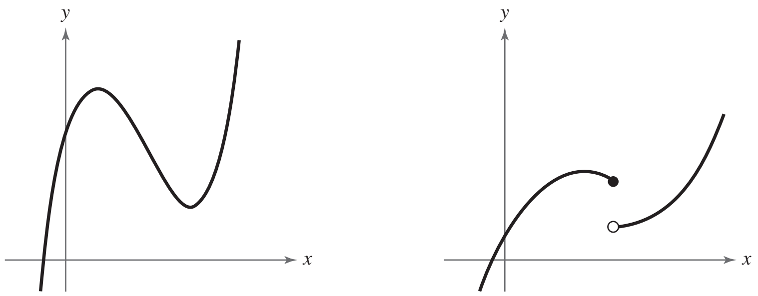
2.2 Polynomial Functions of Higher Degree

Graphs of Polynomial Functions

At this point, you should be able to sketch accurate graphs of polynomial functions of degrees 0, 1, and 2. The graphs of polynomial functions of degree greater than 2 are more difficult to sketch by hand. In this section, however, you will learn how to recognize some of the basic features of the graphs of polynomial functions. Using these features along with point plotting, intercepts, and symmetry, you should be able to make reasonably accurate sketches *by hand*.

The graph of a polynomial function is **continuous.** Essentially, this means that the graph of a polynomial function has no breaks, holes, or gaps, as shown in Figure 2.6. Informally, you can say that a function is continuous when its graph can be drawn with a pencil without lifting the pencil from the paper.

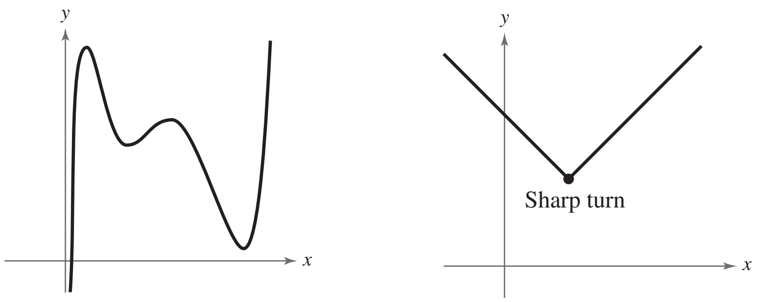


(a) Polynomial functions have continuous graphs.

(b) Functions with graphs that are not continuous are not polynomial functions.

Figure 2.6

Another feature of the graph of a polynomial function is that it has only smooth, rounded turns, as shown in Figure 2.7(a). It cannot have a sharp turn, such as the one shown in Figure 2.7(b).



(a) Polynomial functions have graphs with smooth, rounded turns.

(b) Functions with graphs that have sharp turns are not polynomial functions.

Figure 2.7

The graphs of polynomial functions of degree 1 are lines, and those of functions of degree 2 are parabolas. The graphs of all polynomial functions are smooth and continuous. A polynomial function of degree n has the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$

where n is a positive integer and $a_n \neq 0$.

What you should learn

- Use transformations to sketch graphs of polynomial functions.
- Use the Leading Coefficient Test to determine the end behavior of graphs of polynomial functions.
- Find and use zeros of polynomial functions as sketching aids.
- Use the Intermediate Value Theorem to help locate zeros of polynomial functions.

Why you should learn it

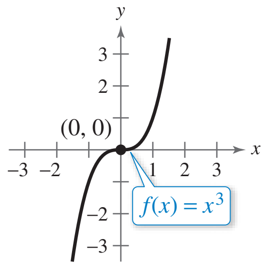
You can use polynomial functions to model various aspects of nature, such as the growth of a red oak tree, as shown in Exercise 112 on page 111.

The polynomial functions that have the simplest graphs are monomials of the form $f(x) = x^n$, where n is an integer greater than zero. The greater the value of n, the flatter the graph near the origin. When n is even, the graph is similar to the graph of $f(x) = x^2$ and touches the x-axis at the x-intercept. When n is odd, the graph is similar to the graph of $f(x) = x^3$ and crosses the x-axis at the x-intercept. Polynomial functions of the form $f(x) = x^n$ are often referred to as **power functions.**

Library of Parent Functions: Cubic Function

The basic characteristics of the *parent cubic function* $f(x) = x^3$ are summarized below and on the inside cover of this text.

Graph of $f(x) = x^3$
Domain: $(-\infty, \infty)$
Range: $(-\infty, \infty)$
Intercept: $(0, 0)$
Increasing on $(-\infty, \infty)$
Odd function
Origin symmetry



EXAMPLE 1 Library of Parent Functions: $f(x) = x^3$

See LarsonPrecalculus.com for an interactive version of this type of example.

Sketch the graphs of (a) $g(x) = -x^3$, (b) $h(x) = x^3 + 1$, and (c) $k(x) = (x - 1)^3$ by hand.

Solution

a. With respect to the graph of $f(x) = x^3$, the graph of g is obtained by a *reflection* in the x-axis, as shown in Figure 2.8.

b. With respect to the graph of $f(x) = x^3$, the graph of h is obtained by a vertical shift one unit *upward*, as shown in Figure 2.9.

c. With respect to the graph of $f(x) = x^3$, the graph of k is obtained by a horizontal shift one unit *to the right*, as shown in Figure 2.10.

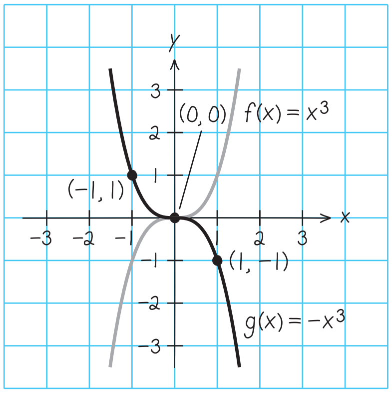


Figure 2.8

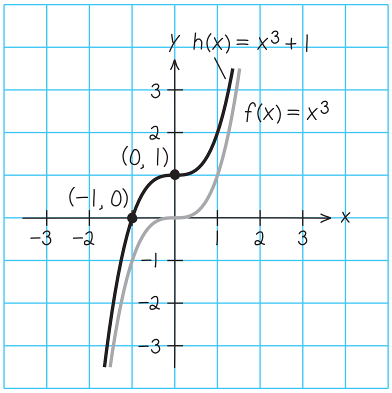


Figure 2.9

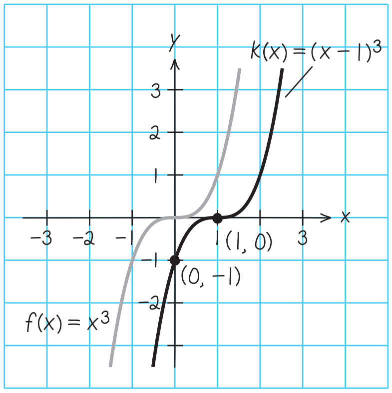


Figure 2.10

Explore the Concept

Use a graphing utility to graph $y = x^n$ for $n = 2$, 4, and 8. (Use the viewing window $-1.5 \le x \le 1.5$ and $-1 \le y \le 6$.) Compare the graphs. In the interval $(-1, 1)$, which graph is on the bottom? Outside the interval $(-1, 1)$, which graph is on the bottom?

Use a graphing utility to graph $y = x^n$ for $n = 3$, 5, and 7. (Use the viewing window $-1.5 \le x \le 1.5$ and $-4 \le y \le 4$.) Compare the graphs. In the intervals $(-\infty, -1)$ and $(0, 1)$, which graph is on the bottom? In the intervals $(-1, 0)$ and $(1, \infty)$, which graph is on the bottom?

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graphs of (a) $g(x) = -x^3 + 2$, (b) $h(x) = x^3 - 5$, and (c) $k(x) = (x + 2)^3$ by hand.

The Leading Coefficient Test

In Example 1, note that all three graphs eventually rise or fall without bound as x moves to the right. Whether the graph of a polynomial eventually rises or falls can be determined by the polynomial function's degree (even or odd) and by its leading coefficient, as indicated in the **Leading Coefficient Test.**

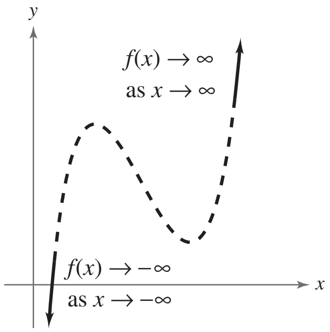
Leading Coefficient Test

As x moves without bound to the left or to the right, the graph of the polynomial function

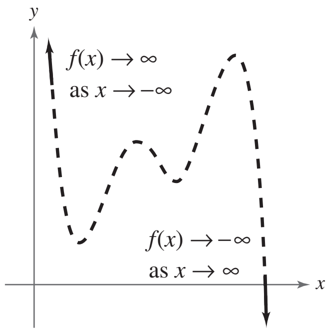
$$f(x) = a_n x^n + \cdot \cdot \cdot + a_1 x + a_0, \quad a_n \neq 0$$

eventually rises or falls in the following manner.

1. When n is odd:

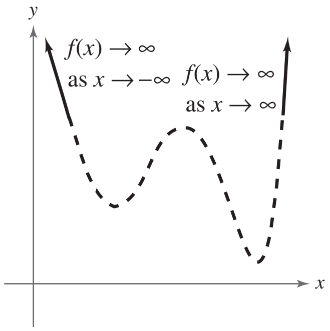


If the leading coefficient is positive ($a_n > 0$), then the graph falls to the left and rises to the right.

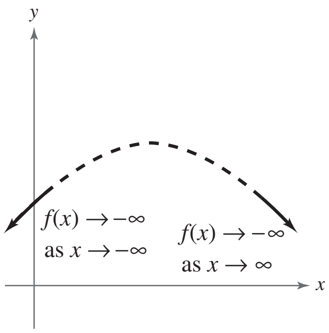


If the leading coefficient is negative ($a_n < 0$), then the graph rises to the left and falls to the right.

2. When n is even:



If the leading coefficient is positive ($a_n > 0$), then the graph rises to the left and right.



If the leading coefficient is negative ($a_n < 0$), then the graph falls to the left and right.

Note that the dashed portions of the graphs indicate that the test determines only the right-hand and left-hand behavior of the graph.

As you continue to study polynomial functions and their graphs, you will notice that the degree of a polynomial plays an important role in determining other characteristics of the polynomial function and its graph.

Explore the Concept

For each function, identify the degree of the function and whether the degree of the function is even or odd. Identify the leading coefficient and whether the leading coefficient is positive or negative. Use a graphing utility to graph each function. Describe the relationship between the degree and sign of the leading coefficient of the function, and the right- and left-hand behavior of the graph of the function.

a. $y = x^3 - 2x^2 - x + 1$

b. $y = 2x^5 + 2x^2 - 5x + 1$

c. $y = -2x^5 - x^2 + 5x + 3$

d. $y = -x^3 + 5x - 2$

e. $y = 2x^2 + 3x - 4$

f. $y = x^4 - 3x^2 + 2x - 1$

g. $y = -x^2 + 3x + 2$

h. $y = -x^6 - x^2 - 5x + 4$

Remark

The notation "$f(x) \to -\infty$ as $x \to -\infty$" indicates that the graph falls to the left. The notation "$f(x) \to \infty$ as $x \to \infty$" indicates that the graph rises to the right.

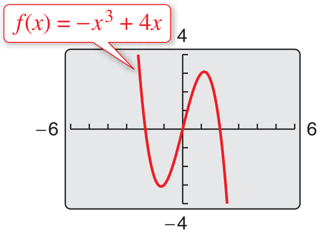
A review of the shapes of the graphs of polynomial functions of degrees 0, 1, and 2 may be used to illustrate the Leading Coefficient Test.

EXAMPLE 2 Applying the Leading Coefficient Test

Use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of $f(x) = -x^3 + 4x$.

Solution

Because the degree is odd and the leading coefficient is negative, the graph rises to the left and falls to the right, as shown in the figure.



A good test of students' understanding is to present a graph of a polynomial function without giving its equation, and ask the students what they can tell you about the polynomial's degree and leading coefficient by looking at the graph. You might want to display a few such graphs on an overhead projector during class for practice.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of $f(x) = 2x^3 - 3x^2 + 5$.

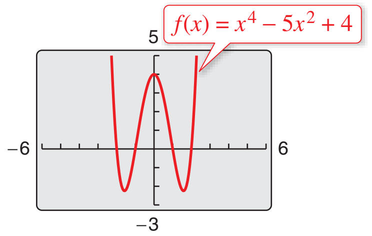
EXAMPLE 3 Applying the Leading Coefficient Test

Use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of each polynomial function.

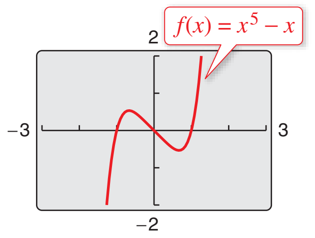
a. $f(x) = x^4 - 5x^2 + 4$ **b.** $f(x) = x^5 - x$

Solution

a. Because the degree is even and the leading coefficient is positive, the graph rises to the left and right, as shown in the figure.



b. Because the degree is odd and the leading coefficient is positive, the graph falls to the left and rises to the right, as shown in the figure.



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Use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of each polynomial function.

a. $f(x) = -x^4 + 2x^2 + 6$ **b.** $f(x) = -x^5 + 3x^4 - x$

In Examples 2 and 3, note that the Leading Coefficient Test tells you only whether the graph *eventually* rises or falls to the right or left. Other characteristics of the graph, such as intercepts and minimum and maximum points, must be determined by other tests.

Explore the Concept

For each of the graphs in Examples 2 and 3, count the number of zeros of the polynomial function and the number of relative extrema, and compare these numbers with the degree of the polynomial. What do you observe?

Zeros of Polynomial Functions

It can be shown that for a polynomial function f of degree n, the following statements are true.

1. The function f has at most n real zeros. (You will study this result in detail in Section 2.5 on the Fundamental Theorem of Algebra.)
2. The graph of f has at most $n - 1$ relative **extrema** (relative minima or maxima).

