10. $f(x) = x^3 + 9x;\ 0,\ -3i,\ 3i$

11. $f(x) = 3x^4 - 48;\ -2,\ 2,\ -2i,\ 2i$

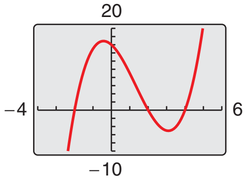
12. $f(x) = 2x^5 - 2x;\ 0,\ 1,\ -1\ -i,\ i$

Comparing the Zeros and the *x*-Intercepts **In Exercises 13–16, find all the zeros of the function. Is there a relationship between the number of real zeros and the number of *x*-intercepts of the graph? Explain.**

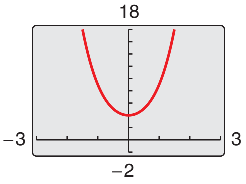
13. $f(x) = x^3 - 4x^2 + x - 4$



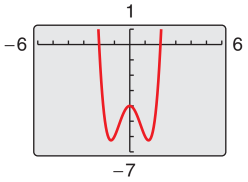
14. $f(x) = x^3 - 4x^2 - 4x + 16$



15. $f(x) = x^4 + 4x^2 + 4$



16. $f(x) = x^4 - 3x^2 - 4$



Finding the Zeros of a Polynomial Function **In Exercises 17–36, find all the zeros of the function and write the polynomial as a product of linear factors. Use a graphing utility to verify your results graphically. (If possible, use the graphing utility to verify the imaginary zeros.)**

17. $h(x) = x^2 - 4x + 1$

18. $g(x) = x^2 + 10x + 23$

19. $f(x) = x^2 - 12x + 26$

20. $f(x) = x^2 + 6x - 2$

21. $f(x) = x^2 + 25$

22. $f(x) = x^2 + 36$

23. $f(x) = 16x^4 - 81$

24. $f(y) = 81y^4 - 625$

25. $f(z) = z^2 - z + 56$

26. $h(x) = x^2 - 4x - 3$

27. $f(x) = x^4 + 10x^2 + 9$

28. $f(x) = x^4 + 29x^2 + 100$

29. $f(x) = 3x^3 - 5x^2 + 48x - 80$

30. $f(x) = 3x^3 - 2x^2 + 75x - 50$

31. $f(t) = t^3 - 3t^2 - 15t + 125$

32. $f(x) = x^3 + 11x^2 + 39x + 29$

33. $f(x) = 5x^3 - 9x^2 + 28x + 6$

34. $f(s) = 3s^3 - 4s^2 + 8s + 8$

35. $g(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$

36. $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

Using the Zeros to Find *x*-Intercepts **In Exercises 37–44, (a) find all zeros of the function, (b) write the polynomial as a product of linear factors, and (c) use your factorization to determine the *x*-intercepts of the graph of the function. Use a graphing utility to verify that the real zeros are the only *x*-intercepts.**

37. $f(x) = x^2 - 14x + 46$

38. $f(x) = x^2 - 12x + 34$

39. $f(x) = 2x^3 - 3x^2 + 8x - 12$

40. $f(x) = 2x^3 - 5x^2 + 18x - 45$

41. $f(x) = x^3 - 11x + 150$

42. $f(x) = x^3 + 10x^2 + 33x + 34$

43. $f(x) = x^4 + 25x^2 + 144$

44. $f(x) = x^4 - 8x^3 + 17x^2 - 8x + 16$

Finding a Polynomial with Given Zeros **In Exercises 45–50, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)**

45. $5, i, -i$

46. $3, 4i, -4i$

47. $1, 1, 2i$

48. $-3, -3, i$

49. $0, -5, 1 + \sqrt{2}i$

50. $0, 4, 1 + \sqrt{2}i$

Finding a Polynomial with Given Zeros **In Exercises 51–54, a polynomial function f with real coefficients has the given degree, zeros, and solution point. Write the function (a) in completely factored form and (b) in polynomial form.**

	Degree	*Zeros*	*Solution Point*
51.	4	$-1, 2, i$	$f(1) = 8$
52.	4	$1, -4, \sqrt{3}i$	$f(0) = -6$
53.	3	$-1, 2 + \sqrt{5}i$	$f(2) = 45$
54.	3	$-2, 2 + 2\sqrt{2}i$	$f(-1) = -34$

Factoring a Polynomial **In Exercises 55–58, write the polynomial (a) as the product of factors that are irreducible over the *rationals,* (b) as the product of linear and quadratic factors that are irreducible over the *reals,* and (c) in completely factored form.**

55. $f(x) = x^4 - 6x^2 - 7$

56. $f(x) = x^4 + 6x^2 - 27$

57. $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$

(*Hint:* One factor is $x^2 - 6$.)

58. $f(x) = x^4 - 3x^3 - x^2 - 12x - 20$

(*Hint:* One factor is $x^2 + 4$.)

Finding the Zeros of a Polynomial Function **In Exercises 59–64, use the given zero to find all the zeros of the function.**

	Function	*Zero*
59.	$f(x) = 2x^3 + 3x^2 + 50x + 75$	$5i$
60.	$f(x) = x^3 + x^2 + 9x + 9$	$3i$
61.	$g(x) = x^3 - 7x^2 - x + 87$	$5 + 2i$
62.	$g(x) = 4x^3 + 23x^2 + 34x - 10$	$-3 + i$
63.	$h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - \sqrt{3}i$
64.	$f(x) = 25x^3 - 55x^2 - 54x - 18$	$\frac{1}{5}(-2 + \sqrt{2}i)$

Using a Graph to Locate the Real Zeros **In Exercises 65–68, (a) use a graphing utility to find the real zeros of the function, and then (b) use the real zeros to find the exact values of the imaginary zeros.**

65. $f(x) = x^4 + 3x^3 - 5x^2 - 21x + 22$

66. $f(x) = x^3 + 4x^2 + 14x + 20$

67. $h(x) = 8x^3 - 14x^2 + 18x - 9$

68. $f(x) = 25x^3 - 55x^2 - 54x - 18$

69. ***Why you should learn it*** *(p. 135)* A football is kicked off the ground with an initial upward velocity of 48 feet per second. The football's height h (in feet) is given by

$$h(t) = -16t^2 + 48t, \quad 0 \le t \le 3$$

where t is the time (in seconds). Does the football reach a height of 50 feet? Explain.

70. Marketing The demand equation for a microwave is $p = 140 - 0.001x$, where p is the unit price (in dollars) of the microwave and x is the number of units produced and sold. The cost equation for the microwave is $C = 40x + 150{,}000$, where C is the total cost (in dollars) and x is the number of units produced. The total profit P obtained by producing and selling x units is given by $P = R - C = xp - C$. Is there a price p that yields a profit of \$3 million? Explain.

Conclusions

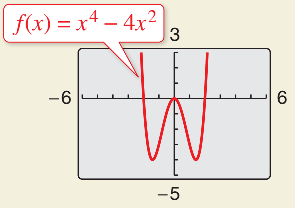
True or False? **In Exercises 71 and 72, decide whether the statement is true or false. Justify your answer.**

71. It is possible for a third-degree polynomial function with integer coefficients to have no real zeros.

72. If $[x + (4 + 3i)]$ is a factor of a polynomial function f with real coefficients, then $[x - (4 + 3i)]$ is also a factor of f.

73. Writing Compile a list of all the various techniques for factoring a polynomial that have been covered so far in the text. Give an example illustrating each technique and write a paragraph discussing when the use of each technique is appropriate.

74. HOW DO YOU SEE IT? Describe a translation of the graph that will result in a function with (a) four distinct real zeros, (b) two real zeros, each of multiplicity 2, (c) two real zeros and two imaginary zeros, and (d) four imaginary zeros.



Cumulative Mixed Review

Identifying the Vertex of a Quadratic Function **In Exercises 75–78, describe the graph of the function and identify the vertex.**

75. $f(x) = x^2 - 7x - 8$

76. $f(x) = -x^2 + x + 6$

77. $f(x) = 6x^2 + 5x - 6$

78. $f(x) = 4x^2 + 2x - 12$

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2.6 Rational Functions and Asymptotes

Introduction to Rational Functions

A **rational function** can be written in the form

$$f(x) = \frac{N(x)}{D(x)}$$

where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

In general, the *domain* of a rational function of x includes all real numbers except x-values that make the denominator zero. Much of the discussion of rational functions will focus on their graphical behavior near these x-values.

What you should learn

- Find the domains of rational functions.
- Find vertical and horizontal asymptotes of graphs of rational functions.
- Use rational functions to model and solve real-life problems.

Why you should learn it

Rational functions are convenient in modeling a wide variety of real-life problems, such as environmental scenarios. For instance, Exercise 49 on page 150 shows how to determine the cost of supplying recycling bins to the population of a rural township.

EXAMPLE 1 Finding the Domain of a Rational Function

See LarsonPrecalculus.com for an interactive version of this type of example.

Find the domain of $f(x) = 1/x$ and discuss the behavior of f near any excluded x-values.

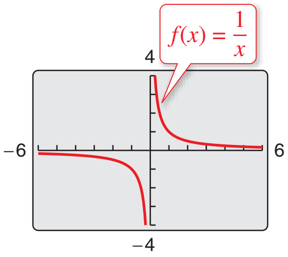
Solution

Because the denominator is zero when $x = 0$, the domain of f is all real numbers except $x = 0$. To determine the behavior of f near this excluded value, evaluate $f(x)$ to the left and right of $x = 0$, as indicated in the following tables.

x	-1	-0.5	-0.1	-0.01	-0.001	$\to 0$
$f(x)$	-1	-2	-10	-100	-1000	$\to -\infty$

x	$0 \leftarrow$	0.001	0.01	0.1	0.5	1
$f(x)$	$\infty \leftarrow$	1000	100	10	2	1

From the table, note that as x approaches 0 *from the left*, $f(x)$ decreases without bound. In contrast, as x approaches 0 *from the right*, $f(x)$ increases without bound. Because $f(x)$ decreases without bound from the left and increases without bound from the right, you can conclude that f is not continuous. The graph of f is shown in the figure.



Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the domain of $f(x) = \dfrac{3x}{x - 1}$ and discuss the behavior of f near any excluded x-values. ■

Explore the Concept

Use the *table* and *trace* features of a graphing utility to verify that the function $f(x) = 1/x$ in Example 1 is not continuous.

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Vertical and Horizontal Asymptotes

In Example 1, the behavior of f near $x = 0$ is denoted as follows.

$$\underbrace{f(x) \to -\infty \text{ as } x \to 0^-}_{\text{$f(x)$ decreases without bound as x approaches 0 from the left.}} \qquad \underbrace{f(x) \to \infty \text{ as } x \to 0^+}_{\text{$f(x)$ increases without bound as x approaches 0 from the right.}}$$

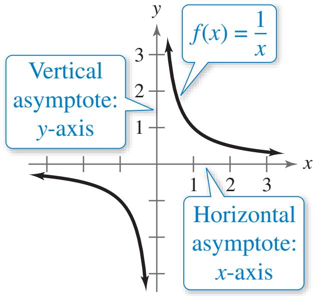


Figure 2.22

The line $x = 0$ is a **vertical asymptote** of the graph of f, as shown in Figure 2.22. From this figure, you can see that the graph of f also has a **horizontal asymptote**—the line $y = 0$. This means the values of $f(x) = 1/x$ approach zero as x increases or decreases without bound.

$$\underbrace{f(x) \to 0 \text{ as } x \to -\infty}_{\text{$f(x)$ approaches 0 as x decreases without bound.}} \qquad \underbrace{f(x) \to 0 \text{ as } x \to \infty}_{\text{$f(x)$ approaches 0 as x increases without bound.}}$$

Definition of Vertical and Horizontal Asymptotes

1. The line $x = a$ is a **vertical asymptote** of the graph of f when

$$f(x) \to \infty \quad \text{or} \quad f(x) \to -\infty$$

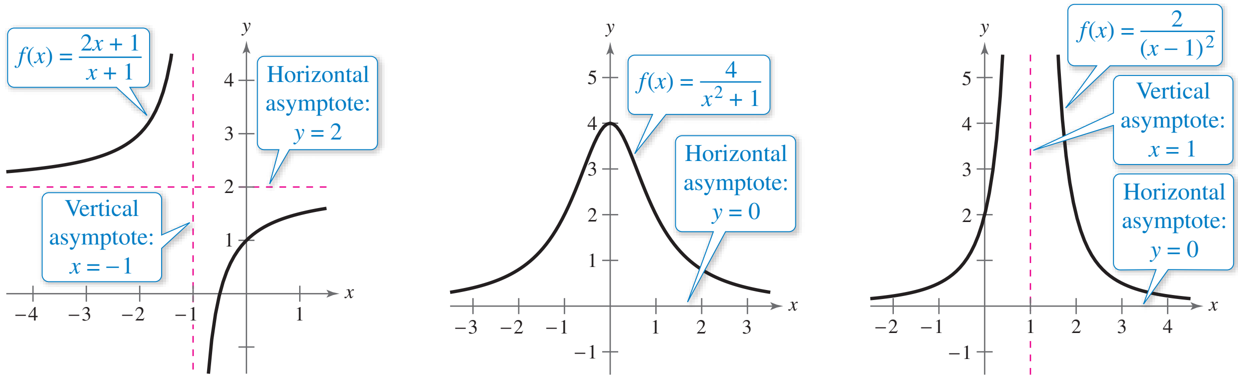
as $x \to a$, either from the right or from the left.

2. The line $y = b$ is a **horizontal asymptote** of the graph of f when

$$f(x) \to b$$

as $x \to \infty$ or $x \to -\infty$.

Figure 2.23 shows the vertical and horizontal asymptotes of the graphs of three rational functions. Note in Figure 2.23 that eventually (as $x \to \infty$ or $x \to -\infty$) the distance between the horizontal asymptote and the points on the graph must approach zero. [The graphs shown in Figures 2.22 and 2.23(a) are **hyperbolas.** You will study hyperbolas in Section 9.3.]



(a) (b) (c)

Figure 2.23

Explore the Concept

Use a table of values to determine whether the functions in Figure 2.23 are continuous. When the graph of a function has an asymptote, can you conclude that the function is not continuous? Explain.

Vertical and Horizontal Asymptotes of a Rational Function

Let f be the rational function

$$f(x) = \frac{N(x)}{D(x)} = \frac{a_nx^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_1x + a_0}{b_mx^m + b_{m-1}x^{m-1} + \cdot \cdot \cdot + b_1x + b_0}$$

where $N(x)$ and $D(x)$ have no common factors.

1. The graph of f has *vertical* asymptotes at the zeros of the denominator, $D(x)$.

2. The graph of f has at most one *horizontal* asymptote determined by comparing the degrees of $N(x)$ and $D(x)$.

a. If $n < m$, then the graph of f has the line $y = 0$ (the x-axis) as a horizontal asymptote.

b. If $n = m$, then the graph of f has the line

$$y = \frac{a_n}{b_m}$$

as a horizontal asymptote, where a_n is the leading coefficient of the numerator and b_m is the leading coefficient of the denominator.

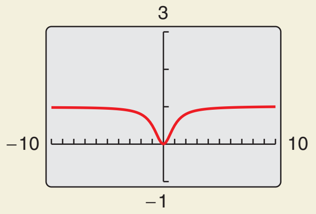
c. If $n > m$, then the graph of f has no horizontal asymptote.

What's Wrong?

You use a graphing utility to graph

$$y_1 = \frac{2x^3 + 1000x^2 + x}{x^3 + 1000x^2 + x + 1000}$$

as shown in the figure. You use the graph to conclude that the graph of y_1 has the line $y = 1$ as a horizontal asymptote. What's wrong?



EXAMPLE 2 Finding Vertical and Horizontal Asymptotes

Find all asymptotes of the graph of each rational function.

a. $f(x) = \dfrac{2x}{3x^2 + 1}$ **b.** $f(x) = \dfrac{2x^2}{x^2 - 1}$

Solution

a. For this rational function, the degree of the numerator is *less than* the degree of the denominator, so the graph has the line $y = 0$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$3x^2 + 1 = 0$ Set denominator equal to zero.

Because this equation has no real solutions, you can conclude that the graph has no vertical asymptote. The graph of the function is shown in Figure 2.24.

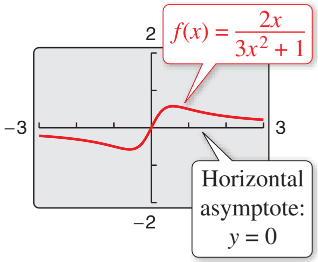


Figure 2.24

b. For this rational function, the degree of the numerator is *equal* to the degree of the denominator. The leading coefficient of the numerator is 2 and the leading coefficient of the denominator is 1, so the graph has the line $y = 2$ as a horizontal asymptote. To find any vertical asymptotes, set the denominator equal to zero and solve the resulting equation for x.

$x^2 - 1 = 0$ Set denominator equal to zero.

$(x + 1)(x - 1) = 0$ Factor.

$x + 1 = 0 \Rightarrow x = -1$ Set 1st factor equal to 0.

$x - 1 = 0 \Rightarrow x = 1$ Set 2nd factor equal to 0.

This equation has two real solutions, $x = -1$ and $x = 1$, so the graph has the lines $x = -1$ and $x = 1$ as vertical asymptotes, as shown in Figure 2.25.

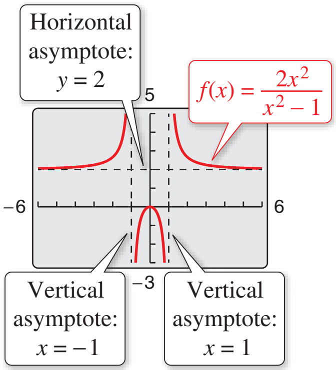


Figure 2.25

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all vertical and horizontal asymptotes of the graph of $f(x) = \dfrac{5x^2}{x^2 - 1}$.

Values for which a rational function is undefined (the denominator is zero) result in a vertical asymptote or a hole in the graph, as shown in Example 3.

EXAMPLE 3 Finding Asymptotes and Holes

Find all asymptotes and holes in the graph of

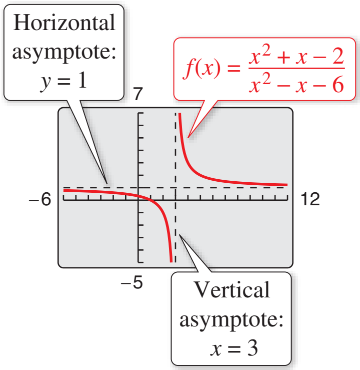
$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6}.$$

Solution

For this rational function, the degree of the numerator is *equal* to the degree of the denominator. The leading coefficients of the numerator and denominator are both 1, so the graph has the line $y = 1$ as a horizontal asymptote. To find any vertical asymptotes, first factor the numerator and denominator as follows.

$$f(x) = \frac{x^2 + x - 2}{x^2 - x - 6} = \frac{(x - 1)\cancel{(x + 2)}}{\cancel{(x + 2)}(x - 3)} = \frac{x - 1}{x - 3}, \quad x \neq -2$$

By setting the denominator $x - 3$ (of the simplified function) equal to zero, you can determine that the graph has the line $x = 3$ as a vertical asymptote, as shown in the figure. To find any holes in the graph, note that the function is undefined at $x = -2$ and $x = 3$. Because $x = -2$ is not a vertical asymptote of the function, there is a hole in the graph at $x = -2$. To find the y-coordinate of the hole, substitute $x = -2$ into the simplified form of the function.



$$y = \frac{x - 1}{x - 3} = \frac{-2 - 1}{-2 - 3} = \frac{3}{5}$$

So, the graph of the rational function has a hole at $\left(-2, \frac{3}{5}\right)$.

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Find all asymptotes and holes in the graph of $f(x) = \dfrac{x^2 - 25}{x^2 + 5x}$.

EXAMPLE 4 Finding a Function's Domain and Asymptotes

For the function f, find (a) the domain of f, (b) the vertical asymptote of f, and (c) the horizontal asymptote of f.

$$f(x) = \frac{3x^3 + 7x^2 + 2}{-2x^3 + 16}$$

Solution

a. Because the denominator is zero when $-2x^3 + 16 = 0$, solve this equation to determine that the domain of f is all real numbers except $x = 2$.

b. Because the denominator of f has a zero at $x = 2$, and 2 is not a zero of the numerator, the graph of f has the vertical asymptote $x = 2$.

c. Because the degrees of the numerator and denominator are the same, and the leading coefficient of the numerator is 3 and the leading coefficient of the denominator is -2, the horizontal asymptote of f is $y = -\frac{3}{2}$.

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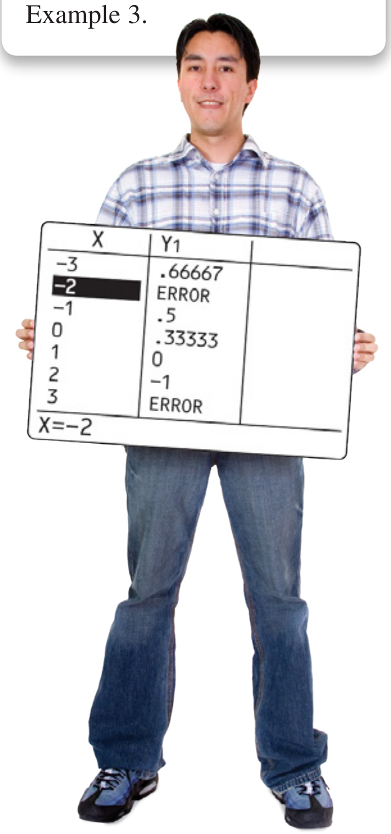
Repeat Example 4 using the function $f(x) = \dfrac{3x^2 + x - 5}{x^2 + 1}$.

Technology Tip

Graphing utilities are limited in their resolution and therefore may not show a break or hole in the graph. You can use the *table* feature of a graphing utility to verify the values of x at which a function is not defined. Try doing this for the function in Example 3.



X	Y1
-3	.66667
-2	ERROR
-1	.5
0	.33333
1	0
2	-1
3	ERROR

X=-2

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Application

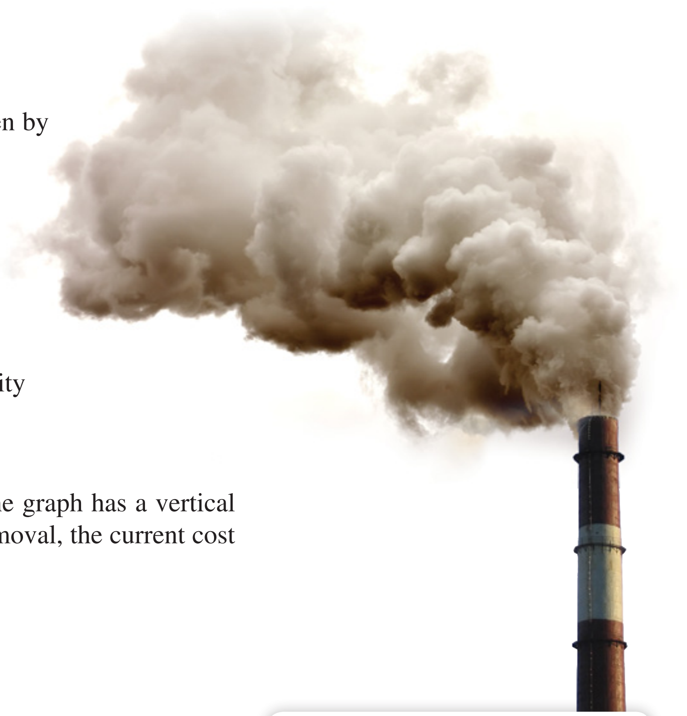
There are many examples of asymptotic behavior in real life. For instance, Example 5 shows how a vertical asymptote can be used to analyze the cost of removing pollutants from smokestack emissions.

EXAMPLE 5 Cost-Benefit Model

A utility company burns coal to generate electricity. The cost C (in dollars) of removing $p\%$ of the smokestack pollutants is given by

$$C = \frac{80{,}000p}{100 - p}$$

for $0 \le p < 100$. Use a graphing utility to graph this function. You are a member of a state legislature that is considering a law that would require utility companies to remove 90% of the pollutants from their smokestack emissions. The current law requires 85% removal. How much additional cost would the utility company incur as a result of the new law?



Solution

The graph of this function is shown in Figure 2.26. Note that the graph has a vertical asymptote at $p = 100$. Because the current law requires 85% removal, the current cost to the utility company is

$$C = \frac{80{,}000(85)}{100 - 85} \approx \$453{,}333. \qquad \text{Evaluate } C \text{ when } p = 85.$$

The cost to remove 90% of the pollutants would be

$$C = \frac{80{,}000(90)}{100 - 90} \approx \$720{,}000. \qquad \text{Evaluate } C \text{ when } p = 90.$$

So, the new law would require the utility company to spend an additional

$$720{,}000 - 453{,}333 = \$266{,}667. \qquad \text{Subtract 85\% removal cost from 90\% removal cost.}$$

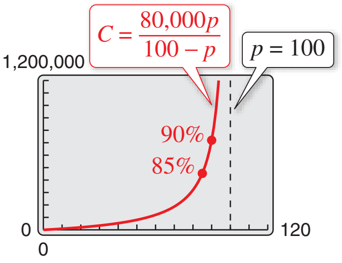


Figure 2.26

Explore the Concept

The *table* feature of a graphing utility can be used to estimate vertical and horizontal asymptotes of rational functions. Use the *table* feature to find any vertical or horizontal asymptotes of

$$f(x) = \frac{2x}{x + 1}.$$

Write a statement explaining how you found the asymptote(s) using the table.

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The cost C (in millions of dollars) of removing $p\%$ of the industrial and municipal pollutants discharged into a river is given by

$$C = \frac{255p}{100 - p}, \quad 0 \le p < 100.$$

a. Find the costs of removing 20%, 45%, and 80% of the pollutants.

b. According to the model, would it be possible to remove 100% of the pollutants? Explain.

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2.6 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. Functions of the form $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial, are called ________ .

2. If $f(x) \to \pm\infty$ as $x \to a$ from the left (or right), then $x = a$ is a ________ of the graph of f.

3. What feature of the graph of $y = \dfrac{9}{x-3}$ can you find by solving $x - 3 = 0$?

4. Is $y = \dfrac{2}{3}$ a horizontal asymptote of the function $f(x) = \dfrac{2x}{3x^2 - 5}$?

Procedures and Problem Solving

Finding the Domain of a Rational Function **In Exercises 5–10, (a) find the domain of the function, (b) complete each table, and (c) discuss the behavior of f near any excluded x-values.**

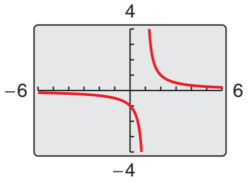
x	$f(x)$
0.5	
0.9	
0.99	
0.999	

x	$f(x)$
1.5	
1.1	
1.01	
1.001	

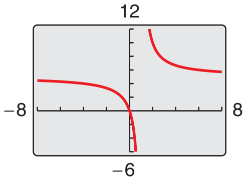
x	$f(x)$
5	
10	
100	
1000	

x	$f(x)$
−5	
−10	
−100	
−1000	

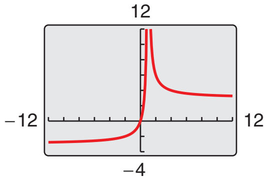
5. $f(x) = \dfrac{1}{x-1}$



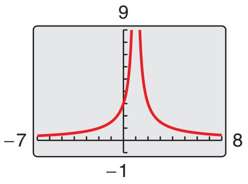
6. $f(x) = \dfrac{5x}{x-1}$



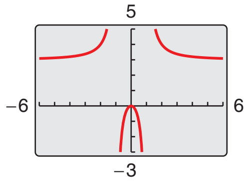
7. $f(x) = \dfrac{3x}{|x-1|}$



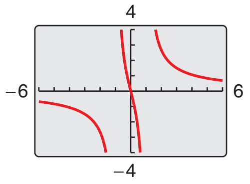
8. $f(x) = \dfrac{3}{|x-1|}$



9. $f(x) = \dfrac{3x^2}{x^2 - 1}$

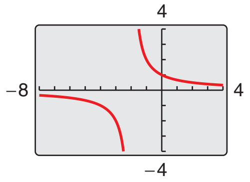


10. $f(x) = \dfrac{4x}{x^2 - 1}$

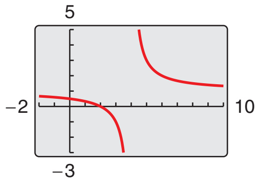


Identifying Graphs of Rational Functions **In Exercises 11–16, match the function with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

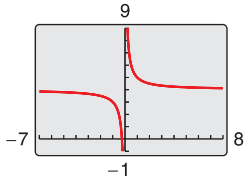
(a)



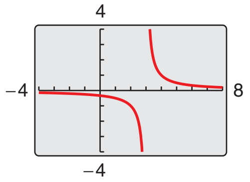
(b)



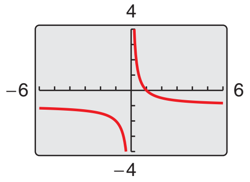
(c)



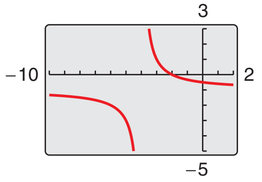
(d)



(e)



(f)



11. $f(x) = \dfrac{2}{x+2}$

12. $f(x) = \dfrac{1}{x-3}$

13. $f(x) = \dfrac{4x+1}{x}$

14. $f(x) = \dfrac{1-x}{x}$

15. $f(x) = \dfrac{x-2}{x-4}$

16. $f(x) = -\dfrac{x+2}{x+4}$

Finding Vertical and Horizontal Asymptotes **In Exercises 17–20, find any asymptotes of the graph of the rational function. Verify your answers by using a graphing utility to graph the function.**

17. $f(x) = \dfrac{1}{x^2}$

18. $f(x) = \dfrac{3}{(x-2)^3}$

19. $f(x) = \dfrac{2x^2}{x^2 + x - 6}$

20. $f(x) = \dfrac{x^2 - 4x}{x^2 - 4}$

Finding Asymptotes and Holes **In Exercises 21–24, find any asymptotes and holes in the graph of the rational function. Verify your answers by using a graphing utility.**

21. $f(x) = \dfrac{x(2+x)}{2x - x^2}$

22. $f(x) = \dfrac{x^2 + 2x + 1}{2x^2 - x - 3}$

23. $f(x) = \dfrac{x^2 - 16}{x^2 + 8x}$

24. $f(x) = \dfrac{3 - 14x - 5x^2}{3 + 7x + 2x^2}$

Finding a Function's Domain and Asymptotes **In Exercises 25–28, (a) find the domain of the function, (b) decide whether the function is continuous, and (c) identify any horizontal and vertical asymptotes. Verify your answer to part (a) both graphically by using a graphing utility and numerically by creating a table of values.**

25. $f(x) = \dfrac{5x^2 - 2x - 6}{x^2 + 4}$

26. $f(x) = \dfrac{3x^2 + 1}{x^2 + x + 9}$

27. $f(x) = \dfrac{x^2 + 3x - 4}{-x^3 + 27}$

28. $f(x) = \dfrac{4x^3 - x^2 + 3}{3x^3 + 24}$

Algebraic-Graphical-Numerical **In Exercises 29–32, (a) determine the domains of f and g, (b) find any vertical asymptotes and holes in the graphs of f and g, (c) compare f and g by completing the table, (d) use a graphing utility to graph f and g, and (e) explain why the differences in the domains of f and g are not shown in their graphs.**

29. $f(x) = \dfrac{x^2 - 16}{x - 4}, \quad g(x) = x + 4$

x	1	2	3	4	5	6	7
$f(x)$							
$g(x)$							

30. $f(x) = \dfrac{x^2 - 9}{x - 3}, \quad g(x) = x + 3$

x	0	1	2	3	4	5	6
$f(x)$							
$g(x)$							

31. $f(x) = \dfrac{x^2 - 1}{x^2 - 2x - 3}, \quad g(x) = \dfrac{x - 1}{x - 3}$

x	-2	-1	0	1	2	3	4
$f(x)$							
$g(x)$							

32. $f(x) = \dfrac{x^2 - 4}{x^2 - 3x + 2}, \quad g(x) = \dfrac{x + 2}{x - 1}$

x	-3	-2	-1	0	1	2	3
$f(x)$							
$g(x)$							

Exploration **In Exercises 33–36, determine the value that the function f approaches as the magnitude of x increases. Is $f(x)$ greater than or less than this value when x is positive and large in magnitude? What about when x is negative and large in magnitude?**

33. $f(x) = 4 - \dfrac{1}{x}$

34. $f(x) = 2 + \dfrac{1}{x - 3}$

35. $f(x) = \dfrac{2x - 1}{x - 3}$

36. $f(x) = \dfrac{2x - 1}{x^2 + 1}$

Finding the Zeros of a Rational Function **In Exercises 37–44, find the zeros (if any) of the rational function. Use a graphing utility to verify your answer.**

37. $f(x) = \dfrac{x^2 - 9}{x^2 + 5}$

38. $h(x) = \dfrac{x^3 + 8}{x^2 - 11}$

39. $g(x) = 1 + \dfrac{6}{x - 3}$

40. $f(x) = 3 - \dfrac{-12}{x^2 + 2}$

41. $h(x) = \dfrac{x^2 - x - 20}{x^2 + 7}$

42. $g(x) = \dfrac{x^2 - 8x + 12}{x^2 + 4}$

43. $f(x) = \dfrac{x^2 + 4x - 21}{x^2 - 4x + 3}$

44. $h(x) = \dfrac{2x^2 + 11x + 5}{3x^2 + 13x - 10}$

45. Biology The game commission introduces 100 deer into newly acquired state game lands. The population N of the herd is given by

$$N = \frac{20(5 + 3t)}{1 + 0.04t}, \quad t \geq 0$$

where t is the time in years.

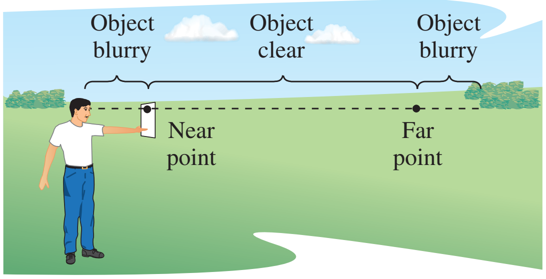
(a) Use a graphing utility to graph the model.