Recall that a zero of a function f is a number x for which $f(x) = 0$. Finding the zeros of polynomial functions is one of the most important problems in algebra. You have already seen that there is a strong interplay between graphical and algebraic approaches to this problem. Sometimes you can use information about the graph of a function to help find its zeros. In other cases, you can use information about the zeros of a function to find a good viewing window.

Real Zeros of Polynomial Functions

When f is a polynomial function and a is a real number, the following statements are equivalent.

1. $x = a$ is a *zero* of the function f.
2. $x = a$ is a *solution* of the polynomial equation $f(x) = 0$.
3. $(x - a)$ is a *factor* of the polynomial $f(x)$.
4. $(a, 0)$ is an *x-intercept* of the graph of f.

Finding zeros of polynomial functions is closely related to factoring and finding x-intercepts, as demonstrated in Examples 4, 5, and 6.

EXAMPLE 4 Finding Zeros of a Polynomial Function

Find all real zeros of $f(x) = x^3 - x^2 - 2x$.

Algebraic Solution

$f(x) = x^3 - x^2 - 2x$ Write original function.

$0 = x^3 - x^2 - 2x$ Substitute 0 for $f(x)$.

$0 = x(x^2 - x - 2)$ Remove common monomial factor.

$0 = x(x - 2)(x + 1)$ Factor completely.

So, the real zeros are $x = 0$, $x = 2$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(2, 0)$, and $(-1, 0)$.

Check

$(0)^3 - (0)^2 - 2(0) = 0$ $x = 0$ is a zero. ✓

$(2)^3 - (2)^2 - 2(2) = 0$ $x = 2$ is a zero. ✓

$(-1)^3 - (-1)^2 - 2(-1) = 0$ $x = -1$ is a zero. ✓

Graphical Solution

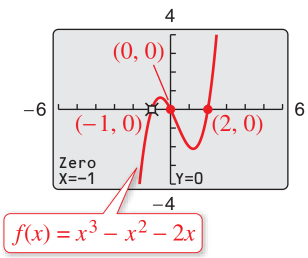
The graph of f has the x-intercepts

$$(0, 0),\quad (2, 0),\quad \text{and} \quad (-1, 0)$$

as shown in the figure. So, the real zeros of f are

$$x = 0,\quad x = 2,\quad \text{and}\quad x = -1.$$

Use the *zero* or *root* feature of a graphing utility to verify these zeros.



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Find all real zeros of $f(x) = x^3 + x^2 - 6x$.

EXAMPLE 5 Analyzing a Polynomial Function

Find all real zeros and relative extrema of $f(x) = -2x^4 + 2x^2$.

Solution

$0 = -2x^4 + 2x^2$ Substitute 0 for $f(x)$.

$0 = -2x^2(x^2 - 1)$ Remove common monomial factor.

$0 = -2x^2(x - 1)(x + 1)$ Factor completely.

So, the real zeros are $x = 0$, $x = 1$, and $x = -1$, and the corresponding x-intercepts are $(0, 0)$, $(1, 0)$, and $(-1, 0)$, as shown in Figure 2.11. Using the *minimum* and *maximum* features of a graphing utility, you can approximate the three relative extrema to be $(-0.71, 0.5)$, $(0, 0)$, and $(0.71, 0.5)$.

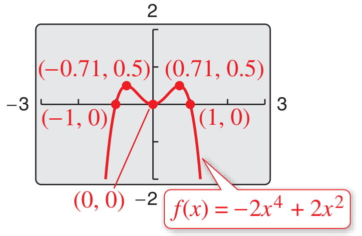

Figure 2.11

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all real zeros and relative extrema of $f(x) = x^3 + 2x^2 - 3x$.

Repeated Zeros

For a polynomial function, a factor of $(x - a)^k$, $k > 1$, yields a **repeated zero** $x = a$ of **multiplicity** k.

1. If k is odd, then the graph *crosses* the x-axis at $x = a$.
2. If k is even, then the graph *touches* the x-axis (but does not cross the x-axis) at $x = a$.

Remark

In Example 5, note that because k is even, the factor $-2x^2$ yields the repeated zero $x = 0$. The graph touches (but does not cross) the x-axis at $x = 0$, as shown in Figure 2.11.

EXAMPLE 6 Analyzing a Polynomial Function

To find all real zeros of $f(x) = x^5 - 3x^3 - x^2 - 4x - 1$, use the *zero* feature of a graphing utility. From Figure 2.12, the zeros are $x \approx -1.86$, $x \approx -0.25$, and $x \approx 2.11$. Note that this fifth-degree polynomial factors as

$$f(x) = x^5 - 3x^3 - x^2 - 4x - 1 = (x^2 + 1)(x^3 - 4x - 1).$$

The three zeros obtained above are the zeros of the cubic factor $x^3 - 4x - 1$. The quadratic factor $x^2 + 1$ has no real zeros but does have two *imaginary* zeros. You will learn more about imaginary zeros in Section 2.5.

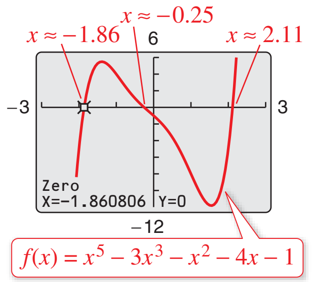

Figure 2.12

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all real zeros of $f(x) = x^5 - 2x^4 + x^3 - 2x^2$.

EXAMPLE 7 Finding a Polynomial Function with Given Zeros

Find a polynomial function with zeros $-\frac{1}{2}$, 3, and 3. (There are many correct solutions.)

Solution

Note that the zero $x = -\frac{1}{2}$ corresponds to either $\left(x + \frac{1}{2}\right)$ or $(2x + 1)$. To avoid fractions, choose the second factor and write

$$f(x) = (2x + 1)(x - 3)^2 = (2x + 1)(x^2 - 6x + 9) = 2x^3 - 11x^2 + 12x + 9.$$

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Find a polynomial function with zeros -2, -1, 1, and 2. (There are many correct solutions.)

Activities

1. Find all real zeros of $f(x) = 6x^4 - 33x^3 - 18x^2$.
   Answer: $-\frac{1}{2}$, 0, 6
2. Determine the right-hand and left-hand behavior of $f(x) = 6x^4 - 33x^3 - 18x^2$.
   Answer: The graph rises to the left and right.
3. Find a polynomial function of degree 3 that has zeros of 0, 2, and $-\frac{1}{3}$.
   Answer: $f(x) = 3x^3 - 5x^2 - 2x$

Note in Example 7 that there are many polynomial functions with the indicated zeros. In fact, multiplying the function by any real number does not change the zeros of the function. For instance, multiply the function from Example 7 by $\frac{1}{2}$ to obtain

$$f(x) = x^3 - \frac{11}{2}x^2 + 6x + \frac{9}{2}.$$

Then find the zeros of the function. You will obtain the zeros $-\frac{1}{2}$, 3, and 3, as given in Example 7.

EXAMPLE 8 Sketching the Graph of a Polynomial Function

Sketch the graph of

$$f(x) = 3x^4 - 4x^3$$

by hand.

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is positive and the degree is even, you know that the graph eventually rises to the left and to the right (see Figure 2.13).

2. *Find the Real Zeros of the Polynomial.* By factoring

$$f(x) = 3x^4 - 4x^3 = x^3(3x - 4)$$

you can see that the real zeros of f are $x = 0$ (of odd multiplicity 3) and $x = \frac{4}{3}$ (of odd multiplicity 1). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{4}{3}, 0\right)$. Add these points to your graph, as shown in Figure 2.13.

3. *Plot a Few Additional Points.* To sketch the graph by hand, find a few additional points, as shown in the table. Be sure to choose points between the zeros and to the left and right of the zeros. Then plot the points (see Figure 2.14).

x	-1	0.5	1	1.5
$f(x)$	7	-0.31	-1	1.69

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.14. Because both zeros are of odd multiplicity, you know that the graph should cross the x-axis at $x = 0$ and $x = \frac{4}{3}$. When you are unsure of the shape of a portion of the graph, plot some additional points.

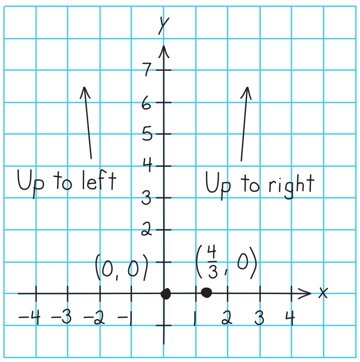

Figure 2.13

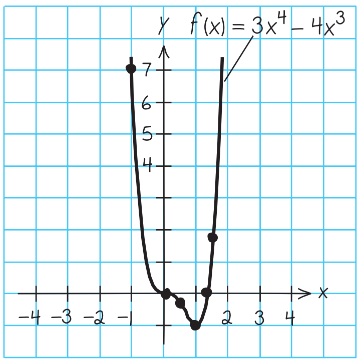

Figure 2.14

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Sketch the graph of $f(x) = 2x^3 - 6x^2$ by hand.

Technology Tip

Because it is easy to make mistakes when entering functions into a graphing utility, it is important to understand the basic shapes of graphs and to be able to graph simple polynomials *by hand*. For instance, suppose you had entered the function in Example 8 as $y = 3x^5 - 4x^3$. From the graph, what mathematical principles would alert you to the fact that you had made a mistake?

Partner Activity

Multiply three, four, or five distinct linear factors to obtain the equation of a polynomial function of degree 3, 4, or 5. Exchange equations with your partner and sketch, by hand, the graph of the equation that your partner wrote. When you are finished, use a graphing utility to check each other's work.

EXAMPLE 9 Sketching the Graph of a Polynomial Function

Sketch the graph of

$$f(x) = -2x^3 + 6x^2 - \tfrac{9}{2}x.$$

Solution

1. *Apply the Leading Coefficient Test.* Because the leading coefficient is negative and the degree is odd, you know that the graph eventually rises to the left and falls to the right (see Figure 2.15).

2. *Find the Real Zeros of the Polynomial.* By factoring

$$\begin{aligned} f(x) &= -2x^3 + 6x^2 - \tfrac{9}{2}x \\ &= -\tfrac{1}{2}x(4x^2 - 12x + 9) \\ &= -\tfrac{1}{2}x(2x - 3)^2 \end{aligned}$$

you can see that the real zeros of f are $x = 0$ (of odd multiplicity 1) and $x = \frac{3}{2}$ (of even multiplicity 2). So, the x-intercepts occur at $(0, 0)$ and $\left(\frac{3}{2}, 0\right)$. Add these points to your graph, as shown in Figure 2.15.

3. *Plot a Few Additional Points.* To sketch the graph, find a few additional points, as shown in the table. Then plot the points (see Figure 2.16).

x	-0.5	0.5	1	2
$f(x)$	4	-1	-0.5	-1

4. *Draw the Graph.* Draw a continuous curve through the points, as shown in Figure 2.16. As indicated by the multiplicities of the zeros, the graph crosses the x-axis at $(0, 0)$ and touches (but does not cross) the x-axis at $\left(\frac{3}{2}, 0\right)$.

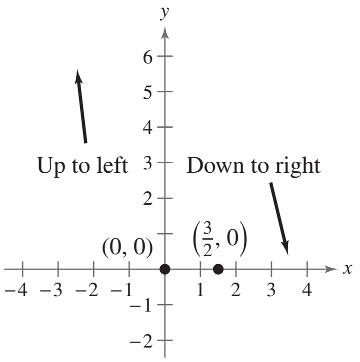


Figure 2.15

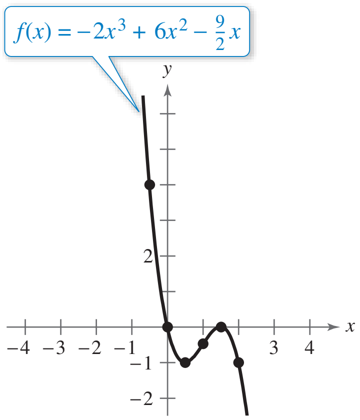


Figure 2.16

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Sketch the graph of $f(x) = -\frac{1}{4}x^4 + \frac{3}{2}x^3 - \frac{9}{4}x^2$.

Technology Tip

Remember that when using a graphing utility to verify your graphs, you may need to adjust your viewing window in order to see all the features of the graph.

Remark

Observe in Example 9 that the sign of $f(x)$ is positive to the left of and negative to the right of the zero $x = 0$. Similarly, the sign of $f(x)$ is negative to the left and to the right of the zero $x = \frac{3}{2}$. This suggests that if a zero of a polynomial function is of *odd* multiplicity, then the sign of $f(x)$ changes from one side of the zero to the other side. If a zero is of *even* multiplicity, then the sign of $f(x)$ does not change from one side of the zero to the other side. The following table helps to illustrate this result.

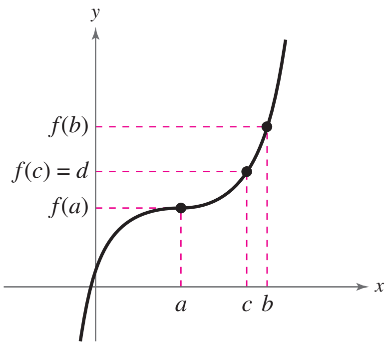
x	-0.5	0	0.5
$f(x)$	4	0	-1
Sign	$+$		$-$

x	1	$\frac{3}{2}$	2
$f(x)$	-0.5	0	-1
Sign	$-$		$-$

This sign analysis may be helpful in graphing polynomial functions.

The Intermediate Value Theorem

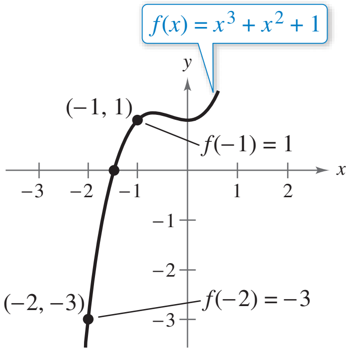
The **Intermediate Value Theorem** implies that if $(a, f(a))$ and $(b, f(b))$ are two points on the graph of a polynomial function such that $f(a) \neq f(b)$, then for any number d between $f(a)$ and $f(b)$, there must be a number c between a and b such that $f(c) = d$. (See figure shown at the right.)



Intermediate Value Theorem

Let a and b be real numbers such that $a < b$. If f is a polynomial function such that $f(a) \neq f(b)$, then in the interval $[a, b]$, f takes on every value between $f(a)$ and $f(b)$.