(b) Find the populations when $t = 5$, $t = 10$, and $t = 25$.

(c) What is the limiting size of the herd as time increases? Explain.

46. MODELING DATA

The endpoints of the interval over which distinct vision is possible are called the *near point* and *far point* of the eye (see figure). With increasing age, these points normally change. The table shows the approximate near points y (in inches) for various ages x (in years).



Age, x	Near point, y
16	3.0
32	4.7
44	9.8
50	19.7
60	39.7

(a) Find a rational model for the data. Take the reciprocals of the near points to generate the points

$$\left(x, \frac{1}{y}\right).$$

Use the *regression* feature of a graphing utility to find a linear model for the data. The resulting line has the form

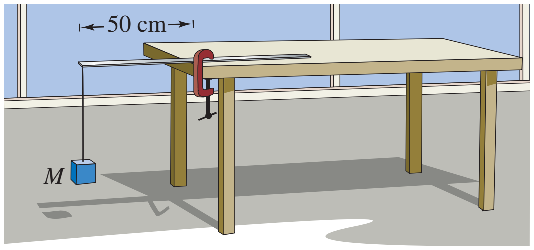
$$\frac{1}{y} = ax + b.$$

Solve for y.

(b) Use the *table* feature of the graphing utility to create a table showing the predicted near point based on the model for each of the ages in the original table.

(c) Do you think the model can be used to predict the near point for a person who is 70 years old? Explain.

47. Physics Consider a physics laboratory experiment designed to determine an unknown mass. A flexible metal meter stick is clamped to a table with 50 centimeters overhanging the edge (see figure). Known masses M ranging from 200 grams to 2000 grams are attached to the end of the meter stick. For each mass, the meter stick is displaced vertically and then allowed to oscillate. The average time t (in seconds) of one oscillation for each mass is recorded in the table.



Mass, M	Time, t
200	0.450
400	0.597
600	0.712
800	0.831
1000	0.906
1200	1.003
1400	1.088
1600	1.126
1800	1.218
2000	1.338

Spreadsheet at LarsonPrecalculus.com

A model for the data is given by

$$t = \frac{38M + 16{,}965}{10(M + 5000)}.$$

(a) Use the *table* feature of a graphing utility to create a table showing the estimated time based on the model for each of the masses shown in the table. What can you conclude?

(b) Use the model to approximate the mass of an object when the average time for one oscillation is 1.056 seconds.

48. Business The sales S (in thousands of units) of a tablet computer during the nth week after the tablet is released are given by

$$S = \frac{150n}{n^2 + 100}, \quad n \geq 0.$$

(a) Use a graphing utility to graph the sales function.

(b) Find the sales in week 5, week 10, and week 20.

(c) According to this model, will sales ever drop to zero units? Explain.

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49. ***Why you should learn it*** *(p. 142)* The cost C (in dollars) of supplying recycling bins to $p\%$ of the population of a rural township is given by

$$C = \frac{25{,}000p}{100 - p}, \quad 0 \le p < 100.$$

(a) Use a graphing utility to graph the cost function.

(b) Find the costs of supplying bins to 15%, 50%, and 90% of the population.

(c) According to the model, would it be possible to supply bins to 100% of the population? Explain.

Conclusions

True or False? **In Exercises 50 and 51, determine whether the statement is true or false. Justify your answer.**

50. A rational function can have infinitely many vertical asymptotes.

51. A rational function must have at least one vertical asymptote.

52. **Writing a Rational Function** Write a rational function f that has the specified characteristics. (There are many correct answers.)

(a) Vertical asymptote: $x = 2$

Horizontal asymptote: $y = 0$

Zero: $x = 1$

(b) Vertical asymptote: $x = -1$

Horizontal asymptote: $y = 0$

Zero: $x = 2$

(c) Vertical asymptotes: $x = -2, x = 1$

Horizontal asymptote: $y = 2$

Zeros: $x = 3, x = -3$

(d) Vertical asymptotes: $x = -1, x = 2$

Horizontal asymptote: $y = -2$

Zeros: $x = -2, x = 3$

(e) Vertical asymptotes: $x = 0, x = \pm 3$

Horizontal asymptote: $y = 3$

Zeros: $x = -1, x = 1, x = 2$

53. **Think About It** A real zero of the numerator of a rational function f is $x = c$. Must $x = c$ also be a zero of f? Explain.

54. **Think About It** When the graph of a rational function f has a vertical asymptote at $x = 4$, can f have a common factor of $(x - 4)$ in the numerator and denominator? Explain.

55. **Exploration** Use a graphing utility to compare the graphs of y_1 and y_2.

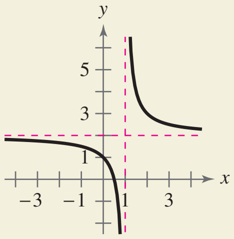
$$y_1 = \frac{3x^3 - 5x^2 + 4x - 5}{2x^2 - 6x + 7}, \quad y_2 = \frac{3x^3}{2x^2}$$

Start with a viewing window of $-5 \le x \le 5$ and $-10 \le y \le 10$, and then zoom out. Make a conjecture about how the graph of a rational function f is related to the graph of $y = a_nx^n/b_mx^m$, where a_nx^n is the leading term of the numerator of f and b_mx^m is the leading term of the denominator of f.

56. **HOW DO YOU SEE IT?** The graph of a rational function

$$f(x) = \frac{N(x)}{D(x)}$$

is shown below. Determine which of the statements about the function is false. Justify your answer.



(a) $D(1) = 0$.

(b) The degrees of $N(x)$ and $D(x)$ are equal.

(c) The ratio of the leading coefficients of $N(x)$ and $D(x)$ is 1.

Cumulative Mixed Review

Finding the Equation of a Line **In Exercises 57–60, write the general form of the equation of the line that passes through the points.**

57. $(3, 2), (0, -1)$

58. $(-6, 1), (4, -5)$

59. $(2, 7), (3, 10)$

60. $(0, 0), (-9, 4)$

Long Division of Polynomials **In Exercises 61–64, divide using long division.**

61. $(x^2 + 5x + 6) \div (x - 4)$

62. $(x^2 - 10x + 15) \div (x - 3)$

63. $(2x^4 + x^2 - 11) \div (x^2 + 5)$

64. $(4x^5 + 3x^3 - 10) \div (2x + 3)$

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2.7 Graphs of Rational Functions

The Graph of a Rational Function

To sketch the graph of a rational function, use the following guidelines.

Guidelines for Graphing Rational Functions

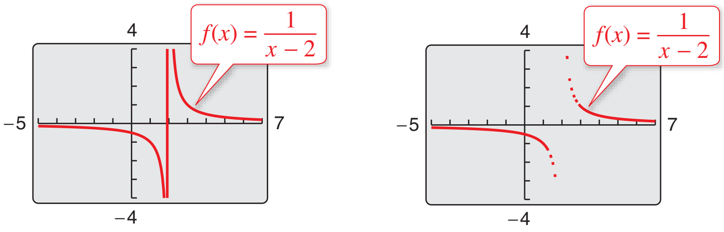
Let $f(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials and $D(x)$ is not the zero polynomial.

1. Simplify f, if possible. Any restrictions on the domain of f not in the simplified function should be listed.
2. Find and plot the y-intercept (if any) by evaluating $f(0)$.
3. Find the zeros of the numerator (if any) by setting the numerator equal to zero. Then plot the corresponding x-intercepts.
4. Find the zeros of the denominator (if any) by setting the denominator equal to zero. Then sketch the corresponding vertical asymptotes using dashed vertical lines and plot the corresponding holes using open circles.
5. Find and sketch any other asymptotes of the graph using dashed lines.
6. Plot at least one point *between* and one point *beyond* each x-intercept and vertical asymptote.
7. Use smooth curves to complete the graph between and beyond the vertical asymptotes, excluding any points where f is not defined.

When graphing simple rational functions, testing for symmetry can be useful. For instance, the graph of $f(x) = 1/x$ is symmetrical with respect to the origin, and the graph of $g(x) = 1/x^2$ is symmetrical with respect to the y-axis.

Technology Tip

Some graphing utilities have difficulty graphing rational functions that have vertical asymptotes. Often, the utility will connect parts of the graph that are not supposed to be connected. Notice that the graph in Figure 2.27(a) should consist of two *unconnected* portions—one to the left of $x = 2$ and the other to the right of $x = 2$. To eliminate this problem, you can try changing the *mode* of the graphing utility to *dot mode*. The problem with this mode is that the graph is then represented as a collection of dots rather than as a smooth curve, as shown in Figure 2.27(b).



(a) Connected mode **(b) Dot mode**

Figure 2.27

What you should learn

- Analyze and sketch graphs of rational functions.
- Sketch graphs of rational functions that have slant asymptotes.
- Use graphs of rational functions to model and solve real-life problems.

Why you should learn it

The graph of a rational function provides a good indication of the behavior of a mathematical model. Exercise 89 on page 159 models the concentration of a chemical in the bloodstream after injection.

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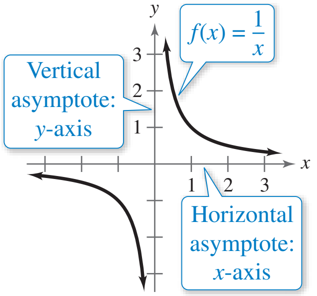
Library of Parent Functions: Rational Function

The simplest type of rational function is the *parent rational function* $f(x) = 1/x$, also known as the *reciprocal function*. The basic characteristics of the parent rational function are summarized below and on the inside cover of this text.

Graph of $f(x) = \frac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$
Range: $(-\infty, 0) \cup (0, \infty)$
No intercepts
Decreasing on $(-\infty, 0)$ and $(0, \infty)$
Odd function
Origin symmetry
Vertical asymptote: y-axis
Horizontal asymptote: x-axis



$f(x) = \frac{1}{x}$

Vertical asymptote: y-axis

Horizontal asymptote: x-axis

EXAMPLE 1 Library of Parent Functions: $f(x) = 1/x$

See LarsonPrecalculus.com for an interactive version of this type of example.

Sketch the graph of each function by hand and compare it with the graph of $f(x) = 1/x$.

a. $g(x) = \dfrac{-1}{x+2}$

b. $h(x) = \dfrac{1}{x-1} + 3$

Solution

a. With respect to the graph of $f(x) = 1/x$, the graph of g is obtained by a *reflection* in the y-axis and a horizontal shift two units *to the left*, as shown in Figure 2.28. Confirm this with a graphing utility.

b. With respect to the graph of $f(x) = 1/x$, the graph of h is obtained by a horizontal shift one unit *to the right* and a vertical shift three units *upward*, as shown in Figure 2.29. Confirm this with a graphing utility.

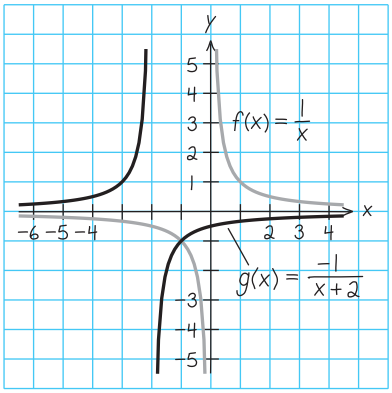


Figure 2.28

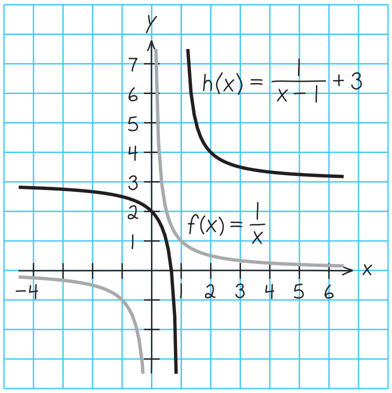


Figure 2.29

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the graph of each function by hand and compare it with the graph of $f(x) = 1/x$.

a. $g(x) = \dfrac{1}{x-4}$ **b.** $h(x) = \dfrac{1}{x+3} - 1$

Explore the Concept

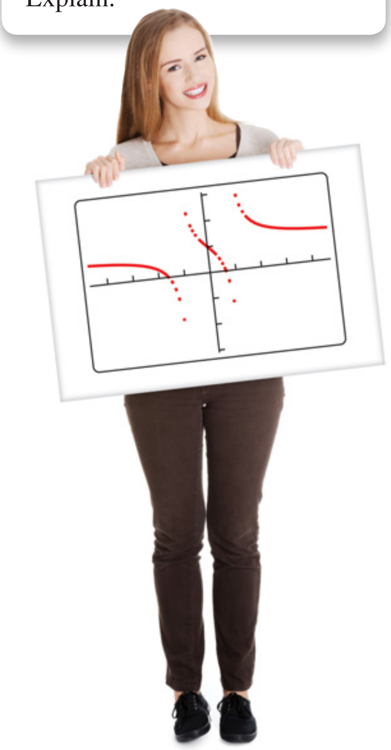
Use a graphing utility to graph

$$f(x) = 1 + \frac{1}{x - \frac{1}{x}}.$$

Set the graphing utility to *dot* mode and use a decimal viewing window. Use the *trace* feature to find three "holes" or "breaks" in the graph. Do all three holes or breaks represent zeros of the denominator

$$x - \frac{1}{x}?$$

Explain.



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In Examples 2–6, note that the vertical asymptotes are included in the tables of additional points. This is done to emphasize numerically the behavior of the graph of the function.

EXAMPLE 2 Sketching the Graph of a Rational Function

Sketch the graph of $g(x) = \dfrac{3}{x - 2}$ by hand.

Solution

y-intercept: $\left(0, -\frac{3}{2}\right)$, because $g(0) = -\frac{3}{2}$

x-intercept: None, because $3 \neq 0$

Vertical asymptote: $x = 2$, zero of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

x	-4	1	2	3	5
$g(x)$	-0.5	-3	Undefined	3	1

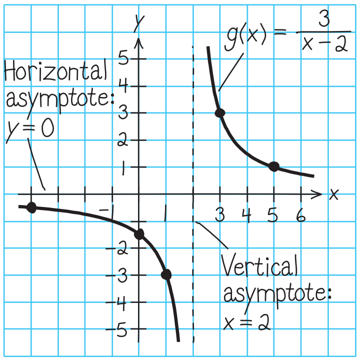

Figure 2.30

By plotting the intercept, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.30. Confirm this with a graphing utility.

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Sketch the graph of $f(x) = \dfrac{1}{x + 3}$ by hand.

Note that the graph of g in Example 2 is a vertical stretch and a right shift of the graph of

$$f(x) = \frac{1}{x}$$

because

$$g(x) = \frac{3}{x - 2} = 3\left(\frac{1}{x - 2}\right) = 3f(x - 2).$$

EXAMPLE 3 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{2x - 1}{x}$ by hand.

Solution

y-intercept: None, because $x = 0$ is not in the domain

x-intercept: $\left(\frac{1}{2}, 0\right)$, because $2x - 1 = 0$ when $x = \frac{1}{2}$

Vertical asymptote: $x = 0$, zero of denominator

Horizontal asymptote: $y = 2$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

x	-4	-1	0	$\frac{1}{4}$	4
$f(x)$	2.25	3	Undefined	-2	1.75

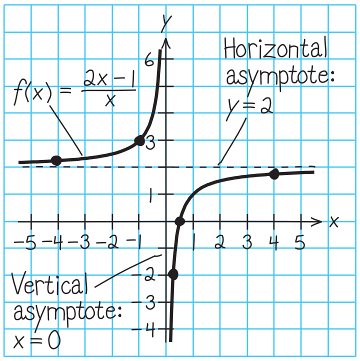


Figure 2.31

By plotting the intercept, asymptotes, and a few additional points, you can obtain the graph shown in Figure 2.31. Confirm this with a graphing utility.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $f(x) = \dfrac{2x + 3}{x + 1}$ by hand.

EXAMPLE 4 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x}{x^2 - x - 2}$.

Solution

Factor the denominator to determine the zeros of the denominator.

$$f(x) = \frac{x}{x^2 - x - 2}$$

$$= \frac{x}{(x + 1)(x - 2)}$$

y-intercept: $(0, 0)$, because $f(0) = 0$

x-intercept: $(0, 0)$

Vertical asymptotes: $x = -1$, $x = 2$, zeros of denominator

Horizontal asymptote: $y = 0$, because degree of $N(x) <$ degree of $D(x)$

Additional points:

x	-3	-1	-0.5	1	2	3
$f(x)$	-0.3	Undefined	0.4	-0.5	Undefined	0.75

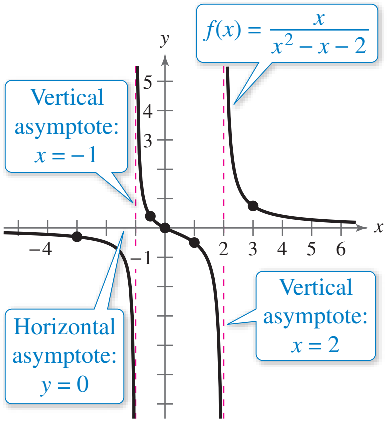


Figure 2.32

The graph is shown in Figure 2.32.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $f(x) = \dfrac{3x}{x^2 + x - 2}$.

EXAMPLE 5 Sketching the Graph of a Rational Function

Sketch the graph of $f(x) = \dfrac{x^2 - 9}{x^2 - 2x - 3}$.

Solution

By factoring the numerator and denominator, you have

$$f(x) = \frac{x^2 - 9}{x^2 - 2x - 3} = \frac{\cancel{(x - 3)}(x + 3)}{\cancel{(x - 3)}(x + 1)} = \frac{x + 3}{x + 1}, \quad x \neq 3.$$

y-intercept: $(0, 3)$, because $f(0) = 3$

x-intercept: $(-3, 0)$, because $x + 3 = 0$ when $x = -3$

Vertical asymptote: $x = -1$, zero of (simplified) denominator

Hole: $\left(3, \frac{3}{2}\right)$, f is not defined at $x = 3$

Horizontal asymptote: $y = 1$, because degree of $N(x) =$ degree of $D(x)$

Additional points:

x	-5	-2	-1	-0.5	1	3	4
$f(x)$	0.5	-1	Undefined	5	2	Undefined	1.4

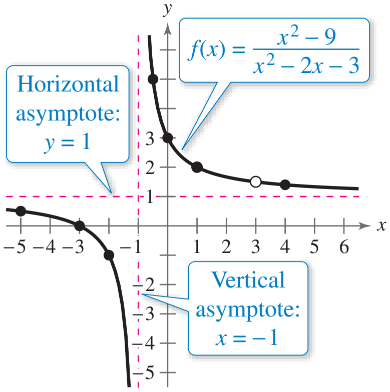


Figure 2.33 ***Hole at*** $x = 3$

The graph is shown in Figure 2.33.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $f(x) = \dfrac{x^2 - 4}{x^2 - x - 6}$.

Slant Asymptotes

Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, then the graph of the function has a **slant** (or **oblique**) **asymptote.** For example, the graph of

$$f(x) = \frac{x^2 - x}{x + 1}$$

has a slant asymptote, as shown in Figure 2.34. To find the equation of a slant asymptote, use long division. For instance, by dividing $x + 1$ into $x^2 - x$, you have

$$f(x) = \frac{x^2 - x}{x + 1} = \underbrace{x - 2}_{\substack{\text{Slant asymptote} \\ (y = x - 2)}} + \frac{2}{x + 1}.$$

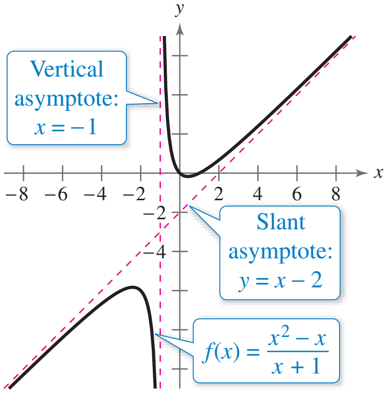


Figure 2.34

As x increases or decreases without bound, the remainder term $2/(x + 1)$ approaches 0, so the graph of f approaches the line $y = x - 2$, as shown in Figure 2.34.

Explore the Concept

Do you think it is possible for the graph of a rational function to cross its horizontal asymptote or its slant asymptote? Use the graphs of the following functions to investigate this question. Write a summary of your conclusion. Explain your reasoning.

$$f(x) = \frac{x}{x^2 + 1}$$

$$g(x) = \frac{2x}{3x^2 - 2x + 1}$$

$$h(x) = \frac{x^3}{x^2 + 1}$$

EXAMPLE 6 A Rational Function with a Slant Asymptote

Sketch the graph of $f(x) = \dfrac{x^2 - x - 2}{x - 1}$.

Solution

First, write $f(x)$ in two different ways. Factoring the numerator

$$f(x) = \frac{x^2 - x - 2}{x - 1} = \frac{(x - 2)(x + 1)}{x - 1}$$

enables you to recognize the x-intercepts. Long division

$$f(x) = \frac{x^2 - x - 2}{x - 1} = x - \frac{2}{x - 1}$$

enables you to recognize that the line $y = x$ is a slant asymptote of the graph.

y-intercept: (0, 2), because $f(0) = 2$

x-intercepts: $(-1, 0)$ and $(2, 0)$

Vertical asymptote: $x = 1$, zero of denominator

Horizontal asymptote: None, because degree of $N(x) >$ degree of $D(x)$

Slant asymptote: $y = x$

Additional points:

x	-2	0.5	1	1.5	3
$f(x)$	-1.33	4.5	Undefined	-2.5	2

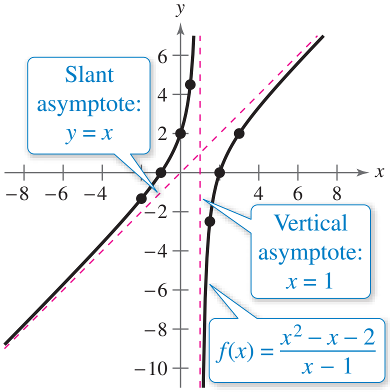


Figure 2.35

The graph is shown in Figure 2.35.

✓ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Sketch the graph of $f(x) = \dfrac{3x^2 + 1}{x}$.

Application

EXAMPLE 7 Publishing

A rectangular page is designed to contain 48 square inches of print. The margins on each side of the page are $1\frac{1}{2}$ inches wide. The margins at the top and bottom are each 1 inch deep. What should the dimensions of the page be so that the minimum amount of paper is used?

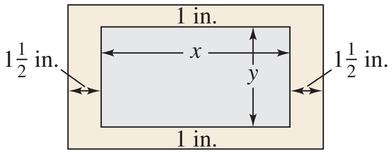


Figure 2.36

Graphical Solution

Let A be the area to be minimized. From Figure 2.36, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

The graph of this rational function is shown in Figure 2.37. Because x represents the width of the printed area, you need consider only the portion of the graph for which x is positive. Using the *minimum* feature of a graphing utility, you can approximate the minimum value of A to occur when $x \approx 8.5$ inches. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be

$$x + 3 \approx 11.5 \text{ inches by } y + 2 \approx 7.6 \text{ inches.}$$

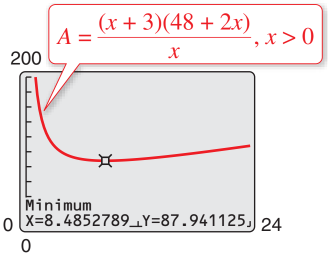


Figure 2.37

Numerical Solution

Let A be the area to be minimized. From Figure 2.36, you can write

$$A = (x + 3)(y + 2).$$

The printed area inside the margins is modeled by $48 = xy$ or $y = 48/x$. To find the minimum area, rewrite the equation for A in terms of just one variable by substituting $48/x$ for y.

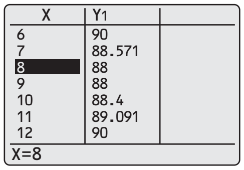
$$A = (x + 3)\left(\frac{48}{x} + 2\right) = \frac{(x + 3)(48 + 2x)}{x}, \quad x > 0$$

Use the *table* feature of a graphing utility to create a table of values for the function

$$y_1 = \frac{(x + 3)(48 + 2x)}{x}$$

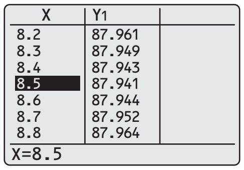
beginning at $x = 1$. From the table, you can see that the minimum value of y_1 occurs when x is somewhere between 8 and 9, as shown in Figure 2.38. To approximate the minimum value of y_1 to one decimal place, change the table to begin at $x = 8$ and set the table step to 0.1. The minimum value of y_1 occurs when $x \approx 8.5$, as shown in Figure 2.39. The corresponding value of y is $48/8.5 \approx 5.6$ inches. So, the dimensions should be

$$x + 3 \approx 11.5 \text{ inches by } y + 2 \approx 7.6 \text{ inches.}$$



```
  X    | Y1
6      | 90
7      | 88.571
8      | 88
9      | 88
10     | 88.4
11     | 89.091
12     | 90
X=8
```

Figure 2.38



```
  X    | Y1
8.2    | 87.961
8.3    | 87.949
8.4    | 87.943
8.5    | 87.941
8.6    | 87.944
8.7    | 87.952
8.8    | 87.964
X=8.5
```

Figure 2.39

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Rework Example 7 when the margins on each side are 2 inches wide and the page contains 40 square inches of print.

If you go on to take a course in calculus, you will learn an analytic technique for finding the exact value of x that produces a minimum area in Example 7. In this case, that value is $x = 6\sqrt{2} \approx 8.485$.

2.7 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank(s).

1. For the rational function $f(x) = N(x)/D(x)$, if the degree of $N(x)$ is exactly one more than the degree of $D(x)$, then the graph of f has a _______ (or oblique) _______ .

2. The graph of $f(x) = 1/x$ has a _______ asymptote at $x = 0$.

3. Does the graph of $f(x) = \dfrac{x^3 - 1}{x^2 + 2}$ have a slant asymptote?

4. Using long division, you find that $f(x) = \dfrac{x^2 + 1}{x + 1} = x - 1 + \dfrac{2}{x + 1}$. What is the slant asymptote of the graph of f?

Procedures and Problem Solving

Library of Parent Functions: $f(x) = 1/x$ **In Exercises 5–8, sketch the graph of the function g and describe how the graph is related to the graph of $f(x) = 1/x$.**

5. $g(x) = \dfrac{-1}{x} + 2$

6. $g(x) = \dfrac{1}{x - 6}$

7. $g(x) = \dfrac{1}{x - 3} - 1$

8. $g(x) = \dfrac{-1}{x + 2} - 4$

Describing a Transformation of $f(x) = 2/x$ **In Exercises 9–12, use a graphing utility to graph $f(x) = 2/x$ and the function g in the same viewing window. Describe the relationship between the two graphs.**

9. $g(x) = f(x) + 1$

10. $g(x) = f(x - 1)$

11. $g(x) = -f(x)$

12. $g(x) = \frac{1}{2}f(x + 2)$

Describing a Transformation of $f(x) = 3/x^2$ **In Exercises 13–16, use a graphing utility to graph $f(x) = 3/x^2$ and the function g in the same viewing window. Describe the relationship between the two graphs.**

13. $g(x) = f(x) - 2$

14. $g(x) = -2f(x)$

15. $g(x) = f(x - 2)$

16. $g(x) = \frac{1}{4}f(x)$

Sketching the Graph of a Rational Function **In Exercises 17–32, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, and holes. Use a graphing utility to verify your graph.**

17. $f(x) = \dfrac{1}{x + 2}$

18. $f(x) = \dfrac{1}{x - 6}$

19. $C(x) = \dfrac{5 + 2x}{1 + x}$

20. $P(x) = \dfrac{1 - 3x}{1 - x}$

21. $f(t) = \dfrac{1 - 2t}{t}$

22. $g(x) = \dfrac{1}{x + 2} + 2$

23. $f(x) = \dfrac{x^2}{x^2 - 4}$

24. $g(x) = \dfrac{x}{x^2 - 9}$

25. $g(x) = \dfrac{4(x + 1)}{x(x - 4)}$

26. $h(x) = \dfrac{2}{x^2(x - 3)}$

27. $f(x) = \dfrac{3x}{x^2 - x - 2}$

28. $f(x) = \dfrac{2x}{x^2 + x - 2}$

29. $f(x) = \dfrac{x^2 + 3x}{x^2 + x - 6}$

30. $g(x) = \dfrac{5(x + 4)}{x^2 + x - 12}$

31. $f(x) = \dfrac{x^2 - 1}{x + 1}$

32. $f(x) = \dfrac{x^2 - 16}{x - 4}$

Finding the Domain and Asymptotes **In Exercises 33–42, use a graphing utility to graph the function. Determine its domain and identify any vertical or horizontal asymptotes.**

33. $f(x) = \dfrac{2 + x}{1 - x}$

34. $f(x) = \dfrac{3 - x}{2 - x}$

35. $g(x) = \dfrac{3x - 4}{-x}$

36. $h(x) = \dfrac{2x - 1}{x + 5}$

37. $g(x) = \dfrac{5}{x^2 + 1}$

38. $g(x) = -\dfrac{x}{(x - 2)^2}$

39. $f(x) = \dfrac{x + 1}{x^2 - x - 6}$

40. $f(x) = \dfrac{x + 4}{x^2 + x - 6}$

41. $f(x) = \dfrac{20x}{x^2 + 1} - \dfrac{1}{x}$

42. $f(x) = 5\left(\dfrac{1}{x - 4} - \dfrac{1}{x + 2}\right)$

Exploration **In Exercises 43–48, use a graphing utility to graph the function. What do you observe about its asymptotes?**

43. $h(x) = \dfrac{6x}{\sqrt{x^2 + 1}}$

44. $f(x) = -\dfrac{x}{\sqrt{9 + x^2}}$

45. $g(x) = \dfrac{4|x - 2|}{x + 1}$

46. $f(x) = -\dfrac{8|3 + x|}{x - 2}$

47. $f(x) = \dfrac{4(x - 1)^2}{x^2 - 4x + 5}$

48. $g(x) = \dfrac{3x^4 - 5x + 3}{x^4 + 1}$

A Rational Function with a Slant Asymptote **In Exercises 49–56, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, and slant asymptotes.**

49. $f(x) = \dfrac{2x^2 + 1}{x}$

50. $g(x) = \dfrac{1 - x^2}{x}$

51. $h(x) = \dfrac{x^2}{x - 1}$

52. $f(x) = \dfrac{x^3}{x^2 - 1}$

53. $g(x) = \dfrac{x^3}{2x^2 - 8}$

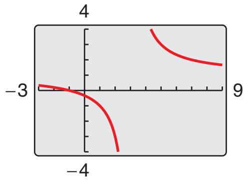
54. $f(x) = \dfrac{x^3}{x^2 + 4}$

55. $f(x) = \dfrac{x^3 + 2x^2 + 4}{2x^2 + 1}$

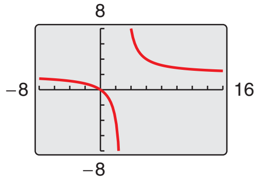
56. $f(x) = \dfrac{2x^2 - 5x + 5}{x - 2}$

Finding the x-Intercepts **In Exercises 57–60, use the graph to estimate any x-intercepts of the rational function. Set $y = 0$ and solve the resulting equation to confirm your result.**

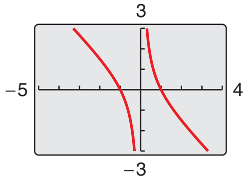
57. $y = \dfrac{x + 1}{x - 3}$



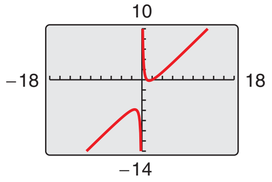
58. $y = \dfrac{2x}{x - 3}$



59. $y = \dfrac{1}{x} - x$



60. $y = x - 3 + \dfrac{2}{x}$



Finding the Domain and Asymptotes **In Exercises 61–64, use a graphing utility to graph the rational function. Determine the domain of the function and identify any asymptotes.**

61. $y = \dfrac{2x^2 + x}{x + 1}$

62. $y = \dfrac{x^2 + 5x + 8}{x + 3}$

63. $y = \dfrac{1 + 3x^2 - x^3}{x^2}$

64. $y = \dfrac{12 - 2x - x^2}{2(4 + x)}$

Finding Asymptotes and Holes **In Exercises 65–70, find all vertical asymptotes, horizontal asymptotes, slant asymptotes, and holes in the graph of the function. Then use a graphing utility to verify your results.**

65. $f(x) = \dfrac{x^2 - 5x + 4}{x^2 - 4}$

66. $f(x) = \dfrac{x^2 - 2x - 8}{x^2 - 9}$

67. $f(x) = \dfrac{2x^2 - 5x + 2}{2x^2 - x - 6}$

68. $f(x) = \dfrac{3x^2 - 8x + 4}{2x^2 - 3x - 2}$

69. $f(x) = \dfrac{2x^3 - x^2 - 2x + 1}{x^2 + 3x + 2}$

70. $f(x) = \dfrac{2x^3 + x^2 - 8x - 4}{x^2 - 3x + 2}$

Finding x-Intercepts Graphically **In Exercises 71–82, use a graphing utility to graph the function and determine any x-intercepts. Set $y = 0$ and solve the resulting equation to confirm your result.**

71. $y = \dfrac{1}{x + 5} + \dfrac{4}{x}$

72. $y = \dfrac{1}{x - 2} - \dfrac{5}{x}$

73. $y = \dfrac{2}{x + 2} - \dfrac{3}{x - 1}$

74. $y = \dfrac{6}{x + 3} - \dfrac{1}{x + 4}$

75. $y = x - \dfrac{2}{x + 1}$

76. $y = 2x - \dfrac{8}{x}$

77. $y = x + 2 - \dfrac{1}{x + 1}$

78. $y = 2x - 1 + \dfrac{1}{x - 2}$

79. $y = x + 1 + \dfrac{2}{x - 1}$

80. $y = x + 2 + \dfrac{2}{x + 2}$

81. $y = x + 3 - \dfrac{2}{2x - 1}$

82. $y = x - 1 - \dfrac{2}{2x - 3}$

83. **Chemistry** A 1000-liter tank contains 50 liters of a 25% brine solution. You add x liters of a 75% brine solution to the tank.