This theorem helps you locate the real zeros of a polynomial function in the following way. If you can find a value $x = a$ at which a polynomial function is positive, and another value $x = b$ at which it is negative, then you can conclude that the function has at least one real zero between these two values. For example, the function $f(x) = x^3 + x^2 + 1$ is negative when $x = -2$ and positive when $x = -1$. So, it follows from the Intermediate Value Theorem that f must have a real zero somewhere between -2 and -1, as shown in the figure. By continuing this line of reasoning, you can approximate any real zeros of a polynomial function to any desired accuracy.



The function f must have a real zero somewhere between -2 and -1.

EXAMPLE 10 Approximating the Zeros of a Function

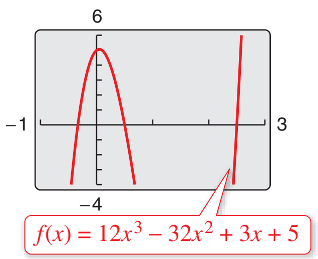
Find three intervals of length 1 in which the polynomial

$$f(x) = 12x^3 - 32x^2 + 3x + 5$$

is guaranteed to have a zero.

Graphical Solution

From the figure, you can see that the graph of f crosses the x-axis three times—between -1 and 0, between 0 and 1, and between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.



Numerical Solution

From the table, you can see that $f(-1)$ and $f(0)$ differ in sign. So, you can conclude from the Intermediate Value Theorem that the function has a zero between -1 and 0. Similarly, $f(0)$ and $f(1)$ differ in sign, so the function has a zero between 0 and 1. Likewise, $f(2)$ and $f(3)$ differ in sign, so the function has a zero between 2 and 3. So, you can conclude that the function has zeros in the intervals $(-1, 0)$, $(0, 1)$, and $(2, 3)$.

X	Y1	
−2	−225	
−1	−42	
0	5	
1	−12	
2	−21	
3	50	
4	273	
X=−1		

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find three intervals of length 1 in which the polynomial $f(x) = x^3 - 4x^2 + 1$ is guaranteed to have a zero.

2.2 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. The graph of a polynomial function is ________ , so it has no breaks, holes, or gaps.

2. A polynomial function of degree n has at most ________ real zeros and at most ________ relative extrema.

3. When $x = a$ is a zero of a polynomial function f, the following statements are true.

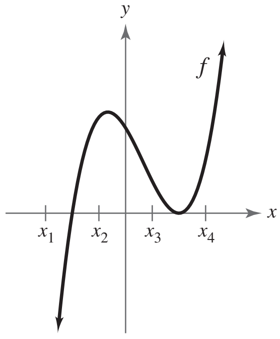
(a) $x = a$ is a ________ of the polynomial equation $f(x) = 0$.

(b) ________ is a factor of the polynomial $f(x)$.

(c) The point ________ is an x-intercept of the graph of f.

4. If a zero of a polynomial function f is of even multiplicity, then the graph of f ________ the x-axis, and if the zero is of odd multiplicity, then the graph of f ________ the x-axis.

For Exercises 5–8, the graph shows the right-hand and left-hand behavior of a polynomial function f.



5. Can f be a fourth-degree polynomial function?

6. Can the leading coefficient of f be negative?

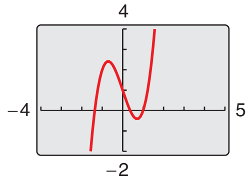
7. The graph shows that $f(x_1) < 0$. What other information shown in the graph allows you to apply the Intermediate Value Theorem to guarantee that f has a zero in the interval $[x_1, x_2]$?

8. Is the repeated zero of f in the interval $[x_3, x_4]$ of even or odd multiplicity?

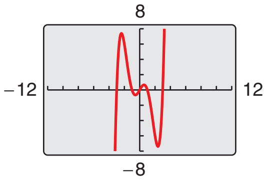
Procedures and Problem Solving

Identifying Graphs of Polynomial Functions **In Exercises 9–16, match the polynomial function with its graph. [The graphs are labeled (a) through (h).]**

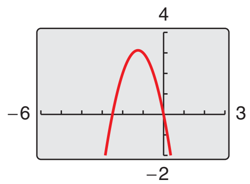
(a)



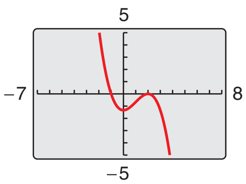
(b)



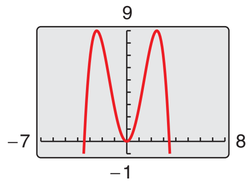
(c)



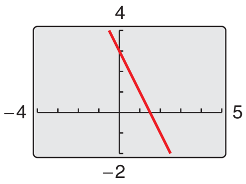
(d)



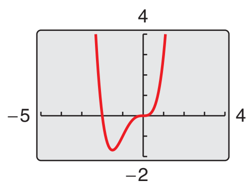
(e)



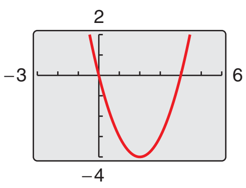
(f)



(g)



(h)



9. $f(x) = -2x + 3$

10. $f(x) = x^2 - 4x$

11. $f(x) = -2x^2 - 5x$

12. $f(x) = 2x^3 - 3x + 1$

13. $f(x) = -\frac{1}{4}x^4 + 3x^2$

14. $f(x) = -\frac{1}{3}x^3 + x^2 - \frac{4}{3}$

15. $f(x) = x^4 + 2x^3$

16. $f(x) = \frac{1}{5}x^5 - 2x^3 + \frac{9}{5}x$

Library of Parent Functions **In Exercises 17–22, sketch the graph of $f(x) = x^3$ and the graph of the function g. Describe the transformation from f to g.**

17. $g(x) = (x - 3)^3$

18. $g(x) = x^3 - 3$

19. $g(x) = -x^3 + 4$

20. $g(x) = (x - 2)^3 - 3$

21. $g(x) = -(x - 3)^3$

22. $g(x) = (x + 4)^3 + 1$

Comparing End Behavior **In Exercises 23–28, use a graphing utility to graph the functions f and g in the same viewing window. Zoom out far enough to see the right-hand and left-hand behavior of each graph. Do the graphs of f and g have the same right-hand and left-hand behavior? Explain why or why not.**

23. $f(x) = 3x^3 - 9x + 1, \quad g(x) = 3x^3$

24. $f(x) = -\frac{1}{3}(x^3 - 3x + 2), \quad g(x) = -\frac{1}{3}x^3$

25. $f(x) = -(x^4 - 4x^3 + 16x), \quad g(x) = -x^4$

26. $f(x) = 3x^4 - 6x^2, \quad g(x) = 3x^4$

27. $f(x) = -2x^3 + 4x^2 - 1, \quad g(x) = 2x^3$

28. $f(x) = -(x^4 - 6x^2 - x + 10), \quad g(x) = x^4$

Applying the Leading Coefficient Test In Exercises 29–36, use the Leading Coefficient Test to describe the right-hand and left-hand behavior of the graph of the polynomial function. Use a graphing utility to verify your results.

29. $f(x) = 2x^4 - 3x + 1$

30. $h(x) = 1 - x^6$

31. $g(x) = 5 - \frac{7}{2}x - 3x^2$

32. $f(x) = \frac{1}{3}x^3 + 5x$

33. $f(x) = \dfrac{6x^5 - 2x^4 + 4x^2 - 5x}{3}$

34. $f(x) = \dfrac{3x^7 - 2x^5 + 5x^3 + 6x^2}{4}$

35. $h(t) = -\frac{2}{3}(t^2 - 5t + 3)$

36. $f(s) = -\frac{7}{8}(s^3 + 5s^2 - 7s + 1)$

Finding Zeros of a Polynomial Function In Exercises 37–48, (a) find the zeros algebraically, (b) use a graphing utility to graph the function, and (c) use the graph to approximate any zeros and compare them with those from part (a).

37. $f(x) = 3x^2 - 12x + 3$

38. $g(x) = 5x^2 - 10x - 5$

39. $g(t) = \frac{1}{2}t^4 - \frac{1}{2}$

40. $y = \frac{1}{4}x^3(x^2 - 9)$

41. $f(x) = x^5 + x^3 - 6x$

42. $g(t) = t^5 - 6t^3 + 9t$

43. $f(x) = 2x^4 - 2x^2 - 40$

44. $f(x) = 5x^4 + 15x^2 + 10$

45. $f(x) = x^3 - 4x^2 - 25x + 100$

46. $y = 4x^3 + 4x^2 - 7x + 2$

47. $y = 4x^3 - 20x^2 + 25x$

48. $y = x^5 - 5x^3 + 4x$

Finding Zeros and Their Multiplicities In Exercises 49–58, find all the real zeros of the polynomial function. Determine the multiplicity of each zero. Use a graphing utility to verify your results.

49. $f(x) = x^2 - 25$

50. $f(x) = 49 - x^2$

51. $h(t) = t^2 - 6t + 9$

52. $f(x) = x^2 + 10x + 25$

53. $f(x) = x^2 + x - 2$

54. $f(x) = 2x^2 - 14x + 24$

55. $f(t) = t^3 - 4t^2 + 4t$

56. $f(x) = x^4 - x^3 - 20x^2$

57. $f(x) = \frac{1}{2}x^2 + \frac{5}{2}x - \frac{3}{2}$

58. $f(x) = \frac{5}{3}x^2 + \frac{8}{3}x - \frac{4}{3}$

Analyzing a Polynomial Function In Exercises 59–64, use a graphing utility to graph the function and approximate (accurate to three decimal places) any real zeros and relative extrema.

59. $f(x) = 2x^4 - 6x^2 + 1$

60. $f(x) = -\frac{3}{8}x^4 - x^3 + 2x^2 + 5$

61. $f(x) = x^5 + 3x^3 - x + 6$

62. $f(x) = -3x^3 - 4x^2 + x - 3$

63. $f(x) = -2x^4 + 5x^2 - x - 1$

64. $f(x) = 3x^5 - 2x^2 - x + 1$

Finding a Polynomial Function with Given Zeros In Exercises 65–74, find a polynomial function that has the given zeros. (There are many correct answers.)

65. 0, 7

66. $-2, 5$

67. $0, -2, -4$

68. 0, 1, 6

69. $4, -3, 3, 0$

70. $-2, -1, 0, 1, 2$

71. $1 + \sqrt{2}, 1 - \sqrt{2}$

72. $4 + \sqrt{3}, 4 - \sqrt{3}$

73. $2, 2 + \sqrt{5}, 2 - \sqrt{5}$

74. $3, 2 + \sqrt{7}, 2 - \sqrt{7}$

Finding a Polynomial Function with Given Zeros In Exercises 75–80, find a polynomial function with the given zeros, multiplicities, and degree. (There are many correct answers.)

75. Zero: -2, multiplicity: 2
Zero: -1, multiplicity: 1
Degree: 3

76. Zero: 3, multiplicity: 1
Zero: 2, multiplicity: 3
Degree: 4

77. Zero: -4, multiplicity: 2
Zero: 3, multiplicity: 2
Degree: 4

78. Zero: 5, multiplicity: 3
Zero: 0, multiplicity: 2
Degree: 5

79. Zero: -1, multiplicity: 2
Zero: -2, multiplicity: 1
Degree: 3
Rises to the left,
Falls to the right

80. Zero: 1, multiplicity: 2
Zero: 4, multiplicity: 2
Degree: 4
Falls to the left,
Falls to the right

Sketching a Polynomial with Given Conditions In Exercises 81–84, sketch the graph of a polynomial function that satisfies the given conditions. If not possible, explain your reasoning. (There are many correct answers.)

81. Third-degree polynomial with two real zeros and a negative leading coefficient

82. Fourth-degree polynomial with three real zeros and a positive leading coefficient

83. Fifth-degree polynomial with three real zeros and a positive leading coefficient

84. Fourth-degree polynomial with two real zeros and a negative leading coefficient

Sketching the Graph of a Polynomial Function In Exercises 85–94, sketch the graph of the function by (a) applying the Leading Coefficient Test, (b) finding the zeros of the polynomial, (c) plotting sufficient solution points, and (d) drawing a continuous curve through the points.

85. $f(x) = x^3 - 9x$

86. $g(x) = x^4 - 4x^2$

87. $f(x) = x^3 - 3x^2$

88. $f(x) = 3x^3 - 24x^2$

89. $f(x) = -x^4 + 9x^2 - 20$

90. $f(x) = -x^6 + 7x^3 + 8$

91. $f(x) = x^3 + 3x^2 - 9x - 27$

92. $h(x) = x^5 - 4x^3 + 8x^2 - 32$

93. $g(t) = -\frac{1}{4}t^4 + 2t^2 - 4$

94. $g(x) = \frac{1}{10}(x^4 - 4x^3 - 2x^2 + 12x + 9)$

Approximating the Zeros of a Function **In Exercises 95–100, (a) use the Intermediate Value Theorem and a graphing utility to find graphically any intervals of length 1 in which the polynomial function is guaranteed to have a zero, and (b) use the *zero* or *root* feature of the graphing utility to approximate the real zeros of the function. Verify your answers in part (a) by using the *table* feature of the graphing utility.**

95. $f(x) = x^3 - 3x^2 + 3$

96. $f(x) = -2x^3 - 6x^2 + 3$

97. $g(x) = 3x^4 + 4x^3 - 3$

98. $h(x) = x^4 - 10x^2 + 2$

99. $f(x) = x^4 - 3x^3 - 4x - 3$

100. $f(x) = x^3 - 4x^2 - 2x + 10$

Identifying Symmetry and x-Intercepts **In Exercises 101–108, use a graphing utility to graph the function. Identify any symmetry with respect to the x-axis, y-axis, or origin. Determine the number of x-intercepts of the graph.**

101. $f(x) = x^2(x + 6)$

102. $h(x) = x^3(x - 3)^2$

103. $g(t) = -\frac{1}{2}(t - 4)^2(t + 4)^2$

104. $g(x) = \frac{1}{8}(x + 1)^2(x - 3)^3$

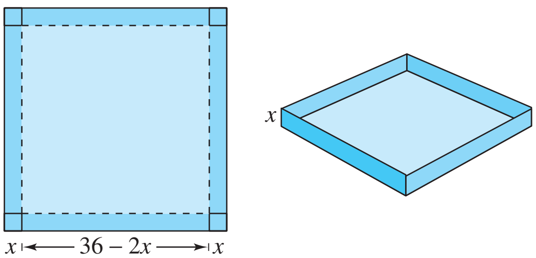
105. $f(x) = x^3 - 4x$

106. $f(x) = x^4 - 2x^2$

107. $g(x) = \frac{1}{5}(x + 1)^2(x - 3)(2x - 9)$

108. $h(x) = \frac{1}{5}(x + 2)^2(3x - 5)^2$

109. Geometry An open box is to be made from a square piece of material 36 centimeters on a side by cutting equal squares with sides of length x from the corners and turning up the sides (see figure).



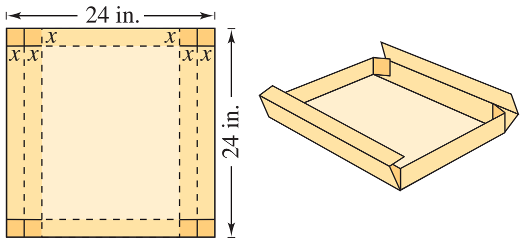
(a) Verify that the volume of the box is given by the function $V(x) = x(36 - 2x)^2$.

(b) Determine the domain of the function V.

(c) Use the *table* feature of a graphing utility to create a table that shows various box heights x and the corresponding volumes V. Use the table to estimate a range of dimensions within which the maximum volume is produced.

(d) Use the graphing utility to graph V and use the range of dimensions from part (c) to find the x-value for which $V(x)$ is maximum.

110. Geometry An open box with locking tabs is to be made from a square piece of material 24 inches on a side. This is done by cutting equal squares from the corners and folding along the dashed lines, as shown in the figure.



(a) Verify that the volume of the box is given by the function $V(x) = 8x(6 - x)(12 - x)$.

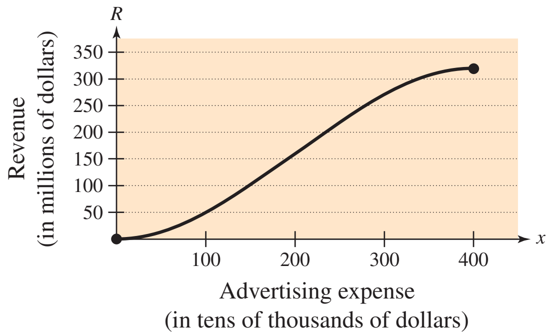
(b) Determine the domain of the function V.

(c) Sketch the graph of the function and estimate the value of x for which $V(x)$ is maximum.

111. Marketing The total revenue R (in millions of dollars) for a company is related to its advertising expense by the function

$$R = 0.00001(-x^3 + 600x^2), \quad 0 \le x \le 400$$

where x is the amount spent on advertising (in tens of thousands of dollars). Use the graph of the function shown in the figure to estimate the point on the graph at which the function is increasing most rapidly. This point is called the **point of diminishing returns** because any expense above this amount will yield less return per dollar invested in advertising.



112. *Why you should learn it* *(p. 100)* The growth of a red oak tree is approximated by the function

$$G = -0.003t^3 + 0.137t^2 + 0.458t - 0.839$$

where G is the height of the tree (in feet) and $t (2 \le t \le 32)$ is its age (in years). Use a graphing utility to graph the function and estimate the age of the tree when it is growing most rapidly. This point is called the **point of diminishing returns** because the increase in growth will be less with each additional year. (*Hint:* Use a viewing window in which $0 \le x \le 35$ and $0 \le y \le 60$.)

113. MODELING DATA

The U.S. production of crude oil y_1 (in quadrillions of British thermal units) and of solar and photovoltaic energy y_2 (in trillions of British thermal units) are shown in the table for the years 2004 through 2013, where t represents the year, with $t = 4$ corresponding to 2004. These data can be approximated by the models

$$y_1 = 0.00281t^4 - 0.0850t^3 + 1.027t^2 - 5.71t + 22.7$$

and

$$y_2 = 0.618t^3 - 10.80t^2 + 66.2t - 71.$$

(Source: Energy Information Administration)

DATA

Year, t	y_1	y_2
4	11.6	63
5	11.0	63
6	10.8	68
7	10.7	76
8	10.6	89
9	11.3	98
10	11.6	126
11	12.0	171
12	13.8	234
13	15.7	321

Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to plot the data and graph the model for y_1 in the same viewing window. How closely does the model represent the data?

(b) Extend the viewing window of the graphing utility to show the right-hand behavior of the model y_1. Would you use the model to estimate the production of crude oil in 2015? in 2020? Explain.

(c) Repeat parts (a) and (b) for y_2.

Conclusions

True or False? **In Exercises 114–120, determine whether the statement is true or false. Justify your answer.**

114. It is possible for a sixth-degree polynomial to have only one zero.

115. It is possible for a fifth-degree polynomial to have no real zeros.

116. It is possible for a polynomial with an even degree to have a range of $(-\infty, \infty)$.

117. The graph of the function $f(x) = x^6 - x^7$ rises to the left and falls to the right.

118. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ crosses the x-axis at $x = 1$.

119. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ touches, but does not cross, the x-axis.

120. The graph of the function $f(x) = 2x(x - 1)^2(x + 3)^3$ rises to the left and falls to the right.

121. Exploration Use a graphing utility to graph

$$y_1 = x + 2 \quad \text{and} \quad y_2 = (x + 2)(x - 1).$$

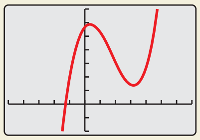
Predict the shape of the graph of

$$y_3 = (x + 2)(x - 1)(x - 3).$$

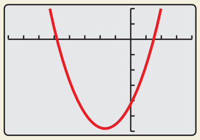
Use the graphing utility to verify your answer.

122. HOW DO YOU SEE IT? For each graph, describe a polynomial function that could represent the graph. (Indicate the degree of the function and the sign of its leading coefficient.)

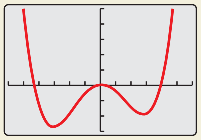
(a)



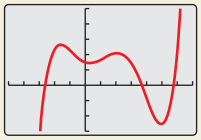
(b)



(c)



(d)



Cumulative Mixed Review

Evaluating Combinations of Functions **In Exercises 123–128, let $f(x) = 14x - 3$ and $g(x) = 8x^2$. Find the indicated value.**

123. $(f + g)(-4)$

124. $(g - f)(3)$

125. $(fg)\left(-\frac{4}{7}\right)$

126. $\left(\frac{f}{g}\right)(-1.5)$

127. $(f \circ g)(-1)$

128. $(g \circ f)(0)$

Solving Inequalities **In Exercises 129–132, solve the inequality and sketch the solution on the real number line. Use a graphing utility to verify your solution graphically.**

129. $3(x - 5) < 4x - 7$

130. $2x^2 - x \geq 1$

131. $\dfrac{5x - 2}{x - 7} \leq 4$

132. $|x + 8| - 1 \geq 15$

2.3 Real Zeros of Polynomial Functions

Long Division of Polynomials

In the graph of $f(x) = 6x^3 - 19x^2 + 16x - 4$ shown in Figure 2.17, it appears that $x = 2$ is a zero of f. Because $f(2) = 0$, you know that $x = 2$ is a zero of the polynomial function f, and that $(x - 2)$ is a factor of $f(x)$. This means that there exists a second-degree polynomial $q(x)$ such that $f(x) = (x - 2) \cdot q(x)$. To find $q(x)$, you can use **long division of polynomials.**

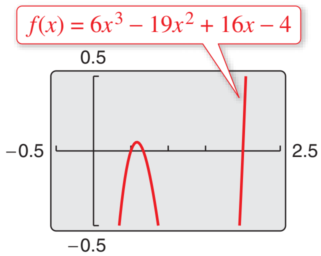


Figure 2.17

What you should learn

- Use long division to divide polynomials by other polynomials.
- Use synthetic division to divide polynomials by binomials of the form $(x - k)$.
- Use the Remainder and Factor Theorems.
- Use the Rational Zero Test to determine possible rational zeros of polynomial functions.
- Use Descartes's Rule of Signs and the Upper and Lower Bound Rules to find zeros of polynomials.

Why you should learn it

The Remainder Theorem can be used to determine the number of employees in education and health services in the United States in a given year based on a polynomial model, as shown in Exercise 106 on page 127.

EXAMPLE 1 Long Division of Polynomials

Divide the polynomial $6x^3 - 19x^2 + 16x - 4$ by $x - 2$, and use the result to factor the polynomial completely.

Solution

Think $\frac{6x^3}{x} = 6x^2$.

Think $\frac{-7x^2}{x} = -7x$.

Think $\frac{2x}{x} = 2$.

$$
\begin{array}{r|l}
 & 6x^2 - 7x + 2 \\
\hline
x - 2 & 6x^3 - 19x^2 + 16x - 4
\end{array}
$$

$6x^3 - 12x^2$	Multiply: $6x^2(x - 2)$.
$-7x^2 + 16x$	Subtract.
$-7x^2 + 14x$	Multiply: $-7x(x - 2)$.
$2x - 4$	Subtract.
$2x - 4$	Multiply: $2(x - 2)$.
0	Subtract.

You can see that

$$6x^3 - 19x^2 + 16x - 4 = (x - 2)(6x^2 - 7x + 2)$$

$$= (x - 2)(2x - 1)(3x - 2).$$

Note that this factorization agrees with the graph of f (see Figure 2.17) in that the three x-intercepts occur at $x = 2$, $x = \frac{1}{2}$, and $x = \frac{2}{3}$.

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Divide the polynomial $3x^2 + 19x + 28$ by $x + 4$, and use the result to factor the polynomial completely.

Note that in Example 1, the division process requires $-7x^2 + 14x$ to be subtracted from $-7x^2 + 16x$. So, it is implied that

$$\begin{array}{r} -7x^2 + 16x \\ \underline{-(-7x^2 + 14x)} \end{array} = \begin{array}{r} -7x^2 + 16x \\ \underline{7x^2 - 14x} \end{array}$$

and instead is written simply as

$$\begin{array}{r} -7x^2 + 16x \\ \underline{-7x^2 + 14x} \\ 2x. \end{array}$$

In Example 1, $x - 2$ is a factor of the polynomial

$$6x^3 - 19x^2 + 16x - 4$$

and the long division process produces a remainder of zero. Often, long division will produce a nonzero remainder. For instance, when you divide $x^2 + 3x + 5$ by $x + 1$, you obtain the following.

Note that one of the many uses of polynomial division is to write a function as a sum of terms in order to find slant asymptotes (see Section 2.7). This is a skill that is also used frequently in calculus.

$$
\begin{array}{rl}
x + 2 & \leftarrow \text{Quotient} \\
\text{Divisor} \rightarrow \quad x + 1 \,\overline{)\, x^2 + 3x + 5} & \leftarrow \text{Dividend} \\
\underline{x^2 + x} \phantom{{}+5} & \\
2x + 5 & \\
\underline{2x + 2} & \\
3 & \leftarrow \text{Remainder}
\end{array}
$$

Have students practice identifying the dividend, divisor, quotient, and remainder when dividing polynomials. For instance, in the division problem

$$\frac{x^3 - x + 1}{x - 1} = x^2 + x + \frac{1}{x - 1}$$

the dividend is $x^3 - x + 1$, the divisor is $x - 1$, the quotient is $x^2 + x$, and the remainder is 1.

In fractional form, you can write this result as follows.

$$\overbrace{\frac{\overbrace{x^2 + 3x + 5}^{\text{Dividend}}}{\underbrace{x + 1}_{\text{Divisor}}}} = \overbrace{x + 2}^{\text{Quotient}} + \frac{\overset{\text{Remainder}}{3}}{\underbrace{x + 1}_{\text{Divisor}}}$$

This implies that

$$x^2 + 3x + 5 = (x + 1)(x + 2) + 3 \qquad \text{Multiply each side by } (x + 1).$$

which illustrates the following theorem, called the **Division Algorithm.**

The Division Algorithm

If $f(x)$ and $d(x)$ are polynomials such that $d(x) \neq 0$, and the degree of $d(x)$ is less than or equal to the degree of $f(x)$, then there exist unique polynomials $q(x)$ and $r(x)$ such that

$$\underset{\text{Dividend}}{f(x)} = \underset{\text{Divisor}}{d(x)}\underset{\text{Quotient}}{q(x)} + \underset{\text{Remainder}}{r(x)}$$

where $r(x) = 0$ *or* the degree of $r(x)$ is less than the degree of $d(x)$. If the remainder $r(x)$ is zero, then $d(x)$ *divides evenly* into $f(x)$.

The Division Algorithm can also be written as

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}.$$

In the Division Algorithm, the rational expression $f(x)/d(x)$ is **improper** because the degree of $f(x)$ is greater than or equal to the degree of $d(x)$. On the other hand, the rational expression $r(x)/d(x)$ is **proper** because the degree of $r(x)$ is less than the degree of $d(x)$.

Before you apply the Division Algorithm, follow these steps.

1. Write the dividend and divisor in descending powers of the variable.
2. Insert placeholders with zero coefficients for missing powers of the variable.

Note how these steps are applied in the next two examples.

EXAMPLE 2 Long Division of Polynomials

Divide $8x^3 - 1$ by $2x - 1$.

Solution

Because there is no x^2-term or x-term in the dividend, you need to line up the subtraction by using zero coefficients (or leaving spaces) for the missing terms.

$$\begin{array}{r}
4x^2 + 2x + 1 \\
2x - 1 \overline{)\, 8x^3 + 0x^2 + 0x - 1} \\
\underline{8x^3 - 4x^2} \qquad\qquad \\
4x^2 + 0x \qquad \\
\underline{4x^2 - 2x} \qquad \\
2x - 1 \\
\underline{2x - 1} \\
0
\end{array}$$

So, $2x - 1$ divides evenly into $8x^3 - 1$, and you can write

$$\frac{8x^3 - 1}{2x - 1} = 4x^2 + 2x + 1, \quad x \neq \frac{1}{2}.$$

You can check this result by multiplying.

$$\begin{aligned}(2x - 1)(4x^2 + 2x + 1) &= 8x^3 + 4x^2 + 2x - 4x^2 - 2x - 1 \\ &= 8x^3 - 1\end{aligned}$$

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Divide $x^3 - 2x^2 - 9$ by $x - 3$.