(a) Show that the concentration C (the ratio of brine to the total solution) of the final mixture is given by

$$C = \frac{3x + 50}{4(x + 50)}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the function. As the tank is filled, what happens to the rate at which the concentration of brine increases? What percent does the concentration of brine appear to approach?

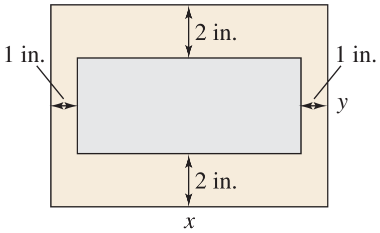
84. **Geometry** A rectangular region of length x and width y has an area of 500 square meters.

(a) Write the width y as a function of x.

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Sketch a graph of the function and determine the width of the rectangle when $x = 30$ meters.

85. **Publishing** A page that is x inches wide and y inches high contains 30 square inches of print (see figure). The margins at the top and bottom are 2 inches deep and the margins on each side are 1 inch wide.



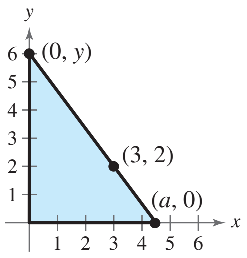
(a) Show that the total area A of the page is given by

$$A = \frac{2x(2x + 11)}{x - 2}.$$

(b) Determine the domain of the function based on the physical constraints of the problem.

(c) Use a graphing utility to graph the area function and approximate the page size such that the minimum amount of paper will be used. Verify your answer numerically using the *table* feature of the graphing utility.

86. **Geometry** A right triangle is formed in the first quadrant by the x-axis, the y-axis, and a line segment through the point $(3, 2)$. (See figure.)



(a) Show that an equation of the line segment is given by

$$y = \frac{2(a - x)}{a - 3}, \quad 0 \le x \le a.$$

(b) Show that the area of the triangle is given by

$$A = \frac{a^2}{a - 3}.$$

(c) Use a graphing utility to graph the area function and estimate the value of a that yields a minimum area. Estimate the minimum area. Verify your answer numerically using the *table* feature of the graphing utility.

87. **Cost Management** The ordering and transportation cost C (in thousands of dollars) for the components used in manufacturing a product is given by

$$C = 100\left(\frac{200}{x^2} + \frac{x}{x + 30}\right), \quad x \ge 1$$

where x is the order size (in hundreds). Use a graphing utility to graph the cost function. From the graph, estimate the order size that minimizes cost.

88. **Cost Management** The cost C of producing x units of a product is given by $C = 0.2x^2 + 10x + 5$, and the average cost per unit is given by

$$\overline{C} = \frac{C}{x} = \frac{0.2x^2 + 10x + 5}{x}, \quad x > 0.$$

Sketch the graph of the average cost function, and estimate the number of units that should be produced to minimize the average cost per unit.

89. ***Why you should learn it*** *(p. 151)* The concentration C of a chemical in the bloodstream t hours after injection into muscle tissue is given by

$$C = \frac{3t^2 + t}{t^3 + 50}, \quad t \ge 0.$$

(a) Determine the horizontal asymptote of the function and interpret its meaning in the context of the problem.

(b) Use a graphing utility to graph the function and approximate the time when the bloodstream concentration is greatest.

(c) Use the graphing utility to determine when the concentration is less than 0.345.

90. **Algebraic-Graphical-Numerical** A driver averaged 50 miles per hour on the round trip between Baltimore, Maryland, and Philadelphia, Pennsylvania, 100 miles away. The average speeds for going and returning were x and y miles per hour, respectively.

(a) Show that $y = \dfrac{25x}{x - 25}$.

(b) Determine the vertical and horizontal asymptotes of the function.

(c) Use a graphing utility to complete the table. What do you observe?

x	30	35	40	45	50	55	60
y							

(d) Use the graphing utility to graph the function.

(e) Is it possible to average 20 miles per hour in one direction and still average 50 miles per hour on the round trip? Explain.

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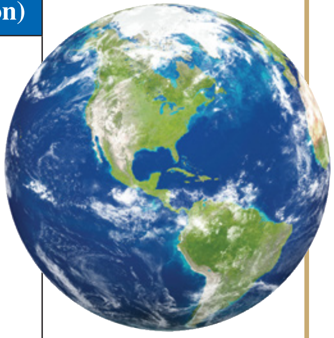
91. MODELING DATA

Data are recorded at 225 monitoring sites throughout the United States to study national trends in air quality. The table shows the mean amount A of carbon monoxide (in parts per million) recorded at these sites in each year from 2003 through 2012. (Source: EPA)

DATA

Year	Amount, A (in parts per million)
2003	2.8
2004	2.6
2005	2.3
2006	2.2
2007	2.0
2008	1.8
2009	1.8
2010	1.6
2011	1.6
2012	1.5

Spreadsheet at LarsonPrecalculus.com

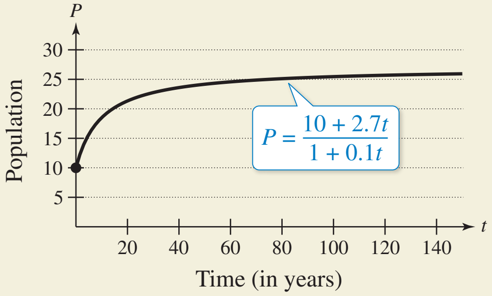


(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let $t = 3$ represent 2003. Use the graphing utility to plot the data and graph the model in the same viewing window.

(b) Find a rational model for the data. Take the reciprocal of A to generate the points $(t, 1/A)$. Use the *regression* feature of the graphing utility to find a linear model for these data. The resulting line has the form $1/A = at + b$. Solve for A. Use the graphing utility to plot the data and graph the rational model in the same viewing window.

(c) Which model do you prefer? Why?

92. **HOW DO YOU SEE IT?** A herd of elk is released onto state game lands. The graph shows the expected population P of the herd, where t is the time (in years) since the initial number of elk were released.



(a) Determine the domain of the function. Explain.

(b) Find the initial number of elk in the herd.

(c) Is there a limit to the size of the herd? If so, what is the expected population?

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Conclusions

True or False? **In Exercises 93 and 94, determine whether the statement is true or false. Justify your answer.**

93. The graph of a rational function is continuous only when the denominator is a constant polynomial.

94. The graph of a rational function can never cross one of its asymptotes.

Think About It **In Exercises 95 and 96, use a graphing utility to graph the function. Explain why there is no vertical asymptote when a superficial examination of the function might indicate that there should be one.**

95. $h(x) = \dfrac{6 - 2x}{3 - x}$

96. $g(x) = \dfrac{x^2 + x - 2}{x - 1}$

97. **Writing** Write a set of guidelines for finding all the asymptotes of a rational function given that the degree of the numerator is not more than 1 greater than the degree of the denominator.

98. **Writing a Rational Function** Write a rational function that has the specified characteristics. (There are many correct answers.)

(a) Vertical asymptote: $x = -2$

Slant asymptote: $y = x + 1$

Zero of the function: $x = 2$

(b) Vertical asymptote: $x = -4$

Slant asymptote: $y = x - 2$

Zero of the function: $x = 3$

Cumulative Mixed Review

Simplifying Exponential Expressions **In Exercises 99–102, simplify the expression.**

99. $\left(\dfrac{x}{8}\right)^{-3}$

100. $(4x^2)^{-2}$

101. $\dfrac{3^{7/6}}{3^{1/6}}$

102. $\dfrac{(x^{-2})(x^{1/2})}{(x^{-1})(x^{5/2})}$

Finding the Domain and Range of a Function **In Exercises 103–106, use a graphing utility to graph the function and find its domain and range.**

103. $f(x) = \sqrt{6 + x^2}$

104. $f(x) = \sqrt{121 - x^2}$

105. $f(x) = -|x + 9|$

106. $f(x) = -x^2 + 9$

107. ***Make a Decision*** To work an extended application analyzing the median sales prices of existing one-family homes, visit this textbook's website at *LarsonPrecalculus.com*. (Data Source: National Association of Realtors)

2.8 Quadratic Models

Classifying Scatter Plots

In real life, many relationships between two variables are parabolic, as in Section 2.1, Example 5. A scatter plot can be used to give you an idea of which type of model will best fit a set of data.

EXAMPLE 1 Classifying Scatter Plots

See LarsonPrecalculus.com for an interactive version of this type of example.

Decide whether each set of data could be better modeled by a linear model,

$$y = ax + b$$

a quadratic model,

$$y = ax^2 + bc + c$$

or neither.

a. (0.9, 1.7), (1.2, 2.0), (1.3, 1.9), (1.4, 2.1), (1.6, 2.5), (1.8, 2.8), (2.1, 3.0), (2.5, 3.4), (2.9, 3.7), (3.2, 3.9), (3.3, 4.1), (3.6, 4.4), (4.0, 4.7), (4.2, 4.8), (4.3, 5.0)

b. (0.9, 3.2), (1.2, 4.0), (1.3, 4.1), (1.4, 4.4), (1.6, 5.1), (1.8, 6.0), (2.1, 7.6), (2.5, 9.8), (2.9, 12.4), (3.2, 14.3), (3.3, 15.2), (3.6, 18.1), (4.0, 22.7), (4.2, 24.9), (4.3, 27.2)

c. (0.9, 1.2), (1.2, 6.5), (1.3, 9.3), (1.4, 11.6), (1.6, 15.2), (1.8, 16.9), (2.1, 14.7), (2.5, 8.1), (2.9, 3.7), (3.2, 5.8), (3.3, 7.1), (3.6, 11.5), (4.0, 20.2), (4.2, 23.7), (4.3, 26.9)

Solution

a. Begin by entering the data into a graphing utility. Then display the scatter plot, as shown in Figure 2.40. From the scatter plot, it appears the data follow a linear pattern. So, the data can be better modeled by a linear function.

b. Enter the data into a graphing utility and then display the scatter plot (see Figure 2.41). From the scatter plot, it appears the data follow a parabolic pattern. So, the data can be better modeled by a quadratic function.

c. Enter the data into a graphing utility and then display the scatter plot (see Figure 2.42). From the scatter plot, it appears the data do not follow either a linear or a parabolic pattern. So, the data cannot be modeled by either a linear function or a quadratic function.

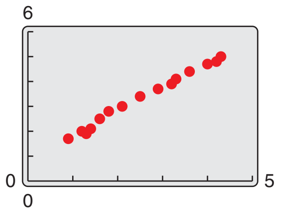


Figure 2.40

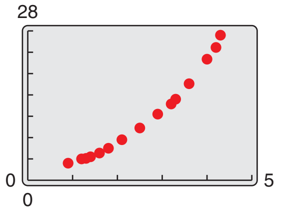


Figure 2.41

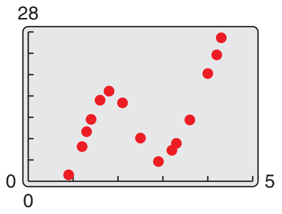


Figure 2.42

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Decide whether the data could be better modeled by a linear model, $y = ax + b$, a quadratic model, $y = ax^2 + bx + c$, or neither.

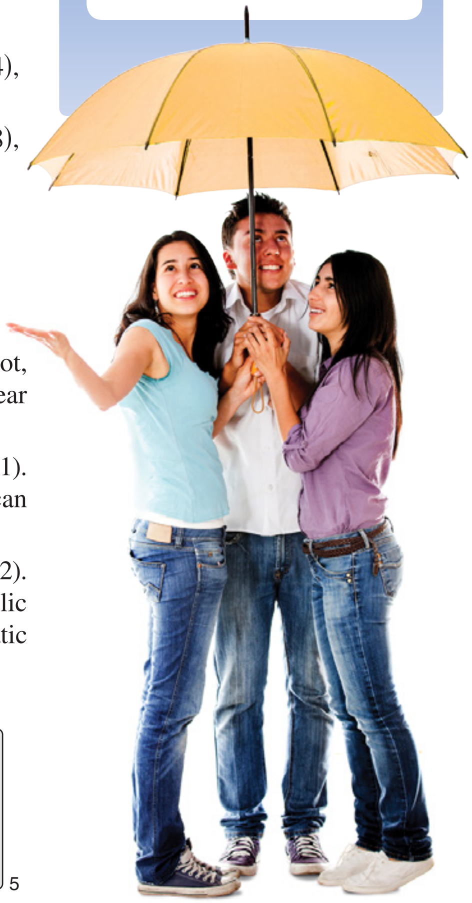
(0, 3480), (5, 2235), (10, 1250), (15, 565), (20, 150), (25, 12), (30, 145), (35, 575), (40, 1275), (45, 2225), (50, 3500), (55, 5010)

What you should learn

- Classify scatter plots.
- Use scatter plots and a graphing utility to find quadratic models for data.
- Choose a model that best fits a set of data.

Why you should learn it

Many real-life situations can be modeled by quadratic equations. For instance, in Exercise 17 on page 165, a quadratic equation is used to model the monthly precipitation for San Francisco, California.



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Fitting a Quadratic Model to Data

In Section 1.7, you created scatter plots of data and used a graphing utility to find the least squares regression lines for the data. You can use a similar procedure to find a model for nonlinear data. Once you have used a scatter plot to determine the type of model that would best fit a set of data, there are several ways that you can actually find the model. Each method is best used with a computer or calculator, rather than with hand calculations.

EXAMPLE 2 Fitting a Quadratic Model to Data

A study was done to compare the speed x (in miles per hour) with the mileage y (in miles per gallon) of an automobile. The results are shown in the table.

Speed, x	Mileage, y
15	22.3
20	25.5
25	27.5
30	29.0
35	28.7
40	29.9
45	30.4
50	30.2
55	30.0
60	28.8
65	27.4
70	25.3
75	23.3

Spreadsheet at LarsonPrecalculus.com

a. Use a graphing utility to create a scatter plot of the data.

b. Use the *regression* feature of the graphing utility to find a model that best fits the data.

c. Approximate the speed at which the mileage is the greatest.

Solution

a. Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 2.43. From the scatter plot, you can see that the data appear to follow a parabolic pattern.

b. Using the *regression* feature of the graphing utility, you can find the quadratic model, as shown in Figure 2.44. So, the quadratic equation that best fits the data is given by

$$y = -0.0082x^2 + 0.75x + 13.5. \qquad \text{Quadratic model}$$

c. Graph the data and the model in the same viewing window, as shown in Figure 2.45. Use the *maximum* feature or the *zoom* and *trace* features of the graphing utility to approximate the speed at which the mileage is greatest. You should obtain a maximum of approximately (46, 31), as shown in Figure 2.45. So, the speed at which the mileage is greatest is about 46 miles per hour.

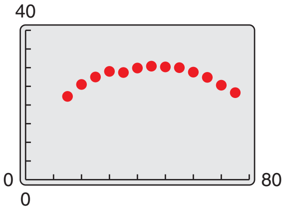
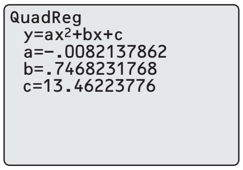
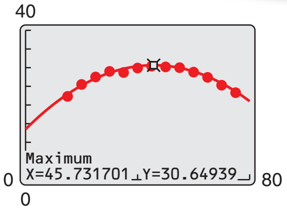


Figure 2.43



```
QuadReg
  y=ax²+bx+c
  a=-.0082137862
  b=.7468231768
  c=13.46223776
```

Figure 2.44



```
Maximum
X=45.731701  Y=30.64939
```

Figure 2.45

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The time y (in seconds) required to attain a speed of x miles per hour from a standing start for an automobile is shown in the table.

Speed, x	0	20	30	40	50	60	70	80
Time, y	0	1.4	2.6	3.8	4.9	6.3	8.0	9.9

Spreadsheet at LarsonPrecalculus.com

a. Use a graphing utility to create a scatter plot of the data.

b. Use the *regression* feature of the graphing utility to find a model that best fits the data.

c. Use the model to estimate how long it takes the automobile, from a standing start, to reach a speed of 55 miles per hour.

Supertrooper | Dreamstime

EXAMPLE 3 Fitting a Quadratic Model to Data

A basketball is dropped from a height of about 5.25 feet. The height of the basketball is recorded 23 times at intervals of about 0.02 second. The results are shown in the table. Use a graphing utility to find a model that best fits the data. Then use the model to predict the time when the basketball will hit the ground.

Time, x	Height, y
0.0	5.23594
0.02	5.20353
0.04	5.16031
0.06	5.09910
0.08	5.02707
0.099996	4.95146
0.119996	4.85062
0.139992	4.74979
0.159988	4.63096
0.179988	4.50132
0.199984	4.35728
0.219984	4.19523
0.23998	4.02958
0.25993	3.84593
0.27998	3.65507
0.299976	3.44981
0.319972	3.23375
0.339961	3.01048
0.359961	2.76921
0.379951	2.52074
0.399941	2.25786
0.419941	1.98058
0.439941	1.63488

Spreadsheet at LarsonPrecalculus.com

Solution

Begin by entering the data into a graphing utility and displaying the scatter plot, as shown in Figure 2.46. From the scatter plot, you can see that the data show a parabolic trend. So, using the *regression* feature of the graphing utility, you can find the quadratic model, as shown in Figure 2.47. The quadratic model that best fits the data is given by $y = -15.449x^2 - 1.30x + 5.2$.