In each of the long division examples presented so far, the divisor has been a first-degree polynomial. The long division algorithm works just as well with polynomial divisors of degree two or more, as shown in Example 3.

EXAMPLE 3 Long Division of Polynomials

Divide $-2 + 3x - 5x^2 + 4x^3 + 2x^4$ by $x^2 + 2x - 3$.

Solution

Begin by writing the dividend in descending powers of x.

$$\begin{array}{r}
2x^2 \qquad\qquad + 1 \\
x^2 + 2x - 3 \overline{)\, 2x^4 + 4x^3 - 5x^2 + 3x - 2} \\
\underline{2x^4 + 4x^3 - 6x^2} \qquad\qquad \\
x^2 + 3x - 2 \\
\underline{x^2 + 2x - 3} \\
x + 1
\end{array}$$

Remind students that when division yields a remainder, it is important that they write the remainder term correctly.

Note that the first subtraction eliminated two terms from the dividend. When this happens, the quotient skips a term. You can write the result as

$$\frac{2x^4 + 4x^3 - 5x^2 + 3x - 2}{x^2 + 2x - 3} = 2x^2 + 1 + \frac{x + 1}{x^2 + 2x - 3}.$$

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Divide $-x^3 + 9x + 6x^4 - x^2 - 3$ by $1 + 3x$.

Synthetic Division

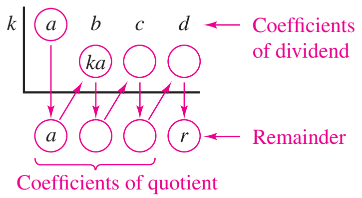
There is a nice shortcut for long division of polynomials when dividing by divisors of the form

$$x - k.$$

The shortcut is called **synthetic division.** The pattern for synthetic division of a cubic polynomial is summarized as follows. (The pattern for higher-degree polynomials is similar.)

Synthetic Division (of a Cubic Polynomial)

To divide $ax^3 + bx^2 + cx + d$ by $x - k$, use the following pattern.



Vertical pattern: Add terms.
Diagonal pattern: Multiply by k.

This algorithm for synthetic division works *only* for divisors of the form $x - k$. Remember that

$$x + k = x - (-k).$$

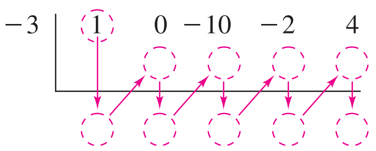
EXAMPLE 4 Using Synthetic Division

Use synthetic division to divide

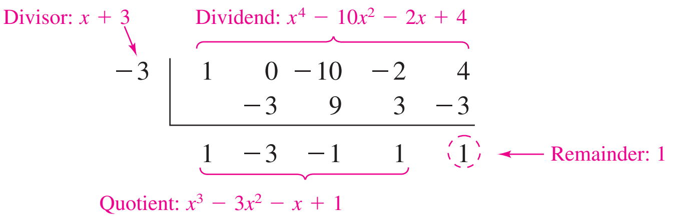
$$x^4 - 10x^2 - 2x + 4 \quad \text{by} \quad x + 3.$$

Solution

You should set up the array as follows. Note that a zero is included for the missing term in the dividend.



Then, use the synthetic division pattern by adding terms in columns and multiplying the results by -3.



So, you have

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}.$$

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Use synthetic division to divide $5x^3 + 8x^2 - x + 6$ by $x + 2$.

Point out to students that they can use a graphing utility to check the answer to a polynomial division problem. When students graph both the original polynomial division problem and the answer in the same viewing window, the graphs should coincide.

Explore the Concept

Evaluate the polynomial $x^4 - 10x^2 - 2x + 4$ when $x = -3$. What do you observe?

The Remainder and Factor Theorems

The remainder obtained in the synthetic division process has an important interpretation, as described in the **Remainder Theorem.**

> **The Remainder Theorem** (See the proof on page 176.)
>
> If a polynomial $f(x)$ is divided by $x - k$, then the remainder is
>
> $r = f(k)$.

The Remainder Theorem tells you that synthetic division can be used to evaluate a polynomial function. That is, to evaluate a polynomial $f(x)$ when $x = k$, divide $f(x)$ by $x - k$. The remainder will be $f(k)$.

EXAMPLE 5 Using the Remainder Theorem

Use the Remainder Theorem to evaluate

$$f(x) = 3x^3 + 8x^2 + 5x - 7$$

when $x = -2$.

Solution

Using synthetic division, you obtain the following.

$$\begin{array}{r|rrrr} -2 & 3 & 8 & 5 & -7 \\ & & -6 & -4 & -2 \\ \hline & 3 & 2 & 1 & -9 \end{array}$$

Because the remainder is $r = -9$, you can conclude that

$$f(-2) = -9. \qquad r = f(k)$$

This means that $(-2, -9)$ is a point on the graph of f. You can check this by substituting $x = -2$ in the original function.

Check

$$\begin{aligned} f(-2) &= 3(-2)^3 + 8(-2)^2 + 5(-2) - 7 \\ &= 3(-8) + 8(4) - 10 - 7 \\ &= -24 + 32 - 10 - 7 \\ &= -9 \end{aligned}$$

Additional Example
Use the Remainder Theorem to evaluate $f(x) = 4x^2 - 10x - 21$ when $x = 5$.
Solution
Using synthetic division, you obtain the following.

$$\begin{array}{r|rrr} 5 & 4 & -10 & -21 \\ & & 20 & 50 \\ \hline & 4 & 10 & 29 \end{array}$$

Because the remainder is 29, you can conclude that $f(5) = 29$.

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Use the Remainder Theorem to find each function value given

$$f(x) = 4x^3 + 10x^2 - 3x - 8.$$

a. $f(-1)$ **b.** $f(4)$
c. $f(\frac{1}{2})$ **d.** $f(-3)$

Another important theorem is the **Factor Theorem.** This theorem states that you can test whether a polynomial has $(x - k)$ as a factor by evaluating the polynomial at $x = k$. If the result is 0, then $(x - k)$ is a factor.

> **The Factor Theorem** (See the proof on page 176.)
>
> A polynomial $f(x)$ has a factor $(x - k)$ if and only if $f(k) = 0$.

EXAMPLE 6 Factoring a Polynomial: Repeated Division

Show that $(x - 2)$ and $(x + 3)$ are factors of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18.$$

Then find the remaining factors of $f(x)$.

Algebraic Solution

Using synthetic division with the factor $(x - 2)$, you obtain the following.

2	2	7	−4	−27	−18
		4	22	36	18
	2	11	18	9	0

0 remainder; $(x - 2)$ is a factor.

Take the result of this division and perform synthetic division again using the factor $(x + 3)$.

−3	2	11	18	9
		−6	−15	−9
	2	5	3	0

$2x^2 + 5x + 3$

0 remainder; $(x + 3)$ is a factor.

Because the resulting quadratic factors as

$$2x^2 + 5x + 3 = (2x + 3)(x + 1)$$

the complete factorization of $f(x)$ is

$$f(x) = (x - 2)(x + 3)(2x + 3)(x + 1).$$

Graphical Solution

From the graph of

$$f(x) = 2x^4 + 7x^3 - 4x^2 - 27x - 18$$

you can see that there are four x-intercepts (see Figure 2.18). These occur at $x = -3$, $x = -\frac{3}{2}$, $x = -1$, and $x = 2$. (Check this algebraically.) This implies that $(x + 3)$, $\left(x + \frac{3}{2}\right)$, $(x + 1)$, and $(x - 2)$ are factors of $f(x)$. $\left[\text{Note that } \left(x + \frac{3}{2}\right) \text{ and } (2x + 3)\right.$ are equivalent factors because they both yield the same zero, $\left. x = -\frac{3}{2}.\right]$

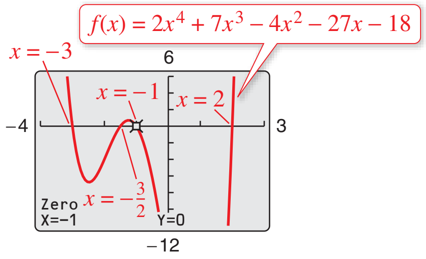

Figure 2.18

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Show that $(x + 3)$ is a factor of $f(x) = x^3 - 19x - 30$. Then find the remaining factors of $f(x)$.

Note in Example 6 that the complete factorization of $f(x)$ implies that f has four real zeros:

$$x = 2, \quad x = -3, \quad x = -\frac{3}{2}, \quad \text{and} \quad x = -1.$$

This is confirmed by the graph of f, which is shown in Figure 2.18.

Using the Remainder in Synthetic Division

In summary, the remainder r, obtained in the synthetic division of a polynomial $f(x)$ by $x - k$, provides the following information.

1. The remainder r gives the *value* of f at $x = k$. That is, $r = f(k)$.
2. If $r = 0$, then $(x - k)$ is a *factor* of $f(x)$.
3. If $r = 0$, then $(k, 0)$ is an *x-intercept* of the graph of f.

Throughout this text, the importance of developing several problem-solving strategies is emphasized. In the exercises for this section, try using more than one strategy to solve several of the exercises. For instance, when you find that $x - k$ divides evenly into $f(x)$, try sketching the graph of f. You should find that $(k, 0)$ is an x-intercept of the graph.

The Rational Zero Test

The **Rational Zero Test** relates the possible rational zeros of a polynomial (having integer coefficients) to the leading coefficient and to the constant term of the polynomial.

> **The Rational Zero Test**
>
> If the polynomial
>
> $$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$$
>
> has integer coefficients, then every rational zero of f has the form
>
> $$\text{Rational zero} = \frac{p}{q}$$
>
> where p and q have no common factors other than 1, p is a factor of the constant term a_0, and q is a factor of the leading coefficient a_n.

To use the Rational Zero Test, first list all rational numbers whose numerators are factors of the constant term and whose denominators are factors of the leading coefficient.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Now that you have formed this list of *possible rational zeros,* use a trial-and-error method to determine which, if any, are actual zeros of the polynomial. Note that when the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term. This case is illustrated in Example 7.

EXAMPLE 7 Rational Zero Test with Leading Coefficient of 1

Find the rational zeros of $f(x) = x^3 + x + 1$.

Solution

Because the leading coefficient is 1, the possible rational zeros are simply the factors of the constant term.

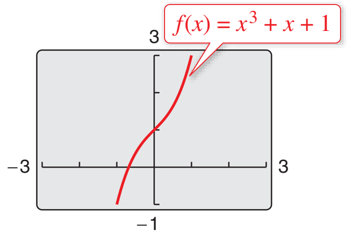
Possible rational zeros: ± 1

By testing these possible zeros, you can see that neither works.

$$f(1) = (1)^3 + 1 + 1 = 3$$

$$f(-1) = (-1)^3 + (-1) + 1 = -1$$

So, you can conclude that the polynomial has *no* rational zeros. Note from the graph of f shown below that f does have one real zero between -1 and 0. However, by the Rational Zero Test, you know that this real zero is *not* a rational number.



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Find the rational zeros of $f(x) = x^3 - 5x^2 + 2x + 8$.

> **Remark**
>
> Use a graphing utility to graph the polynomial
>
> $$y = x^3 - 53x^2 + 103x - 51$$
>
> in a standard viewing window. From the graph alone, it appears that there is only one zero. From the Leading Coefficient Test, you know that because the degree of the polynomial is odd and the leading coefficient is positive, the graph falls to the left and rises to the right. So, the function must have another zero. From the Rational Zero Test, you know that ± 51 might be zeros of the function. When you zoom out several times, you will see a more complete picture of the graph. Your graph should confirm that $x = 51$ is a zero of f.

When the leading coefficient of a polynomial is not 1, the list of possible rational zeros can increase dramatically. In such cases, the search can be shortened in several ways.

1. A graphing utility can be used to speed up the calculations.
2. A graph, drawn either by hand or with a graphing utility, can give good estimates of the locations of the zeros.
3. The Intermediate Value Theorem, along with a table generated by a graphing utility, can give approximations of zeros.
4. The Factor Theorem and synthetic division can be used to test the possible rational zeros.

Finding the first zero is often the most difficult part. After that, the search is simplified by working with the lower-degree polynomial obtained in synthetic division, as shown in Example 8.

EXAMPLE 8 Using the Rational Zero Test

Find the rational zeros of

$$f(x) = 2x^3 + 3x^2 - 8x + 3.$$

Solution

The leading coefficient is 2, and the constant term is 3.

Possible rational zeros: $\dfrac{\text{Factors of } 3}{\text{Factors of } 2} = \dfrac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm\dfrac{1}{2}, \pm\dfrac{3}{2}$

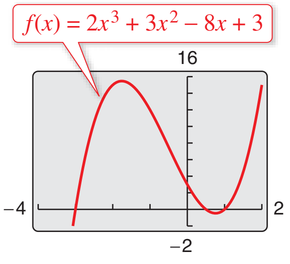
By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$\begin{aligned} f(x) &= (x - 1)(2x^2 + 5x - 3) \\ &= (x - 1)(2x - 1)(x + 3) \end{aligned}$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$, as shown in the figure.



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Find the rational zeros of $f(x) = 2x^4 - 9x^3 - 18x^2 + 71x - 30$.

Remember that when you try to find the rational zeros of a polynomial function with many possible rational zeros, as in Example 8, you must use trial and error. There is no quick algebraic method to determine which of the possibilities is an actual zero; however, sketching a graph may be helpful.

Activities

1. Use synthetic division to determine whether $(x + 3)$ is a factor of $f(x) = 3x^3 + 4x^2 - 18x - 3$.
   Answer: No
2. Divide using long division.
   $\dfrac{4x^5 - x^3 + 2x^2 - x}{2x + 1}$
   Answer:
   $2x^4 - x^3 + x - 1 + \dfrac{1}{2x + 1}$
3. Use the Remainder Theorem to evaluate $f(-3)$ for $f(x) = 2x^3 - 4x^2 + 1$.
   Answer: -89
4. Use the Rational Zero Test to find all the possible rational zeros of $f(x) = 6x^3 - x^2 + 9x + 4$.
   Answer: $\pm 1, \pm 2, \pm 4, \pm\frac{1}{6}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm\frac{4}{3}$

Other Tests for Zeros of Polynomials

You know that an nth-degree polynomial function can have *at most n* real zeros. Of course, many nth-degree polynomials do not have that many real zeros. For instance, $f(x) = x^2 + 1$ has no real zeros, and $f(x) = x^3 + 1$ has only one real zero. The following theorem, called **Descartes's Rule of Signs,** sheds more light on the number of real zeros of a polynomial.

Decartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$.

1. The number of *positive real zeros* of f is either equal to the number of variations in sign of $f(x)$ or less than that number by an even integer.
2. The number of *negative real zeros* of f is either equal to the number of variations in sign of $f(-x)$ or less than that number by an even integer.

A **variation in sign** means that two consecutive coefficients have opposite signs. Missing terms (those with zero coefficients) can be ignored.

When using Descartes's Rule of Signs, a zero of multiplicity k should be counted as k zeros. For instance, the polynomial $x^3 - 3x + 2$ has two variations in sign, and so has either two positive or no positive real zeros. Because

$$x^3 - 3x + 2 = (x - 1)(x - 1)(x + 2)$$

you can see that the two positive real zeros are $x = 1$ of multiplicity 2.

EXAMPLE 9 Using Descartes's Rule of Signs

Determine the possible numbers of positive and negative real zeros of

$$f(x) = 3x^3 - 5x^2 + 6x - 4.$$

Solution

The original polynomial has *three* variations in sign.

+ to −   + to −

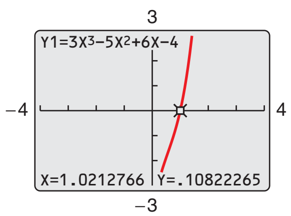
$$f(x) = 3x^3 - 5x^2 + 6x - 4$$

− to +

The polynomial

$$f(-x) = 3(-x)^3 - 5(-x)^2 + 6(-x) - 4$$
$$= -3x^3 - 5x^2 - 6x - 4$$

has no variations in sign. So, from Descartes's Rule of Signs, the polynomial $f(x) = 3x^3 - 5x^2 + 6x - 4$ has either three positive real zeros or one positive real zero, and has no negative real zeros. By using the *trace* feature of a graphing utility, you can see that the function has only one real zero (it is a positive number near $x = 1$), as shown in the figure.



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Determine the possible numbers of positive and negative real zeros of

$$f(x) = -2x^3 + 5x^2 - x + 8.$$

Another test for zeros of a polynomial function is related to the sign pattern in the last row of the synthetic division array. This test can give you an upper or lower bound for the real zeros of f, which can help you eliminate possible real zeros. A real number c is an **upper bound** for the real zeros of f when no zeros are greater than c. Similarly, c is a **lower bound** when no real zeros of f are less than c.

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, then c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), then c is a **lower bound** for the real zeros of f.

EXAMPLE 10 Finding the Zeros of a Polynomial Function

Find all real zeros of $f(x) = 6x^3 - 4x^2 + 3x - 2$.

Solution

The possible real zeros are as follows.

$$\frac{\text{Factors of } -2}{\text{Factors of } 6} = \frac{\pm 1, \pm 2}{\pm 1, \pm 2, \pm 3, \pm 6} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm \frac{2}{3}, \pm 2$$

The original polynomial $f(x)$ has three variations in sign. The polynomial

$$f(-x) = 6(-x)^3 - 4(-x)^2 + 3(-x) - 2$$
$$= -6x^3 - 4x^2 - 3x - 2$$

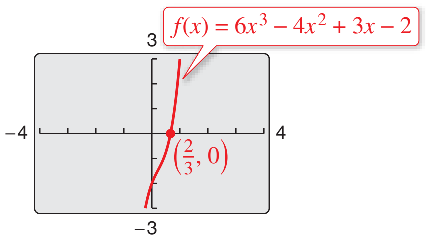
has no variations in sign. So, you can apply Descartes's Rule of Signs to conclude that there are either three positive real zeros or one positive real zero, and no negative real zeros. Trying $x = 1$ produces the following.

$$\begin{array}{r|rrrr} 1 & 6 & -4 & 3 & -2 \\ & & 6 & 2 & 5 \\ \hline & 6 & 2 & 5 & 3 \end{array}$$

So, $x = 1$ is not a zero, but because the last row has all positive entries, you know that $x = 1$ is an upper bound for the real zeros. Therefore, you can restrict the search to zeros between 0 and 1. By trial and error, you can determine that $x = \frac{2}{3}$ is a zero. So,

$$f(x) = \left(x - \frac{2}{3}\right)(6x^2 + 3).$$

Because $6x^2 + 3$ has no real zeros, it follows that $x = \frac{2}{3}$ is the only real zero, as shown in the figure.



Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find all real zeros of

$$f(x) = 8x^3 - 4x^2 + 6x - 3.$$

Explore the Concept

Use a graphing utility to graph the polynomial

$$y_1 = 6x^3 - 4x^2 + 3x - 2.$$

Notice that the graph intersects the x-axis at the point $\left(\frac{2}{3}, 0\right)$. How does this information relate to the real zero found in Example 10? Use the graphing utility to graph

$$y_2 = x^4 - 5x^3 + 3x^2 + x.$$

How many times does the graph intersect the x-axis? How many real zeros does y_2 have?

Here are two additional hints that can help you find the real zeros of a polynomial.

1. When the terms of $f(x)$ have a common monomial factor, it should be factored out before applying the tests in this section. For instance, by writing

   $$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x^3 - 5x^2 + 3x + 1)$$

   you can see that $x = 0$ is a zero of f and that the remaining zeros can be obtained by analyzing the cubic factor.

2. When you are able to find all but two zeros of f, you can always use the Quadratic Formula on the remaining quadratic factor. For instance, after writing

   $$f(x) = x^4 - 5x^3 + 3x^2 + x = x(x - 1)(x^2 - 4x - 1)$$

   you can apply the Quadratic Formula to $x^2 - 4x - 1$ to conclude that the two remaining zeros are $x = 2 + \sqrt{5}$ and $x = 2 - \sqrt{5}$.

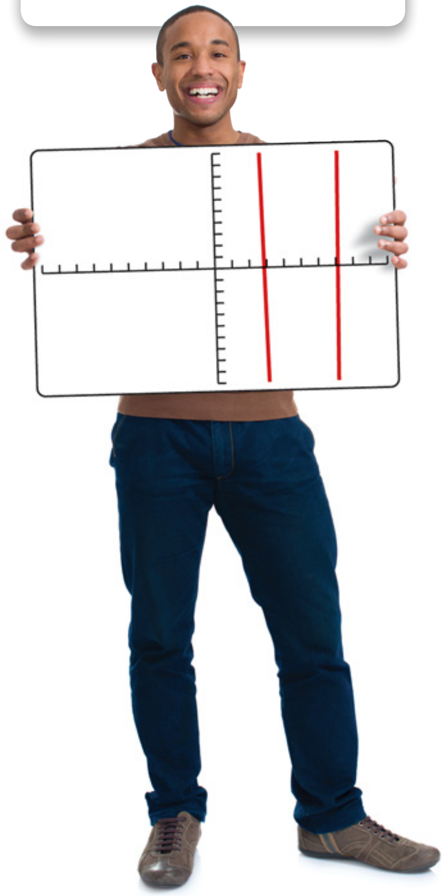
Note how these hints are applied in the next example.

Explore the Concept

Use a graphing utility to graph the polynomial

$$y = x^3 + 4.8x^2 - 127x + 309$$

in a standard viewing window. From the graph, what do the real zeros appear to be? Discuss how the mathematical tools of this section might help you realize that the graph does not show all the important features of the polynomial function. Now use the *zoom* feature to find all the zeros of this function.



EXAMPLE 11 Finding the Zeros of a Polynomial Function

See LarsonPrecalculus.com for an interactive version of this type of example.

Find all real zeros of $f(x) = 10x^4 - 15x^3 - 16x^2 + 12x$.

Solution

Remove the common monomial factor x to write

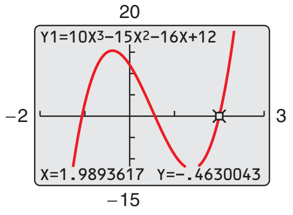
$$f(x) = 10x^4 - 15x^3 - 16x^2 + 12x = x(10x^3 - 15x^2 - 16x + 12).$$

So, $x = 0$ is a zero of f. You can find the remaining zeros of f by analyzing the cubic factor. Because the leading coefficient is 10 and the constant term is 12, there is a long list of possible rational zeros.

Possible rational zeros:

$$\frac{\text{Factors of 12}}{\text{Factors of 10}} = \frac{\pm1, \pm2, \pm3, \pm4, \pm6, \pm12}{\pm1, \pm2, \pm5, \pm10}$$

With so many possibilities (32, in fact), it is worth your time to use a graphing utility to focus on just a few. By using the *trace* feature of a graphing utility, it looks like three reasonable choices are $x = -\frac{6}{5}$, $x = \frac{1}{2}$, and $x = 2$ (see figure). Synthetic division shows that only $x = 2$ works. (You could also use the Factor Theorem to test these choices.)



$$\begin{array}{r|rrrr} 2 & 10 & -15 & -16 & 12 \\ & & 20 & 10 & -12 \\ \hline & 10 & 5 & -6 & 0 \end{array}$$

So, $x = 2$ is one zero and you have

$$f(x) = x(x - 2)(10x^2 + 5x - 6).$$

Using the Quadratic Formula, you find that the two additional zeros are irrational numbers.

$$x = \frac{-5 + \sqrt{265}}{20} \approx 0.56 \quad \text{and} \quad x = \frac{-5 - \sqrt{265}}{20} \approx -1.06$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find all real zeros of $f(x) = 3x^4 - 14x^2 - 4x$.

2.3 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises. For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Two forms of the Division Algorithm are shown below. Identify and label each part.

$$f(x) = d(x)q(x) + r(x) \qquad \frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

In Exercises 2–5, fill in the blank(s).

2. The rational expression $p(x)/q(x)$ is called _______ when the degree of the numerator is greater than or equal to that of the denominator.

3. Every rational zero of a polynomial function with integer coefficients has the form p/q, where p is a factor of the _______ and q is a factor of the _______ .

4. The theorem that can be used to determine the possible numbers of positive real zeros and negative real zeros of a function is called _______ of _______ .

5. A real number c is a(n) _______ bound for the real zeros of f when no zeros are greater than c, and is a(n) _______ bound when no real zeros of f are less than c.

6. How many negative real zeros are possible for a polynomial function f, given that $f(-x)$ has 5 variations in sign?

7. You divide the polynomial $f(x)$ by $(x - 4)$ and obtain a remainder of 7. What is $f(4)$?

8. What value should you write in the circle to check whether $(x - 2)$ is a factor of $f(x) = x^3 + 6x^2 - 5x + 3$?

○ | 1   6   −5   3

Procedures and Problem Solving

Long Division of Polynomials **In Exercises 9–12, use long division to divide and use the result to factor the dividend completely.**

9. $(x^2 + 5x + 6) \div (x + 3)$

10. $(5x^2 - 17x - 12) \div (x - 4)$

11. $(x^3 + 5x^2 - 12x - 36) \div (x + 2)$

12. $(2x^3 - 3x^2 - 50x + 75) \div (2x - 3)$

Long Division of Polynomials **In Exercises 13–22, use long division to divide.**

13. $(x^3 - 4x^2 - 17x + 6) \div (x - 3)$

14. $(4x^3 - 7x^2 - 11x + 5) \div (4x + 5)$

15. $(7x^3 + 3) \div (x + 2)$ **16.** $(8x^4 - 5) \div (2x + 1)$

17. $(5x - 1 + 10x^3 - 2x^2) \div (2x^2 + 1)$

18. $(1 + 3x^2 + x^4) \div (3 - 2x + x^2)$

19. $(x^3 - 9) \div (x^2 + 1)$ **20.** $(x^5 + 7) \div (x^3 - 1)$

21. $\dfrac{2x^3 - 4x^2 - 15x + 5}{(x - 1)^2}$ **22.** $\dfrac{x^4}{(x - 1)^3}$

Using Synthetic Division **In Exercises 23–32, use synthetic division to divide.**

23. $(3x^3 - 17x^2 + 15x - 25) \div (x - 5)$

24. $(5x^3 + 18x^2 + 7x - 6) \div (x + 3)$

25. $(6x^3 + 7x^2 - x + 26) \div (x - 3)$

26. $(2x^3 + 14x^2 - 20x + 7) \div (x + 6)$

27. $(9x^3 - 18x^2 - 16x + 32) \div (x - 2)$

28. $(5x^3 + 6x + 8) \div (x + 2)$

29. $(x^3 + 512) \div (x + 8)$ **30.** $(x^3 - 729) \div (x - 9)$

31. $\dfrac{4x^3 + 16x^2 - 23x - 15}{x + \frac{1}{2}}$ **32.** $\dfrac{3x^3 - 4x^2 + 5}{x - \frac{3}{2}}$

Verifying Quotients **In Exercises 33–36, use a graphing utility to graph the two equations in the same viewing window. Use the graphs to verify that the expressions are equivalent. Verify the results algebraically.**

33. $y_1 = \dfrac{x^2}{x + 2}, \quad y_2 = x - 2 + \dfrac{4}{x + 2}$

34. $y_1 = \dfrac{x^2 + 2x - 1}{x + 3}, \quad y_2 = x - 1 + \dfrac{2}{x + 3}$

35. $y_1 = \dfrac{x^4 - 3x^2 - 1}{x^2 + 5}, \quad y_2 = x^2 - 8 + \dfrac{39}{x^2 + 5}$

36. $y_1 = \dfrac{x^4 + x^2 - 1}{x^2 + 1}, \quad y_2 = x^2 - \dfrac{1}{x^2 + 1}$

Verifying the Remainder Theorem **In Exercises 37–42, write the function in the form $f(x) = (x - k)q(x) + r(x)$ for the given value of k. Use a graphing utility to demonstrate that $f(k) = r$.**