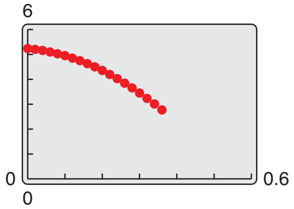
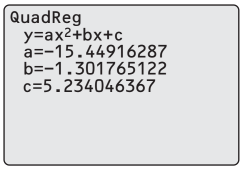


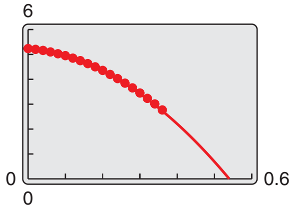
Figure 2.46



```
QuadReg
 y=ax²+bx+c
 a=-15.44916287
 b=-1.301765122
 c=5.234046367
```

Figure 2.47

You can graph the data and the model in the same viewing window to see that the model fits the data well, as shown in the next figure.



Using this model, you can predict the time when the basketball will hit the ground by substituting 0 for y and solving the resulting equation for x.

$$y = -15.449x^2 - 1.30x + 5.2$$
Write original model.

$$0 = -15.449x^2 - 1.30x + 5.2$$
Substitute 0 for y.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
Quadratic Formula

$$= \frac{-(-1.30) \pm \sqrt{(-1.30)^2 - 4(-15.449)(5.2)}}{2(-15.449)}$$
Substitute for a, b, and c.

$$\approx 0.54$$
Choose positive solution.

So, the solution is about 0.54 second. In other words, the basketball will continue to fall for about $0.54 - 0.44 = 0.1$ second more before hitting the ground.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The table shows the annual sales y (in millions of dollars) of a department store chain. Use a graphing utility to find a model that best fits the data. Then use the model to estimate the first year when the annual sales will be less than $190 million.

Year, x	1	2	3	4	5	6
Sales, y	221	222	220	219	216	211

Spreadsheet at LarsonPrecalculus.com

Choosing a Model

Sometimes it is not easy to distinguish from a scatter plot which type of model will best fit the data. You should first find several models for the data, using the *Library of Parent Functions*, and then choose the model that best fits the data by comparing the y-values of each model with the actual y-values.

EXAMPLE 4 Choosing a Model

The table shows the numbers y (in millions) of people ages 16 and older that were not in the U.S. labor force from 2003 through 2013. Use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data. Determine which model better fits the data. (Source: U.S. Department of Labor)

Year	Number of people unemployed (in millions), y
2003	74.7
2004	76.0
2005	76.8
2006	77.4
2007	78.7
2008	79.5
2009	81.7
2010	83.9
2011	86.0
2012	88.3
2013	90.3

Spreadsheet at LarsonPrecalculus.com

Solution

Let x represent the year, with $x = 3$ corresponding to 2003. Begin by entering the data into a graphing utility. Using the *regression* feature, a linear model for the data is

$$y = 1.55x + 68.8$$

and a quadratic model for the data is

$$y = 0.107x^2 - 0.16x + 74.6.$$

Plot the data and the linear model in the same viewing window, as shown in Figure 2.48. Then plot the data and the quadratic model in the same viewing window, as shown in Figure 2.49. To determine which model fits the data better, compare the y-values given by each model with the actual y-values. The model whose y-values are closest to the actual values is the better fit. In this case, the better-fitting model is the quadratic model.

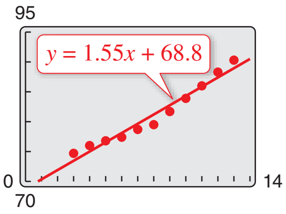


Figure 2.48

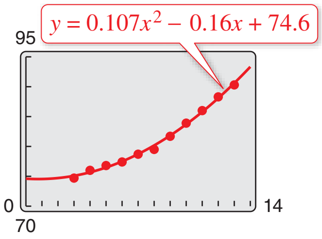


Figure 2.49

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

The table shows the numbers y (in thousands) of full-size, alternative fueled vehicles in use from 2005 through 2011. Use the *regression* feature of a graphing utility to find a linear model and a quadratic model for the data. Determine which model better fits the data. (Source: U.S. Energy Information Administration)

Year	Number of alternative fueled vehicles (in thousands), y
2005	19.2
2006	31.3
2007	44.9
2008	59.8
2009	64.2
2010	72.1
2011	81.3

Spreadsheet at LarsonPrecalculus.com

Technology Tip

When you use the *regression* feature of a graphing utility, the program may output an "r^2-value." This r^2-value is the **coefficient of determination** of the data and gives a measure of how well the model fits the data. The coefficient of determination for the linear model in Example 4 is $r^2 \approx 0.9608$, and the coefficient of determination for the quadratic model is $r^2 \approx 0.9962$. Because the coefficient of determination for the quadratic model is closer to 1, the quadratic model better fits the data.

2.8 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

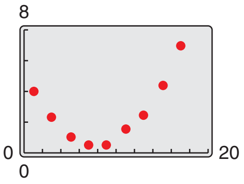
1. What type of model best represents data that follow a parabolic pattern?

2. Which coefficient of determination indicates a better model for a set of data, $r^2 = 0.0365$ or $r^2 = 0.9688$?

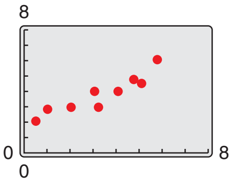
Procedures and Problem Solving

Classifying Scatter Plots **In Exercises 3–8, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, or neither.**

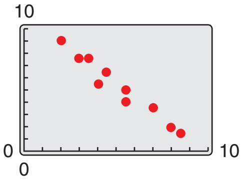
3.



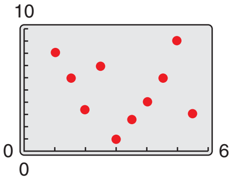
4.



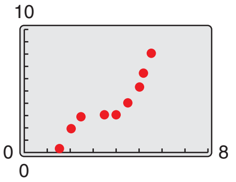
5.



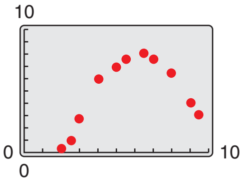
6.



7.



8.



Choosing a Model **In Exercises 9–16, (a) use a graphing utility to create a scatter plot of the data, (b) determine whether the data could be better modeled by a linear model or a quadratic model, (c) use the *regression* feature of the graphing utility to find a model for the data, (d) use the graphing utility to graph the model with the scatter plot from part (a), and (e) create a table comparing the original data with the data given by the model.**

9. (0, 2.1), (1, 2.4), (2, 2.5), (3, 2.8), (4, 2.9), (5, 3.0), (6, 3.0), (7, 3.2), (8, 3.4), (9, 3.5), (10, 3.6)

10. $(-2, 11.0)$, $(-1, 10.7)$, (0, 10.4), (1, 10.3), (2, 10.1), (3, 9.9), (4, 9.6), (5, 9.4), (6, 9.4), (7, 9.2), (8, 9.0)

11. (0, 2795), (5, 1590), (10, 650), $(15, -30)$, $(20, -450)$, $(25, -615)$, $(30, -520)$, $(35, -55)$, (40, 625), (45, 1630), (50, 2845), (55, 4350)

12. (0, 6140), (2, 6815), (4, 7335), (6, 7710), (8, 7915), (10, 7590), (12, 7975), (14, 7700), (16, 7325), (18, 6820), (20, 6125), (22, 5325)

13. (1, 4.0), (2, 6.5), (3, 8.8), (4, 10.6), (5, 13.9), (6, 15.0), (7, 17.5), (8, 20.1), (9, 24.0), (10, 27.1)

14. $(-6, 10.7)$, $(-4, 9.0)$, $(-2, 7.0)$, (0, 5.4), (2, 3.5), (4, 1.7), $(6, -0.1)$, $(8, -1.8)$, $(10, -3.6)$, $(12, -5.3)$

15. (0, 587), (5, 551), (10, 512), (15, 478), (20, 436), (25, 430), (30, 424), (35, 420), (40, 423), (45, 429), (50, 444)

16. (2, 34.3), (3, 33.8), (4, 32.6), (5, 30.1), (6, 27.8), (7, 22.5), (8, 19.1), (9, 14.8), (10, 9.4), (11, 3.7), $(12, -1.6)$

17. ***Why you should learn it*** *(p. 161)* The table shows the monthly normal precipitation P (in inches) for San Francisco, California. (Source: The Weather Channel)

Month	Precipitation, P
January	4.50
February	4.61
March	3.26
April	1.46
May	0.70
June	0.16
July	0.00
August	0.06
September	0.21
October	1.12
November	3.16
December	4.56

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the month, with $t = 1$ corresponding to January.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(c) Use the graphing utility to graph the model with the scatter plot from part (a).

(d) Use the graph from part (c) to determine in which month the normal precipitation in San Francisco is the least.

(e) Use the table to determine the month in which the normal precipitation in San Francisco is the least. Compare your answer with that of part (d).

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18. MODELING DATA

The table shows the annual sales S (in billions of dollars) of pharmacies and drug stores in the United States from 2007 through 2012. (Source: U.S. Census Bureau)

Year	Sales, S (in billions of dollars)
2007	202.3
2008	211.0
2009	217.6
2010	222.8
2011	231.5
2012	233.4

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 2007.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model with the scatter plot from part (a).

(d) Use the model to estimate the first year when the annual sales of pharmacies and drug stores will be less than \$200 billion. Is this a good model for predicting future sales? Explain.

19. MODELING DATA

The table shows the percents P of U.S. households with Internet access from 2007 through 2012. (Source: U.S. Census Bureau)

Year	Percent, P
2007	54.7
2008	61.7
2009	68.7
2010	71.1
2011	71.7
2012	74.8

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 2007.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the model with the scatter plot from part (a).

(d) According to the model, in what year will the percent of U.S. households with Internet access be less than 60%? Is this a good model for making future predictions? Explain.

20. MODELING DATA

The table shows the estimated average numbers of hours H that adults in the United States spent reading newspapers each year from 2003 through 2012. (Source: Statista)

Year	Hours, H
2003	198
2004	195
2005	191
2006	182
2007	176
2008	169
2009	158
2010	155
2011	152
2012	150

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 3$ corresponding to 2003.

(b) A cubic model for the data is

$$H = 0.131t^3 - 2.81t^2 + 12.1t + 183$$

which has an r^2-value of 0.9962. Use the graphing utility to graph the model with the scatter plot from part (a). Is the cubic model a good fit for the data? Explain.

(c) Use the *regression* feature of the graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(d) Use the graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data? Explain.

(e) Which model is a better fit for the data? Explain.

(f) A consumer research company makes projections about the average numbers of hours H^* that adults spent reading newspapers each year from 2013 through 2015. The company's projections are shown in the table. Use the models from parts (b) and (c) to *predict* the average numbers of hours for 2013 through 2015. Explain why your values may differ from those in the table.

Year	2013	2014	2015
H^*	145	143	139

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21. MODELING DATA

The table shows the numbers of U.S. households with televisions (in millions) from 2000 through 2012. (Source: The Nielsen Company)

Year	Televisions, *T* (in millions)
2000	100.8
2001	102.2
2002	105.5
2003	106.7
2004	108.4
2005	109.6
2006	110.2
2007	111.4
2008	112.8
2009	114.5
2010	114.9
2011	115.9
2012	114.7

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 0$ corresponding to 2000.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data and identify the coefficient of determination.

(c) Use the graphing utility to graph the linear model with the scatter plot from part (a).

(d) Use the *regression* feature of the graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(e) Use the graphing utility to graph the quadratic model with the scatter plot from part (a).

(f) Which model is a better fit for the data? Explain.

(g) Use each model to approximate the year when the number of households with televisions will reach 120 million, if possible.

Conclusions

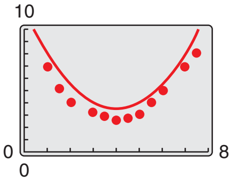
True or False? **In Exercises 22–24, determine whether the statement is true or false. Justify your answer.**

22. The graph of a quadratic model with a negative leading coefficient will have a maximum value at its vertex.

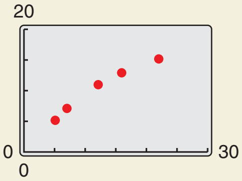
23. The graph of a quadratic model with a positive leading coefficient will have a minimum value at its vertex.

24. Data that are positively correlated are always better modeled by a linear equation than by a quadratic equation.

25. **Writing** Explain why the parabola shown in the figure is not a good fit for the data.



26. **HOW DO YOU SEE IT?** The r^2-values representing the coefficients of determination for the least squares linear model and the least squares quadratic model for the data shown are given below. Which is which? Explain your reasoning.



$r^2 \approx 0.9995$

$r^2 \approx 0.9782$

Cumulative Mixed Review

Compositions of Functions **In Exercises 27–30, find (a) $f \circ g$ and (b) $g \circ f$.**

27. $f(x) = 2x - 1,\ g(x) = x^2 + 3$

28. $f(x) = 5x + 8,\ g(x) = 2x^2 - 1$

29. $f(x) = x^3 - 1,\ g(x) = \sqrt[3]{x + 1}$

30. $f(x) = \sqrt[3]{x + 5},\ g(x) = x^3 - 5$

Testing Whether a Function is One-to-One **In Exercises 31–34, determine algebraically whether the function is one-to-one. If it is, find its inverse function. Verify your answer graphically.**

31. $f(x) = 2x + 5$

32. $f(x) = \dfrac{x - 4}{5}$

33. $f(x) = x^2 + 5,\ x \geq 0$

34. $f(x) = 2x^2 - 3,\ x \geq 0$

Multiplying Complex Conjugates **In Exercises 35–38, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.**

35. $1 - 3i$

36. $-2 + 4i$

37. $-5i$

38. $8i$

2 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
2.1	**Analyze graphs of quadratic functions *(p. 90)*.**	Let a, b, and c be real numbers with $a \neq 0$. The function $f(x) = ax^2 + bx + c$ is called a quadratic function. Its graph is a "U-shaped" curve called a parabola.	1–6
	Write quadratic functions in standard form and use the results to sketch graphs of functions *(p. 93)*.	The quadratic function $f(x) = a(x - h)^2 + k$, $a \neq 0$, is in standard form. The graph of f is a parabola whose axis is the vertical line $x = h$ and whose vertex is the point (h, k). The parabola opens upward when $a > 0$ and opens downward when $a < 0$.	7–12
	Find minimum and maximum values of quadratic functions in real-life applications *(p. 95)*.	Consider $f(x) = ax^2 + bx + c$ with vertex $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$. If $a > 0$, then f has a *minimum* at $x = -b/(2a)$. If $a < 0$, then f has a *maximum* at $x = -b/(2a)$.	13, 14
2.2	**Use transformations to sketch graphs of polynomial functions *(p. 100)*.**	The graph of a polynomial function is continuous (no breaks, holes, or gaps) and has only smooth, rounded turns.	15–20
	Use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions *(p. 102)*.	Consider $f(x) = a_nx^n + \cdot \cdot \cdot + a_1x + a_0$, $a_n \neq 0$. ***n* is odd:** If $a_n > 0$, then the graph falls to the left and rises to the right. If $a_n < 0$, then the graph rises to the left and falls to the right. ***n* is even:** If $a_n > 0$, then the graph rises to the left and right. If $a_n < 0$, then the graph falls to the left and right.	21–26
	Find and use zeros of polynomial functions as sketching aids *(p. 104)*.	If f is a polynomial function and a is a real number, the following are equivalent: (1) $x = a$ is a *zero* of the function f, (2) $x = a$ is a *solution* of the polynomial equation $f(x) = 0$, (3) $(x - a)$ is a *factor* of the polynomial $f(x)$, and (4) $(a, 0)$ is an *x-intercept* of the graph of f.	27–38
	Use the Intermediate Value Theorem to help locate zeros of polynomial functions *(p. 108)*.	Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then, in $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.	39–42
2.3	**Use long division to divide polynomials by other polynomials *(p. 113)*.**	$\dfrac{\overset{\text{Dividend}}{x^2 + 3x + 5}}{\underset{\text{Divisor}}{x + 1}} = \overbrace{x + 2}^{\text{Quotient}} + \dfrac{\overset{\text{Remainder}}{3}}{\underset{\text{Divisor}}{x + 1}}$	43–50
	Use synthetic division to divide polynomials by binomials of the form *x* − *k* *(p. 116)*.	Divisor: $x + 3$ Dividend: $x^4 - 10x^2 - 2x + 4$ $\begin{array}{r\|rrrrr} -3 & 1 & 0 & -10 & -2 & 4 \\ & & -3 & 9 & 3 & -3 \\ \hline & 1 & -3 & -1 & 1 & \boxed{1} \end{array}$ ← Remainder: 1 Quotient: $x^3 - 3x^2 - x + 1$	51–56
	Use the Remainder Theorem and the Factor Theorem *(p. 117)*.	**The Remainder Theorem:** If a polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$. **The Factor Theorem:** A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.	57–62
	Use the Rational Zero Test to determine possible rational zeros of polynomial functions *(p. 119)*.	The Rational Zero Test relates the possible rational zeros of a polynomial to the leading coefficient and to the constant term of the polynomial.	63, 64