	Function	*Value of k*
37.	$f(x) = x^3 - x^2 - 14x + 11$	$k = 4$
38.	$f(x) = 15x^4 + 10x^3 - 6x^2 + 14$	$k = -\frac{2}{3}$

Function	*Value of k*
39. $f(x) = x^3 + 3x^2 - 2x - 14$	$k = \sqrt{2}$
40. $f(x) = x^3 + 2x^2 - 5x - 4$	$k = -\sqrt{5}$
41. $f(x) = 4x^3 - 6x^2 - 12x - 4$	$k = 1 - \sqrt{3}$
42. $f(x) = -3x^3 + 8x^2 + 10x - 8$	$k = 2 + \sqrt{2}$

Using the Remainder Theorem **In Exercises 43–46, use the Remainder Theorem and synthetic division to evaluate the function at each given value. Use a graphing utility to verify your results.**

43. $f(x) = 2x^3 - 7x + 3$

(a) $f(1)$ (b) $f(-2)$ (c) $f\left(\frac{1}{2}\right)$ (d) $f(2)$

44. $g(x) = 2x^6 + 3x^4 - x^2 + 3$

(a) $g(2)$ (b) $g(1)$ (c) $g(3)$ (d) $g(-1)$

45. $h(x) = x^3 - 5x^2 - 7x + 4$

(a) $h(3)$ (b) $h(2)$ (c) $h(-2)$ (d) $h(-5)$

46. $f(x) = 4x^4 - 16x^3 + 7x^2 + 20$

(a) $f(1)$ (b) $f(-2)$ (c) $f(5)$ (d) $f(-10)$

Using the Factor Theorem **In Exercises 47–52, use synthetic division to show that x is a solution of the third-degree polynomial equation, and use the result to factor the polynomial completely. List all the real solutions of the equation.**

Polynomial Equation	*Value of x*
47. $x^3 - 13x - 12 = 0$	$x = 4$
48. $x^3 - 31x + 30 = 0$	$x = -6$
49. $2x^3 - 17x^2 + 12x + 63 = 0$	$x = -\frac{3}{2}$
50. $60x^3 - 89x^2 + 41x - 6 = 0$	$x = \frac{1}{3}$
51. $x^3 + 2x^2 - 3x - 6 = 0$	$x = \sqrt{3}$
52. $x^3 - x^2 - 13x - 3 = 0$	$x = 2 - \sqrt{5}$

Factoring a Polynomial **In Exercises 53–58, (a) verify the given factor(s) of the function f, (b) find the remaining factors of f, (c) use your results to write the complete factorization of f, and (d) list all real zeros of f. Confirm your results by using a graphing utility to graph the function.**

Function	*Factor(s)*
53. $f(x) = 2x^3 + x^2 - 5x + 2$	$(x + 2)$
54. $f(x) = 3x^3 + 2x^2 - 19x + 6$	$(x + 3)$
55. $f(x) = x^4 - 4x^3 - 15x^2 + 58x - 40$	$(x - 5), (x + 4)$
56. $f(x) = 8x^4 - 14x^3 - 71x^2 - 10x + 24$	$(x + 2), (x - 4)$
57. $f(x) = 6x^3 + 41x^2 - 9x - 14$	$(2x + 1)$
58. $f(x) = 2x^3 - x^2 - 10x + 5$	$(2x - 1)$

Using the Rational Zero Test **In Exercises 59–62, use the Rational Zero Test to list all possible rational zeros of f. Then find the rational zeros.**

59. $f(x) = x^3 + 3x^2 - x - 3$

60. $f(x) = x^3 - 4x^2 - 4x + 16$

61. $f(x) = 2x^4 - 17x^3 + 35x^2 + 9x - 45$

62. $f(x) = 4x^5 - 8x^4 - 5x^3 + 10x^2 + x - 2$

Using Descartes's Rule of Signs **In Exercises 63–66, use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of the function.**

63. $f(x) = 2x^4 - x^3 + 6x^2 - x + 5$

64. $f(x) = 3x^4 + 5x^3 - 6x^2 + 8x - 3$

65. $g(x) = 4x^3 - 5x + 8$

66. $g(x) = 2x^3 - 4x^2 - 5$

Finding the Zeros of a Polynomial Function **In Exercises 67–72, (a) use Descartes's Rule of Signs to determine the possible numbers of positive and negative real zeros of f, (b) list the possible rational zeros of f, (c) use a graphing utility to graph f so that some of the possible zeros in parts (a) and (b) can be disregarded, and (d) determine all the real zeros of f.**

67. $f(x) = x^3 + x^2 - 4x - 4$

68. $f(x) = -3x^3 + 20x^2 - 36x + 16$

69. $f(x) = -2x^4 + 13x^3 - 21x^2 + 2x + 8$

70. $f(x) = 4x^4 - 17x^2 + 4$

71. $f(x) = 32x^3 - 52x^2 + 17x + 3$

72. $f(x) = x^4 - x^3 - 29x^2 - x - 30$

Finding the Zeros of a Polynomial Function **In Exercises 73–76, use synthetic division to verify the upper and lower bounds of the real zeros of f. Then find all real zeros of the function.**

73. $f(x) = x^4 - 4x^3 + 15$

Upper bound: $x = 4$

Lower bound: $x = -1$

74. $f(x) = 2x^3 - 3x^2 - 12x + 8$

Upper bound: $x = 4$

Lower bound: $x = -3$

75. $f(x) = x^4 - 4x^3 + 16x - 16$

Upper bound: $x = 5$

Lower bound: $x = -3$

76. $f(x) = 2x^4 - 8x + 3$

Upper bound: $x = 3$

Lower bound: $x = -4$

Occasionally, throughout this text, you will be asked to round to a place value rather than to a number of decimal places.

Rewriting to Use the Rational Zero Test In Exercises 77–80, find the rational zeros of the polynomial function.

77. $P(x) = x^4 - \frac{25}{4}x^2 + 9 = \frac{1}{4}(4x^4 - 25x^2 + 36)$

78. $f(x) = x^3 - \frac{3}{2}x^2 - \frac{23}{2}x + 6 = \frac{1}{2}(2x^3 - 3x^2 - 23x + 12)$

79. $f(x) = x^3 - \frac{1}{4}x^2 - x + \frac{1}{4} = \frac{1}{4}(4x^3 - x^2 - 4x + 1)$

80. $f(z) = z^3 + \frac{11}{6}z^2 - \frac{1}{2}z - \frac{1}{3} = \frac{1}{6}(6z^3 + 11z^2 - 3z - 2)$

A Cubic Polynomial with Two Terms In Exercises 81–84, match the cubic function with the correct number of rational and irrational zeros.

(a) Rational zeros: 0; Irrational zeros: 1

(b) Rational zeros: 3; Irrational zeros: 0

(c) Rational zeros: 1; Irrational zeros: 2

(d) Rational zeros: 1; Irrational zeros: 0

81. $f(x) = x^3 - 1$

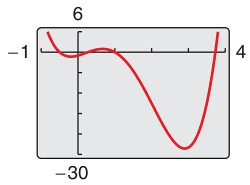
82. $f(x) = x^3 - 2$

83. $f(x) = x^3 - x$

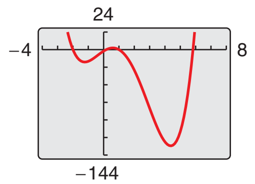
84. $f(x) = x^3 - 2x$

Using a Graph to Help Find Zeros In Exercises 85–88, the graph of $y = f(x)$ is shown. Use the graph as an aid to find all real zeros of the function.

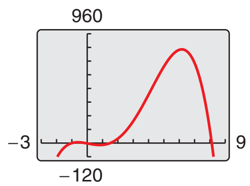
85. $y = 2x^4 - 9x^3 + 5x^2 + 3x - 1$



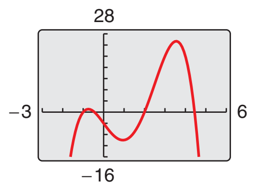
86. $y = x^4 - 5x^3 - 7x^2 + 13x - 2$



87. $y = -2x^4 + 17x^3 - 3x^2 - 25x - 3$



88. $y = -x^4 + 5x^3 - 10x - 4$



Finding the Zeros of a Polynomial Function In Exercises 89–100, find all real zeros of the polynomial function.

89. $f(x) = 5x^4 + 9x^3 - 19x^2 - 3x$

90. $g(x) = 4x^4 - 11x^3 - 22x^2 + 8x$

91. $f(z) = z^4 - z^3 - 2z - 4$

92. $f(x) = 4x^3 + 7x^2 - 11x - 18$

93. $g(y) = 2y^4 + 7y^3 - 26y^2 + 23y - 6$

94. $h(x) = x^5 - x^4 - 3x^3 + 5x^2 - 2x$

95. $f(x) = 4x^4 - 55x^2 - 45x + 36$

96. $z(x) = 6x^4 + 33x^3 - 69x + 30$

97. $g(x) = 8x^4 + 28x^3 + 9x^2 - 9x$

98. $h(x) = x^5 + 5x^4 - 5x^3 - 15x^2 - 6x$

99. $f(x) = 8x^5 + 6x^4 - 37x^3 - 36x^2 + 29x + 30$

100. $g(x) = 4x^5 + 8x^4 - 15x^3 - 23x^2 + 11x + 15$

Using a Rational Zero In Exercises 101–104, (a) use the *zero* or *root* feature of a graphing utility to approximate (accurate to the nearest thousandth) the zeros of the function, (b) determine one of the exact zeros and use synthetic division to verify your result, and (c) factor the polynomial completely.

101. $h(t) = t^3 - 2t^2 - 7t + 2$

102. $f(s) = s^3 - 12s^2 + 40s - 24$

103. $h(x) = x^5 - 7x^4 + 10x^3 + 14x^2 - 24x$

104. $g(x) = 6x^4 - 11x^3 - 51x^2 + 99x - 27$

105. MODELING DATA

The table shows the numbers S of cellular phone subscriptions per 100 people in the United States from 1995 through 2012. (Source: International Telecommunications Union)

DATA

Year	Subscriptions per 100 people, S
1995	12.6
1996	16.2
1997	20.1
1998	24.9
1999	30.6
2000	38.5
2001	44.7
2002	48.9
2003	54.9
2004	62.6
2005	68.3
2006	76.3
2007	82.1
2008	85.2
2009	88.6
2010	91.3
2011	94.7
2012	95.5

Spreadsheet at LarsonPrecalculus.com

The data can be approximated by the model

$$S = -0.0223t^3 + 0.825t^2 - 3.58t + 12.6, \quad 5 \le t \le 22$$

where t represents the year, with $t = 5$ corresponding to 1995.

(a) Use a graphing utility to plot the data and graph the model in the same viewing window.

(b) How well does the model fit the data?

(c) Use the Remainder Theorem to evaluate the model for the year 2020. Is the value reasonable? Explain.

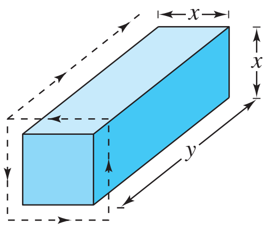
106. ***Why you should learn it*** *(p. 113)* The numbers of employees E (in thousands) in education and health services in the United States from 1960 through 2013 are approximated by $E = -0.088t^3 + 10.77t^2 + 14.6t + 3197$, $0 \le t \le 53$, where t is the year, with $t = 0$ corresponding to 1960. (Source: U.S. Bureau of Labor Statistics)

(a) Use a graphing utility to graph the model over the domain.

(b) Estimate the number of employees in education and health services in 1960. Use the Remainder Theorem to estimate the number in 2010.

(c) Is this a good model for making predictions in future years? Explain.

107. Geometry A rectangular package sent by a delivery service can have a maximum combined length and girth (perimeter of a cross section) of 120 inches (see figure).



(a) Show that the volume of the package is given by the function $V(x) = 4x^2(30 - x)$.

(b) Use a graphing utility to graph the function and approximate the dimensions of the package that yield a maximum volume.

(c) Find values of x such that $V = 13{,}500$. Which of these values is a physical impossibility in the construction of the package? Explain.

108. Environmental Science The number of parts per million of nitric oxide emissions y from a car engine is approximated by $y = -5.05x^3 + 3857x - 38{,}411.25$, $13 \le x \le 18$, where x is the air-fuel ratio.

(a) Use a graphing utility to graph the model.

(b) There are two air-fuel ratios that produce 2400 parts per million of nitric oxide. One is $x = 15$. Use the graph to approximate the other.

(c) Find the second air-fuel ratio from part (b) algebraically. (*Hint:* Use the known value of $x = 15$ and synthetic division.)

Conclusions

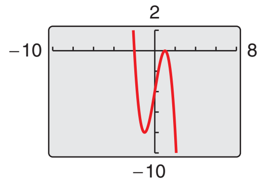
True or False? **In Exercises 109 and 110, determine whether the statement is true or false. Justify your answer.**

109. If $(7x + 4)$ is a factor of some polynomial function f, then $\frac{4}{7}$ is a zero of f.

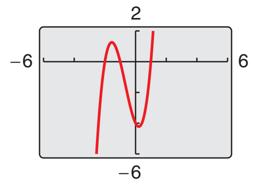
110. The value $x = \frac{1}{7}$ is a zero of the polynomial function $f(x) = 3x^5 - 2x^4 + x^3 - 16x^2 + 3x - 8$.

Think About It **In Exercises 111 and 112, the graph of a cubic polynomial function $y = f(x)$ with integer zeros is shown. Find the factored form of f.**

111.



112.



113. Think About It Let $y = f(x)$ be a fourth-degree polynomial with leading coefficient $a = -1$ and $f(\pm 1) = f(\pm 2) = 0$. Find the factored form of f.

114. Think About It Find the value of k such that $x - 3$ is a factor of $x^3 - kx^2 + 2kx - 12$.

115. Writing Complete each polynomial division. Write a brief description of the pattern that you obtain, and use your result to find a formula for the polynomial division $(x^n - 1)/(x - 1)$. Create a numerical example to test your formula.