	What did you learn?	Explanation and Examples	Review Exercises
2.3	**Use Descartes's Rule of Signs *(p. 121)* and the Upper and Lower Bound Rules *(p. 122)* to find zeros of polynomials.**	Example 9 shows how to use Descartes's Rule of Signs. Example 10 uses Descartes's Rule of Signs and the Upper and Lower Bound Rules.	65–72
2.4	**Use the imaginary unit *i* to write complex numbers *(p. 128)*.**	The imaginary unit i is defined as $i = \sqrt{-1}$, where $i^2 = -1$. If a and b are real numbers, then the number $a + bi$ is a complex number, and it is written in standard form.	73–76
	Add, subtract, and multiply complex numbers *(p. 129)*.	**Sum:** $(a + bi) + (c + di) = (a + c) + (b + d)i$ **Difference:** $(a + bi) - (c + di) = (a - c) + (b - d)i$	77–88
	Use complex conjugates to write the quotient of two complex numbers in standard form *(p. 131)*.	Complex numbers of the forms $a + bi$ and $a - bi$ are complex conjugates. To write $(a + bi)/(c + di)$ in standard form, multiply by $(c - di)/(c - di)$.	89–92
	Find complex solutions of quadratic equations *(p. 132)*.	If a is a positive number, then the principal square root of the negative number $-a$ is defined as $\sqrt{-a} = \sqrt{a}i$.	93–98
2.5	**Use the Fundamental Theorem of Algebra to determine the number of zeros of a polynomial function *(p. 135)*.**	**The Fundamental Theorem of Algebra** If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has at least one zero in the complex number system.	99–102
	Find all zeros of polynomial functions *(p. 136)* and find conjugate pairs of complex zeros *(p. 137)*.	**Complex Zeros Occur in Conjugate Pairs** Let f be a polynomial function that has real coefficients. If $a + bi$ $(b \neq 0)$ is a zero of the function, the conjugate $a - bi$ is also a zero of the function.	103–120
	Find zeros of polynomials by factoring *(p. 138)*.	Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.	121–124
2.6	**Find the domains *(p. 142)* and vertical and horizontal asymptotes *(p. 143)* of rational functions.**	The domain of a rational function of x includes all real numbers except x-values that make the denominator zero.	125–136
	Use rational functions to model and solve real-life problems *(p. 146)*.	A rational function can be used to model the cost of removing a given percent of smokestack pollutants at a utility company that burns coal. (See Example 5.)	137, 138
2.7	**Analyze and sketch graphs of rational functions *(p. 151)*, including functions with slant asymptotes *(p. 155)*.**	Consider a rational function whose denominator is of degree 1 or greater. If the degree of the numerator is exactly *one more* than the degree of the denominator, then the graph of the function has a slant asymptote.	139–152
	Use rational functions to model and solve real-life problems *(p. 156)*.	A rational function can be used to model the area of a page. The model can be used to determine the dimensions of the page that use the minimum amount of paper. (See Example 7.)	153, 154
2.8	**Classify scatter plots *(p. 161)*, find quadratic models for data *(p. 162)*, and choose a model that best fits a set of data *(p. 164)*.**	Sometimes it is not easy to distinguish from a scatter plot which type of model will best fit the data. You should first find several models for the data and then choose the model that best fits the data by comparing the y-values of each model with the actual y-values.	155–160

2 Review Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

2.1

Library of Parent Functions **In Exercises 1–6, sketch the graph of each function and compare it with the graph of $y = x^2$.**

1. $y = x^2 + 1$	**2.** $y = -x^2 + 5$
3. $y = (x - 6)^2$	**4.** $y = -(x + 1)^2$
5. $y = -(x + 5)^2 - 1$	**6.** $y = -(x - 2)^2 + 2$

Identifying the Vertex of a Quadratic Function **In Exercises 7–10, describe the graph of the function and identify the vertex. Then, sketch the graph of the function. Identify any x-intercepts.**

7. $f(x) = \left(x + \frac{3}{2}\right)^2 + 1$

8. $f(x) = (x - 4)^2 - 4$

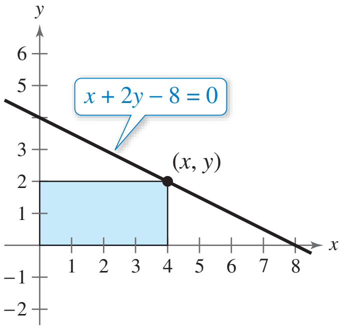
9. $f(x) = \frac{1}{3}(x^2 + 5x - 4)$

10. $f(x) = 3x^2 - 12x + 11$

Writing the Equation of a Parabola in Standard Form **In Exercises 11 and 12, write the standard form of the quadratic function that has the indicated vertex and whose graph passes through the given point. Use a graphing utility to verify your result.**

11. Vertex: $(-3, 4)$	Point: $(0, -5)$
12. Vertex: $(2, -1)$	Point: $(5, 2)$

13. **Geometry** A rectangle is inscribed in the region bounded by the x-axis, the y-axis, and the graph of $x + 2y - 8 = 0$, as shown in the figure.

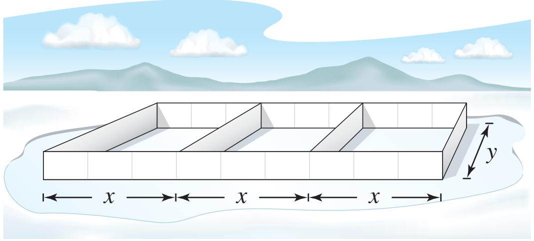


(a) Write the area A of the rectangle as a function of x. Determine the domain of the function in the context of the problem.

(b) Use a graphing utility to graph the area function. Use the graph to approximate the dimensions that will produce a maximum area.

(c) Write the area function in standard form to find algebraically the dimensions that will produce a maximum area. Compare your results with your answer from part (b).

14. **Physical Education** A college has 1500 feet of portable rink boards to form three adjacent outdoor ice rinks, as shown in the figure. Determine the dimensions that will produce the maximum total area of ice surface.



2.2

Library of Parent Functions **In Exercises 15–20, sketch the graph of $f(x) = x^3$ and the graph of the function g. Describe the transformation from f to g.**

15. $g(x) = (x - 5)^3$	**16.** $g(x) = x^3 - 2$
17. $g(x) = -(x + 4)^3$	**18.** $g(x) = (x + 4)^3 - 1$
19. $g(x) = -(x - 2)^3 + 6$	**20.** $g(x) = -(x - 1)^3 - 5$

Comparing End Behavior **In Exercises 21 and 22, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out far enough to see the right-hand and left-hand behavior of each graph. Do the graphs of f and g have the same right-hand and left-hand behavior? Explain why or why not.**

21. $f(x) = \frac{1}{2}x^3 - 2x + 1, \quad g(x) = \frac{1}{2}x^3$

22. $f(x) = -x^4 + 2x^3, \quad g(x) = -x^4$

Applying the Leading Coefficient Test **In Exercises 23–26, use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of the polynomial function.**

23. $f(x) = x^2 - x - 6$

24. $g(x) = -3x^3 + 4x^2 - 1$

25. $h(x) = -\frac{2}{3}(x^4 - 7x^3 + 5)$

26. $f(x) = -x^2 + 4x^5 + 5 - 2x^4$

Finding Zeros of a Polynomial Function **In Exercises 27–32, (a) find the zeros algebraically, (b) use a graphing utility to graph the function, and (c) use the graph to approximate any zeros and compare them with those in part (a).**

27. $g(x) = x^4 - x^3 - 2x^2$	**28.** $h(x) = -2x^3 - x^2 + x$
29. $f(t) = t^3 - 3t$	**30.** $f(x) = -(x + 6)^3 - 8$
31. $f(x) = x(x + 3)^2$	**32.** $f(t) = t^4 - 4t^2$

Finding a Polynomial Function with Given Zeros **In Exercises 33–36, find a polynomial function that has the given zeros. (There are many correct answers.)**

33. $4, -2, -2$

34. $-1, 0, 3, 5$

35. $3, 2 - \sqrt{3}, 2 + \sqrt{3}$

36. $-7, 4 - \sqrt{6}, 4 + \sqrt{6}$

Sketching the Graph of a Polynomial Function **In Exercises 37 and 38, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.**

37. $f(x) = x^4 - 2x^3 - 12x^2 + 18x + 27$

38. $f(x) = 18 + 27x - 2x^2 - 3x^3$

Approximating the Zeros of a Function **In Exercises 39–42, (a) use the Intermediate Value Theorem and a graphing utility to find graphically any intervals of length 1 in which the polynomial function is guaranteed to have a zero, and (b) use the *zero* or *root* feature of the graphing utility to approximate the real zeros of the function. Verify your results in part (a) by using the *table* feature of the graphing utility.**

39. $f(x) = x^3 + 2x^2 - x - 1$

40. $f(x) = x^4 - 6x^2 - 4$

41. $f(x) = 0.24x^3 - 2.6x - 1.4$

42. $f(x) = 2x^4 + \frac{7}{2}x^3 - 2$

2.3

Long Division of Polynomials **In Exercises 43–50, use long division to divide.**

43. $\dfrac{24x^2 - x - 8}{3x - 2}$

44. $\dfrac{4x^2 + 7}{3x - 2}$

45. $\dfrac{x^4 - 3x^2 + 2}{x^2 - 1}$

46. $\dfrac{3x^4 + x^2 - 1}{x^2 - 1}$

47. $(5x^3 - 13x^2 - x + 2) \div (x^2 - 3x + 1)$

48. $(x^4 + x^3 - x^2 + 2x) \div (x^2 + 2x)$

49. $\dfrac{6x^4 + 10x^3 + 13x^2 - 5x + 2}{2x^2 - 1}$

50. $\dfrac{x^4 - 3x^3 + 4x^2 - 6x + 3}{x^2 + 2}$

Using Synthetic Division **In Exercises 51–56, use synthetic division to divide.**

51. $(3x^3 - 10x^2 + 12x - 22) \div (x - 4)$

52. $(2x^3 + 6x^2 - 14x + 9) \div (x - 1)$

53. $(0.25x^4 - 4x^3) \div (x + 2)$

54. $(0.1x^3 + 0.3x^2 - 0.5) \div (x - 5)$

55. $(6x^4 - 4x^3 - 27x^2 + 18x) \div \left(x - \frac{2}{3}\right)$

56. $(2x^3 + 2x^2 - x + 2) \div \left(x - \frac{1}{2}\right)$

Using the Remainder Theorem **In Exercises 57 and 58, use the Remainder Theorem and synthetic division to evaluate the function at each given value. Use a graphing utility to verify your results.**

57. $f(x) = x^4 + 10x^3 - 24x^2 + 20x + 44$

(a) $f(-3)$ (b) $f(-2)$

58. $g(t) = 2t^5 - 5t^4 - 8t + 20$

(a) $g(-4)$ (b) $g\left(\sqrt{2}\right)$

Factoring a Polynomial **In Exercises 59–62, (a) verify the given factor(s) of the function *f*, (b) find the remaining factors of *f*, (c) use your results to write the complete factorization of *f*, and (d) list all real zeros of *f*. Confirm your results by using a graphing utility to graph the function.**

	Function	*Factor(s)*
59.	$f(x) = x^3 + 4x^2 - 25x - 28$	$(x - 4)$
60.	$f(x) = 2x^3 + 11x^2 - 21x - 90$	$(x + 6)$
61.	$f(x) = x^4 - 4x^3 - 7x^2 + 22x + 24$	$(x + 2),$ $(x - 3)$
62.	$f(x) = x^4 - 11x^3 + 41x^2 - 61x + 30$	$(x - 2),$ $(x - 5)$

Using the Rational Zero Test **In Exercises 63 and 64, use the Rational Zero Test to list all possible rational zeros of *f*. Use a graphing utility to verify that all the zeros of *f* are contained in the list.**

63. $f(x) = 4x^3 - 11x^2 + 10x - 3$

64. $f(x) = 10x^3 + 21x^2 - x - 6$

Using Descartes's Rule of Signs **In Exercises 65 and 66, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.**

65. $g(x) = 5x^3 - 6x + 9$

66. $f(x) = 2x^5 - 3x^2 + 2x - 1$

Finding the Zeros of a Polynomial Function **In Exercises 67 and 68, use synthetic division to verify the upper and lower bounds of the real zeros of *f*. Then find all real zeros of the function.**

67. $f(x) = 4x^3 - 3x^2 + 4x - 3$

Upper bound: $x = 1$; Lower bound: $x = -\frac{1}{4}$

68. $f(x) = 2x^3 - 5x^2 - 14x + 8$

Upper bound: $x = 8$; Lower bound: $x = -4$

Finding the Zeros of a Polynomial Function **In Exercises 69–72, find all real zeros of the polynomial function.**

69. $f(x) = x^3 - 4x^2 + x + 6$

70. $f(x) = x^3 + x^2 - 28x - 10$

71. $f(x) = 6x^4 - 25x^3 + 14x^2 + 27x - 18$

72. $f(x) = 5x^4 + 126x^2 + 25$

2.4

Writing a Complex Number in Standard Form **In Exercises 73–76, write the complex number in standard form.**

73. $3 - \sqrt{-16}$

74. $\sqrt{-50} + 8$

75. $-i + 4i^2$

76. $7i - 9i^2$

Operations with Complex Numbers **In Exercises 77–88, perform the operation(s) and write the result in standard form.**

77. $(2 + 13i) + (6 - 5i)$

78. $\left(\frac{1}{2} + \frac{\sqrt{3}}{4}i\right) - \left(\frac{1}{2} - \frac{\sqrt{3}}{4}i\right)$

79. $5i(13 - 8i)$

80. $(1 + 6i)(5 - 2i)$

81. $(10 - 8i)(2 - 3i)$

82. $i(6 + i)(3 - 2i)$

83. $(3 + 7i)^2 + (3 - 7i)^2$

84. $(4 - i)^2 - (4 + i)^2$

85. $\left(\sqrt{-16} + 3\right)\left(\sqrt{-25} - 2\right)$

86. $\left(5 - \sqrt{-4}\right)\left(5 + \sqrt{-4}\right)$

87. $\sqrt{-9} + 3 + \sqrt{-36}$

88. $7 - \sqrt{-81} + \sqrt{-49}$

Writing a Quotient of Complex Numbers in Standard Form **In Exercises 89–92, write the quotient in standard form.**

89. $\dfrac{6 + i}{i}$

90. $\dfrac{4}{-3i}$

91. $\dfrac{3 + 2i}{5 + i}$

92. $\dfrac{1 - 7i}{2 + 3i}$

Complex Solutions of a Quadratic Equation **In Exercises 93–98, solve the quadratic equation.**

93. $x^2 + 16 = 0$

94. $x^2 + 48 = 0$

95. $x^2 + 3x + 6 = 0$

96. $x^2 + 4x + 8 = 0$

97. $3x^2 - 5x + 6 = 0$

98. $5x^2 - 2x + 4 = 0$

2.5

Zeros of a Polynomial Function **In Exercises 99–102, confirm that the function has the indicated zero(s).**

99. $f(x) = x^2 + 6x + 9$; Repeated zero: -3

100. $f(x) = x^2 - 10x + 25$; Repeated zero: 5

101. $f(x) = x^3 + 16x$; $0, -4i, 4i$

102. $f(x) = x^3 + 144x$; $0, -12i, 12i$

Using the Factored Form of a Function **In Exercises 103 and 104, find all the zeros of the function.**

103. $f(x) = -4x(x + 3)$

104. $f(x) = (x - 8)^3(x + 2)$

Finding the Zeros of a Polynomial Function **In Exercises 105–110, find all the zeros of the function and write the polynomial as a product of linear factors. Verify your results by using a graphing utility to graph the function.**

105. $h(x) = x^3 - 7x^2 + 18x - 24$

106. $f(x) = 2x^3 - 5x^2 - 9x + 40$

107. $f(x) = 2x^4 - 5x^3 + 10x - 12$

108. $g(x) = 3x^4 - 4x^3 + 7x^2 + 10x - 4$

109. $f(x) = x^5 + x^4 + 5x^3 + 5x^2$

110. $f(x) = x^5 - 5x^3 + 4x$

Using the Zeros to Find the *x*-Intercepts **In Exercises 111–116, (a) find all the zeros of the function, (b) write the polynomial as a product of linear factors, and (c) use your factorization to determine the *x*-intercepts of the graph of the function. Use a graphing utility to verify that the real zeros are the only *x*-intercepts.**

111. $f(x) = x^3 - 4x^2 + 6x - 4$

112. $f(x) = x^3 - 5x^2 - 7x + 51$

113. $f(x) = -3x^3 - 19x^2 - 4x + 12$

114. $f(x) = 2x^3 - 9x^2 + 22x - 30$

115. $f(x) = x^4 + 34x^2 + 225$

116. $f(x) = x^4 + 10x^3 + 26x^2 + 10x + 25$

Finding a Polynomial with Given Zeros **In Exercises 117–120, find a polynomial function with real coefficients that has the given zeros. (There are many correct answers.)**

117. $5, 3i$

118. $-6, -i$

119. $-2, -2 - 4i$

120. $1, -5 + \sqrt{2}i$

Factoring a Polynomial **In Exercises 121 and 122, write the polynomial (a) as the product of factors that are irreducible over the *rationals,* (b) as the product of linear and quadratic factors that are irreducible over the *reals,* and (c) in completely factored form.**

121. $f(x) = x^4 - 2x^3 + 8x^2 - 18x - 9$

(*Hint:* One factor is $x^2 + 9$.)

122. $f(x) = x^4 - 4x^3 + 3x^2 + 8x - 16$

(*Hint:* One factor is $x^2 - x - 4$.)

Finding the Zeros of a Polynomial Function **In Exercises 123 and 124, use the given zero to find all the zeros of the function.**

	Function	*Zero*
123.	$f(x) = x^3 + 3x^2 + 4x + 12$	$-2i$
124.	$f(x) = 2x^3 - 7x^2 + 14x + 9$	$2 + \sqrt{5}i$

2.6

Finding a Function's Domain and Asymptotes **In Exercises 125–136, (a) find the domain of the function, (b) decide whether the function is continuous, and (c) identify any horizontal and vertical asymptotes.**

125. $f(x) = \dfrac{2 - x}{x + 3}$

126. $f(x) = \dfrac{4x}{x - 8}$

127. $f(x) = \dfrac{2}{x^2 - 3x - 18}$

128. $f(x) = \dfrac{2x^2 + 3}{x^2 + x + 3}$

129. $f(x) = \dfrac{7 + x}{7 - x}$

130. $f(x) = \dfrac{6x}{x^2 - 1}$

131. $f(x) = \dfrac{4x^2}{2x^2 - 3}$

132. $f(x) = \dfrac{3x^2 - 11x - 4}{x^2 + 2}$

133. $f(x) = \dfrac{2x - 10}{x^2 - 2x - 15}$

134. $f(x) = \dfrac{4 - x}{x^3 + 6x^2}$

135. $f(x) = \dfrac{3x^2 - 15}{x^3 - 5x^2 - 24x}$

136. $f(x) = \dfrac{x^2 + 3x + 2}{x^3 - 4x^2}$

137. **Criminology** The cost C (in millions of dollars) for the U.S. government to seize $p\%$ of an illegal drug as it enters the country is given by

$$C = \frac{528p}{100 - p}, \quad 0 \le p < 100.$$

(a) Find the costs of seizing 25%, 50%, and 75% of the illegal drug.

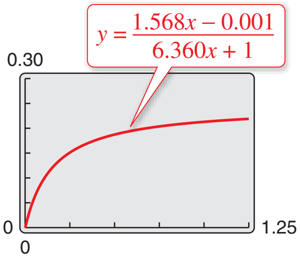
(b) Use a graphing utility to graph the function. Be sure to choose an appropriate viewing window. Explain why you chose the values you used in your viewing window.

(c) According to this model, would it be possible to seize 100% of the drug? Explain.

138. **Biology** A biology class performs an experiment comparing the quantity of food consumed by a certain kind of moth with the quantity supplied. The model for the experimental data is given by

$$y = \frac{1.568x - 0.001}{6.360x + 1}, \quad x > 0$$

where x is the quantity (in milligrams) of food supplied and y is the quantity (in milligrams) eaten (see figure). At what level of consumption will the moth become satiated?



2.7

Finding Asymptotes and Holes **In Exercises 139–142, find all of the vertical, horizontal, and slant asymptotes, and any holes in the graph of the function. Then use a graphing utility to verify your results.**

139. $f(x) = \dfrac{x^2 - 5x + 4}{x^2 - 1}$

140. $f(x) = \dfrac{2x^2 - 7x + 3}{2x^2 - 3x - 9}$

141. $f(x) = \dfrac{3x^2 + 5x - 2}{x + 1}$

142. $f(x) = \dfrac{2x^2 + 5x + 3}{x - 2}$

Sketching the Graph of a Rational Function **In Exercises 143–152, sketch the graph of the rational function by hand. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, slant asymptotes, and holes.**

143. $f(x) = \dfrac{1}{x} + 3$

144. $f(x) = \dfrac{-1}{x + 2}$

145. $f(x) = \dfrac{2x - 1}{x - 5}$

146. $f(x) = \dfrac{x - 3}{x - 2}$

147. $f(x) = \dfrac{2}{(x + 1)^2}$

148. $f(x) = \dfrac{4}{(x - 1)^2}$

149. $f(x) = \dfrac{2x^2}{x^2 - 4}$

150. $f(x) = \dfrac{5x}{x^2 + 1}$

151. $f(x) = \dfrac{x^2 - x + 1}{x - 3}$

152. $f(x) = \dfrac{2x^2 + 7x + 3}{x + 1}$

153. **Biology** A parks and wildlife commission releases 80,000 fish into a lake. After t years, the population N of the fish (in thousands) is given by

$$N = \frac{20(4 + 3t)}{1 + 0.05t}, \quad t \ge 0.$$

(a) Use a graphing utility to graph the function and find the populations when $t = 5$, $t = 10$, and $t = 25$.

(b) What is the maximum number of fish in the lake as time passes? Explain your reasoning.

154. **Publishing** A page that is x inches wide and y inches high contains 30 square inches of print. The top and bottom margins are 2 inches deep and the margins on each side are 2 inches wide.

(a) Draw a diagram that illustrates the problem.

(b) Show that the total area A of the page is given by

$$A = \frac{2x(2x + 7)}{x - 4}.$$

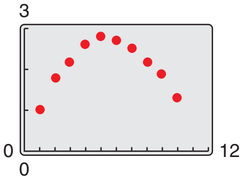
(c) Determine the domain of the function based on the physical constraints of the problem.

(d) Use a graphing utility to graph the area function and approximate the page size such that the minimum amount of paper will be used.

2.8

Classifying Scatter Plots In Exercises 155–158, determine whether the scatter plot could best be modeled by a linear model, a quadratic model, or neither.

155.



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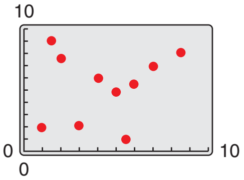

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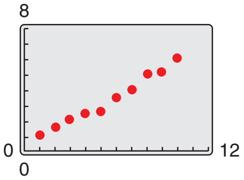

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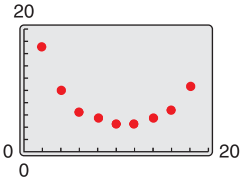

159. MODELING DATA

The table shows the numbers of commercial FM radio stations S in the United States from 2004 through 2013. (Source: Federal Communications Commission)

Year	FM Stations, S
2004	6218
2005	6231
2006	6266
2007	6309
2008	6427
2009	6479
2010	6526
2011	6542
2012	6598
2013	6612

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 4$ corresponding to 2004.

(b) A cubic model for the data is

$$S = -1.088t^3 + 26.61t^2 - 150.9t + 6459$$

Use the graphing utility to graph this model with the scatter plot from part (a).

(c) Use the *regression* feature of the graphing utility to find a quadratic model for the data and identify the coefficient of determination.

(d) Use the graphing utility to graph the quadratic model with the scatter plot from part (a).

(e) Which model is a better fit for the data? Explain.

(f) Use the model you chose in part (e) to predict the number of commercial FM radio stations in 2015.

160. MODELING DATA

The table shows the sales S (in billions of dollars) of Office Depot for each of the years from 2007 through 2013. (Source: Office Depot, Inc.)

Year	Sales, S (in billions of dollars)
2007	15.528
2008	14.496
2009	12.144
2010	11.633
2011	11.489
2012	10.696
2013	11.242

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 2007.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data?

(d) According to the model, what is the first year when Office Depot will have sales greater than $14 billion?

(e) Is this a good model for predicting the sales of Office Depot in future years? Explain.

Conclusions

True or False? In Exercises 161–163, determine whether the statement is true or false. Justify your answer.

161. The graph of $f(x) = \dfrac{2x^3}{x+1}$ has a slant asymptote.

162. A fourth-degree polynomial with real coefficients can have -5, $-8i$, $4i$, and 5 as its zeros.

163. The sum of two complex numbers cannot be a real number.

164. Think About It Describe the domain restrictions of a rational function when the denominator divides evenly into the numerator.

165. Writing Write a paragraph discussing whether every rational function has a vertical asymptote.

166. Error Analysis Describe the error.

~~$\sqrt{-8}\sqrt{-8} = \sqrt{(-8)(-8)} = \sqrt{64} = 8$~~

167. Error Analysis Describe the error.

~~$-i(\sqrt{-4} - 1) = -i(4i - 1) = -4i^2 + i = 4 + i$~~

168. Finding a Power of *i* Write each of the powers of i as i, $-i$, 1, or -1.

(a) i^{40} (b) i^{25} (c) i^{50} (d) i^{67}

2 Chapter Test

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

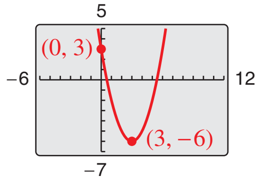

Figure for 2

1. Identify the vertex and intercepts of the graph of $y = x^2 + 5x + 6$.

2. Write an equation in standard form of the parabola shown at the right.

3. Find all the real zeros of $f(x) = 4x^3 - 12x^2 + 9x$. Determine the multiplicity of each zero.

4. Sketch the graph of the function $f(x) = -x^3 + 7x + 6$.

5. Divide using long division: $(2x^3 + 5x - 3) \div (x^2 + 2)$.

6. Divide using synthetic division: $(2x^4 - 5x^2 - 3) \div (x - 2)$.

7. Use synthetic division to evaluate $f(-2)$ for $f(x) = 3x^4 - 6x^2 + 5x - 1$.

In Exercises 8 and 9, list all the possible rational zeros of the function. Use a graphing utility to graph the function and find all the rational zeros.

8. $g(t) = 2t^4 - 3t^3 + 16t - 24$

9. $h(x) = 3x^5 + 2x^4 - 3x - 2$

10. Find all the zeros of the function $f(x) = x^3 - 7x^2 + 11x + 19$ and write the polynomial as a product of linear factors.

In Exercises 11–14, perform the operation(s) and write the result in standard form.

11. $(-8 - 3i) + (-1 - 15i)$

12. $\left(10 + \sqrt{-20}\right) - \left(4 - \sqrt{-14}\right)$

13. $(2 + i)(6 - i)$

14. $(4 + 3i)^2 - (5 + i)^2$

In Exercises 15–17, write the quotient in standard form.

15. $\dfrac{8 + 5i}{i}$

16. $\dfrac{5i}{2 - i}$

17. $(2i - 1) \div (3i + 2)$

In Exercises 18 and 19, solve the quadratic equation.

18. $x^2 + 54 = 0$

19. $x^2 - 2x + 8 = 0$

In Exercises 20–22, sketch the graph of the rational function. As sketching aids, check for intercepts, vertical asymptotes, horizontal asymptotes, and slant asymptotes.

20. $h(x) = \dfrac{4}{x^2} - 1$

21. $g(x) = \dfrac{x^2 + 2}{x - 1}$

22. $f(x) = \dfrac{2x^2 + 9}{5x^2 + 2}$

23. The table shows the amounts A (in billions of dollars) spent on national defense by the United States for the years 2007 through 2013. (Source: U.S. Office of Management and Budget)

(a) Use a graphing utility to create a scatter plot of the data. Let t represent the year, with $t = 7$ corresponding to 2007.

(b) Use the *regression* feature of the graphing utility to find a quadratic model for the data.

(c) Use the graphing utility to graph the quadratic model with the scatter plot from part (a). Is the quadratic model a good fit for the data?

(d) Use the model to estimate the amounts spent on national defense in 2015 and 2020.

(e) Do you believe the model is useful for predicting the amounts spent on national defense for years beyond 2013? Explain.

DATA

Spreadsheet at LarsonPrecalculus.com

Year	National defense, A (in billions of dollars)
2007	551.3
2008	616.1
2009	661.0
2010	693.5
2011	705.6
2012	677.9
2013	633.4

Table for 23

Proofs in Mathematics

These two pages contain proofs of four important theorems about polynomial functions. The first two theorems are from Section 2.3, and the second two theorems are from Section 2.5.

The Remainder Theorem (p. 117)

If a polynomial $f(x)$ is divided by $x - k$, then the remainder is

$$r = f(k).$$

Proof

From the Division Algorithm, you have

$$f(x) = (x - k)q(x) + r(x)$$

and because either $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $x - k$, you know that $r(x)$ must be a constant. That is, $r(x) = r$. Now, by evaluating $f(x)$ at $x = k$, you have

$$\begin{aligned} f(k) &= (k - k)q(k) + r \\ &= (0)q(k) + r \\ &= r. \end{aligned}$$

■

To be successful in algebra, it is important that you understand the connection among the *factors* of a polynomial, the *zeros* of a polynomial function, and the *solutions* or *roots* of a polynomial equation. The Factor Theorem is the basis for this connection.

The Factor Theorem (p. 117)

A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

Proof

Using the Division Algorithm with the factor $(x - k)$, you have

$$f(x) = (x - k)q(x) + r(x).$$

By the Remainder Theorem, $r(x) = r = f(k)$, and you have

$$f(x) = (x - k)q(x) + f(k)$$

where $q(x)$ is a polynomial of lesser degree than $f(x)$. If $f(k) = 0$, then

$$f(x) = (x - k)q(x)$$

and you see that $(x - k)$ is a factor of $f(x)$. Conversely, if $(x - k)$ is a factor of $f(x)$, then division of $f(x)$ by $(x - k)$ yields a remainder of 0. So, by the Remainder Theorem, you have $f(k) = 0$. ■

Linear Factorization Theorem (p. 135)

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, . . . , c_n$ are complex numbers.

Proof

Using the Fundamental Theorem of Algebra, you know that f must have at least one zero, c_1. Consequently, $(x - c_1)$ is a factor of $f(x)$, and you have

$$f(x) = (x - c_1)f_1(x).$$

If the degree of $f_1(x)$ is greater than zero, then apply the Fundamental Theorem again to conclude that f_1 must have a zero c_2, which implies that

$$f(x) = (x - c_1)(x - c_2)f_2(x).$$

It is clear that the degree of $f_1(x)$ is $n - 1$, that the degree of $f_2(x)$ is $n - 2$, and that you can repeatedly apply the Fundamental Theorem n times until you obtain

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where a_n is the leading coefficient of the polynomial $f(x)$. ■

Factors of a Polynomial (p. 138)

Every polynomial of degree $n > 0$ with real coefficients can be written as the product of linear and quadratic factors with real coefficients, where the quadratic factors have no real zeros.

Proof

To begin, you use the Linear Factorization Theorem to conclude that $f(x)$ can be *completely* factored in the form

$$f(x) = d(x - c_1)(x - c_2)(x - c_3) . . . (x - c_n).$$

If each c_i is real, then there is nothing more to prove. If any c_i is complex ($c_i = a + bi$, $b \neq 0$), then, because the coefficients of $f(x)$ are real, you know that the conjugate $c_j = a - bi$ is also a zero. By multiplying the corresponding factors, you obtain

$$\begin{aligned}(x - c_i)(x - c_j) &= [x - (a + bi)][x - (a - bi)]\\ &= x^2 - 2ax + (a^2 + b^2)\end{aligned}$$

where each coefficient is real. ■

The Fundamental Theorem of Algebra

The Linear Factorization Theorem is closely related to the Fundamental Theorem of Algebra. The Fundamental Theorem of Algebra has a long and interesting history. In the early work with polynomial equations, the Fundamental Theorem of Algebra was thought to have been not true, because imaginary solutions were not considered. In fact, in the very early work by mathematicians such as Abu al-Khwarizmi (c. 800 A.D.), negative solutions were also not considered.

Once imaginary numbers were accepted, several mathematicians attempted to give a general proof of the Fundamental Theorem of Algebra. These included Gottfried von Leibniz (1702), Jean d'Alembert (1746), Leonhard Euler (1749), Joseph-Louis Lagrange (1772), and Pierre Simon Laplace (1795). The mathematician usually credited with the first correct proof of the Fundamental Theorem of Algebra is Carl Friedrich Gauss, who published the proof in his doctoral thesis in 1799.

Progressive Summary (Chapters 1–2)

This chart outlines the topics that have been covered so far in this text. Progressive Summary charts appear after Chapters 2, 3, 6, and 9. In each Progressive Summary, new topics encountered for the first time appear in blue.

Algebraic Functions

Polynomial, Rational, Radical

■ Rewriting

Polynomial form ↔ Factored form
Operations with polynomials
Rationalize denominators
Simplify rational expressions
Operations with complex numbers

■ Solving

Equation	*Strategy*
Linear	Isolate variable
Quadratic.	Factor, set to zero Extract square roots Complete the square Quadratic Formula
Polynomial	Factor, set to zero Rational Zero Test
Rational.	Multiply by LCD
Radical	Isolate, raise to power
Absolute value . . .	Isolate, form two equations

■ Analyzing

Graphically
Intercepts
Symmetry
Slope
Asymptotes
End behavior
Minimum values
Maximum values

Algebraically
Domain, Range
Transformations
Composition
Standard forms of equations
Leading Coefficient Test
Synthetic division
Descartes's Rule of Signs

Numerically
Table of values

Transcendental Functions

■ Rewriting

■ Solving

■ Analyzing

Other Topics

■ Rewriting

■ Solving

■ Analyzing