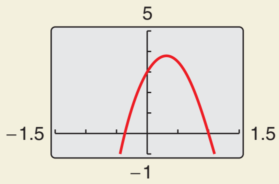
(a) $\dfrac{x^2 - 1}{x - 1} = \square$ (b) $\dfrac{x^3 - 1}{x - 1} = \square$

(c) $\dfrac{x^4 - 1}{x - 1} = \square$

116. HOW DO YOU SEE IT? A graph of $y = f(x)$ is shown, where

$f(x) = 2x^5 - 3x^4 + x^3 - 8x^2 + 5x + 3$ and

$f(-x) = -2x^5 - 3x^4 - x^3 - 8x^2 - 5x + 3$.



(a) How many negative real zeros does f have? Explain.

(b) How many positive real zeros are *possible* for f? Explain. What does this tell you about the eventual right-hand behavior of the graph?

(c) Is $x = -\frac{1}{3}$ a possible rational zero of f? Explain.

(d) Explain how to check whether $\left(x - \frac{3}{2}\right)$ is a factor of f and whether $x = \frac{3}{2}$ is an upper bound for the real zeros of f.

Cumulative Mixed Review

Solving a Quadratic Equation **In Exercises 117–120, use any convenient method to solve the quadratic equation.**

117. $4x^2 - 17 = 0$

118. $25x^2 - 1 = 0$

119. $3x^2 - 11x - 20 = 0$

120. $6x^2 + 4x - 3 = 0$

2.4 Complex Numbers

The Imaginary Unit i

Some quadratic equations have no real solutions. For instance, the quadratic equation $x^2 + 1 = 0$ has no real solution because there is no real number x that can be squared to produce -1. To overcome this deficiency, mathematicians created an expanded system of numbers using the **imaginary unit *i*,** defined as

$$i = \sqrt{-1} \qquad \text{Imaginary unit}$$

where $i^2 = -1$. By adding real numbers to real multiples of this imaginary unit, you obtain the set of **complex numbers.** Each complex number can be written in the **standard form** $a + bi$. For instance, the standard form of the complex number $\sqrt{-9} - 5$ is $-5 + 3i$ because

$$\begin{aligned}\sqrt{-9} - 5 &= \sqrt{3^2(-1)} - 5\\ &= 3\sqrt{-1} - 5\\ &= 3i - 5\\ &= -5 + 3i.\end{aligned}$$

In the standard form $a + bi$, the real number a is called the **real part** of the **complex number** $a + bi$, and the number bi (where b is a real number) is called the **imaginary part** of the complex number.

Definition of a Complex Number

If a and b are real numbers, then the number $a + bi$ is a **complex number,** and it is said to be written in **standard form.** If $b = 0$, then the number $a + bi = a$ is a real number. If $b \neq 0$, then the number $a + bi$ is called an **imaginary number.** A number of the form bi, where $b \neq 0$, is called a **pure imaginary number.**

The set of real numbers is a subset of the set of complex numbers, as shown in Figure 2.19. This is true because every real number a can be written as a complex number using $b = 0$. That is, for every real number a, you can write $a = a + 0i$.

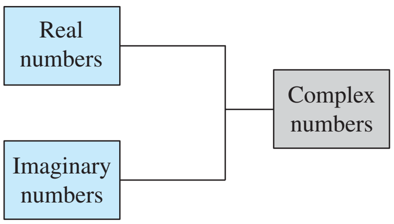


Figure 2.19

Equality of Complex Numbers

Two complex numbers $a + bi$ and $c + di$, written in standard form, are equal to each other

$$a + bi = c + di \qquad \text{Equality of two complex numbers}$$

if and only if $a = c$ and $b = d$.

What you should learn

- Use the imaginary unit i to write complex numbers.
- Add, subtract, and multiply complex numbers.
- Use complex conjugates to write the quotient of two complex numbers in standard form.
- Find complex solutions of quadratic equations.

Why you should learn it

Complex numbers are used to model numerous aspects of the natural world, such as the impedance of an electrical circuit, as shown in Exercise 89 on page 134.

Operations with Complex Numbers

To add (or subtract) two complex numbers, you add (or subtract) the real and imaginary parts of the numbers separately.

Addition and Subtraction of Complex Numbers

If $a + bi$ and $c + di$ are two complex numbers written in standard form, then their sum and difference are defined as follows.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

The **additive identity** in the complex number system is zero (the same as in the real number system). Furthermore, the **additive inverse** of the complex number $a + bi$ is

$$-(a + bi) = -a - bi. \qquad \text{Additive inverse}$$

So, you have

$$(a + bi) + (-a - bi) = 0 + 0i$$
$$= 0.$$

EXAMPLE 1 Adding and Subtracting Complex Numbers

For each operation on complex numbers, you can show the parallel operations on polynomials.

a. $(3 - i) + (2 + 3i) = 3 - i + 2 + 3i$ — Remove parentheses.

$= (3 + 2) + (-i + 3i)$ — Group like terms.

$= 5 + 2i$ — Write in standard form.

b. $(1 + 2i) - (4 + 2i) = 1 + 2i - 4 - 2i$ — Remove parentheses.

$= (1 - 4) + (2i - 2i)$ — Group like terms.

$= -3 + 0i$ — Simplify.

$= -3$ — Write in standard form.

c. $3 - (-2 + 3i) + (-5 + i) = 3 + 2 - 3i - 5 + i$

$= (3 + 2 - 5) + (-3i + i)$

$= 0 - 2i$

$= -2i$

d. $(3 + 2i) + (4 - i) - (7 + i) = 3 + 2i + 4 - i - 7 - i$

$= (3 + 4 - 7) + (2i - i - i)$

$= 0 + 0i$

$= 0$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Perform each operation and write the result in standard form.

a. $(7 + 3i) + (5 - 4i)$

b. $(3 + 4i) - (5 - 3i)$

c. $2i + (-3 - 4i) - (-3 - 3i)$

d. $(5 - 3i) + (3 + 5i) - (8 + 2i)$

In Examples 1(b) and 1(d), note that the sum of complex numbers can be a real number.

Many of the properties of real numbers are valid for complex numbers as well. Here are some examples.

Associative Properties of Addition and Multiplication

Commutative Properties of Addition and Multiplication

Distributive Property of Multiplication over Addition

Notice how these properties are used when two complex numbers are multiplied.

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + bi(c + di) && \text{Distributive Property}\\ &= ac + (ad)i + (bc)i + (bd)i^2 && \text{Distributive Property}\\ &= ac + (ad)i + (bc)i + (bd)(-1) && i^2 = -1\\ &= ac - bd + (ad)i + (bc)i && \text{Commutative Property}\\ &= (ac - bd) + (ad + bc)i && \text{Associative Property}\end{aligned}$$

The procedure above is similar to multiplying two polynomials and combining like terms, as in the FOIL Method.

EXAMPLE 2 Multiplying Complex Numbers

a. $$\begin{aligned}5(-2 + 3i) &= 5(-2) + 5(3i) && \text{Distributive Property}\\ &= -10 + 15i && \text{Simplify.}\end{aligned}$$

b. $$\begin{aligned}(2 - i)(4 + 3i) &= 2(4 + 3i) - i(4 + 3i) && \text{Distributive Property}\\ &= 8 + 6i - 4i - 3i^2 && \text{Distributive Property}\\ &= 8 + 6i - 4i - 3(-1) && i^2 = -1\\ &= 8 + 3 + 6i - 4i && \text{Group like terms.}\\ &= 11 + 2i && \text{Write in standard form.}\end{aligned}$$

c. $$\begin{aligned}(3 + 2i)(3 - 2i) &= 3(3 - 2i) + 2i(3 - 2i) && \text{Distributive Property}\\ &= 9 - 6i + 6i - 4i^2 && \text{Distributive Property}\\ &= 9 - 4(-1) && i^2 = -1\\ &= 9 + 4 && \text{Simplify.}\\ &= 13 && \text{Write in standard form.}\end{aligned}$$

d. $$\begin{aligned}4i(-1 + 5i) &= 4i(-1) + 4i(5i) && \text{Distributive Property}\\ &= -4i + 20i^2 && \text{Simplify.}\\ &= -4i + 20(-1) && i^2 = -1\\ &= -20 - 4i && \text{Write in standard form.}\end{aligned}$$

e. $$\begin{aligned}(3 + 2i)^2 &= (3 + 2i)(3 + 2i) && \text{Property of exponents}\\ &= 3(3 + 2i) + 2i(3 + 2i) && \text{Distributive Property}\\ &= 9 + 6i + 6i + 4i^2 && \text{Distributive Property}\\ &= 9 + 6i + 6i + 4(-1) && i^2 = -1\\ &= 9 + 12i - 4 && \text{Simplify.}\\ &= 5 + 12i && \text{Write in standard form.}\end{aligned}$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Perform each operation and write the result is standard form.

a. $(2 - 4i)(3 + 3i)$

b. $(4 + 5i)(4 - 5i)$

c. $(4 + 2i)^2$

Explore the Concept

Complete the following:

$i^1 = i$	$i^7 =$
$i^2 = -1$	$i^8 =$
$i^3 = -i$	$i^9 =$
$i^4 = 1$	$i^{10} =$
$i^5 =$	$i^{11} =$
$i^6 =$	$i^{12} =$

What pattern do you see? Write a brief description of how you would find i raised to any positive integer power.

Complex Conjugates

Notice in Example 2(c) that the product of two complex numbers can be a real number. This occurs with pairs of complex numbers of the forms $a + bi$ and $a - bi$, called **complex conjugates.**

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - abi + abi - b^2i^2 \\ &= a^2 - b^2(-1) \\ &= a^2 + b^2\end{aligned}$$

EXAMPLE 3 Multiplying Conjugates

Multiply each complex number by its complex conjugate.

a. $1 + i$ **b.** $4 - 3i$

Solution

a. The complex conjugate of $1 + i$ is $1 - i$.

$$(1 + i)(1 - i) = 1^2 - i^2 = 1 - (-1) = 2$$

b. The complex conjugate of $4 - 3i$ is $4 + 3i$.

$$(4 - 3i)(4 + 3i) = 4^2 - (3i)^2 = 16 - 9i^2 = 16 - 9(-1) = 25$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Multiply each complex number by its complex conjugate.

a. $3 + 6i$ **b.** $2 - 5i$

A comparison with the method of rationalizing denominators may be helpful.

To write the quotient of $a + bi$ and $c + di$ in standard form, where c and d are not both zero, multiply the numerator and denominator by the complex conjugate of the *denominator* to obtain

$$\frac{a + bi}{c + di} = \frac{a + bi}{c + di}\left(\frac{c - di}{c - di}\right)$$ Multiply numerator and denominator by complex conjugate of denominator.

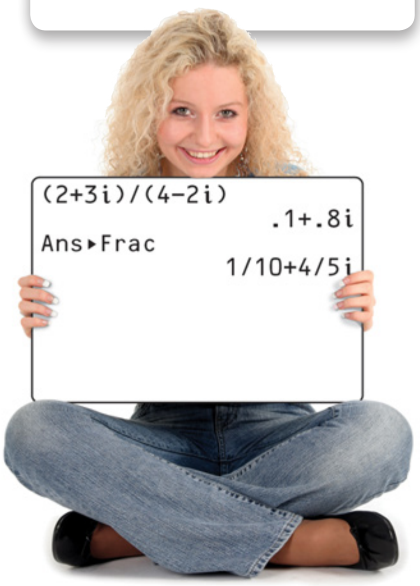
$$= \frac{ac + bd}{c^2 + d^2} + \left(\frac{bc - ad}{c^2 + d^2}\right)i.$$ Standard form

Technology Tip

Some graphing utilities can perform operations with complex numbers. For instance, on some graphing utilities, to divide $2 + 3i$ by $4 - 2i$, use the following keystrokes.

(2 + 3 *i*) ÷

(4 − 2 *i*) ENTER



EXAMPLE 4 Quotient of Complex Numbers in Standard Form

$$\frac{2 + 3i}{4 - 2i} = \frac{2 + 3i}{4 - 2i}\left(\frac{4 + 2i}{4 + 2i}\right)$$ Multiply numerator and denominator by complex conjugate of denominator.

$$= \frac{8 + 4i + 12i + 6i^2}{16 - 4i^2}$$ Expand.

$$= \frac{8 - 6 + 16i}{16 + 4}$$ $i^2 = -1$

$$= \frac{2 + 16i}{20}$$ Simplify.

$$= \frac{1}{10} + \frac{4}{5}i$$ Write in standard form.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write $\dfrac{2 + i}{2 - i}$ in standard form.

Complex Solutions of Quadratic Equations

When using the Quadratic Formula to solve a quadratic equation, you often obtain a result such as $\sqrt{-3}$, which you know is not a real number. By factoring out $i = \sqrt{-1}$, you can write this number in standard form.

$$\sqrt{-3} = \sqrt{3(-1)} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$$

The number $\sqrt{3}i$ is called the *principal square root* of -3.

Principal Square Root of a Negative Number

If a is a positive number, then the **principal square root** of the negative number $-a$ is defined as

$$\sqrt{-a} = \sqrt{a}i.$$

Remark

The definition of principal square root uses the rule

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

for $a > 0$ and $b < 0$. This rule is not valid when *both* a and b are negative. For example,

$$\begin{aligned}\sqrt{-5}\sqrt{-5} &= \sqrt{5(-1)}\sqrt{5(-1)}\\ &= \sqrt{5}i\sqrt{5}i\\ &= \sqrt{25}i^2\\ &= 5i^2\\ &= -5\end{aligned}$$

whereas

$$\sqrt{(-5)(-5)} = \sqrt{25} = 5.$$

To avoid problems with square roots of negative numbers, be sure to convert complex numbers to standard form *before* multiplying.

EXAMPLE 5 Writing Complex Numbers in Standard Form

a. $\sqrt{-3}\sqrt{-12} = \sqrt{3}i\sqrt{12}i = \sqrt{36}i^2 = 6(-1) = -6$

b. $\begin{aligned}\sqrt{-48} - \sqrt{-27} &= \sqrt{48}i - \sqrt{27}i\\ &= 4\sqrt{3}i - 3\sqrt{3}i\\ &= \sqrt{3}i\end{aligned}$

c. $\begin{aligned}\left(-1 + \sqrt{-3}\right)^2 &= \left(-1 + \sqrt{3}i\right)^2\\ &= (-1)^2 - 2\sqrt{3}i + \left(\sqrt{3}\right)^2(i^2)\\ &= 1 - 2\sqrt{3}i + 3(-1)\\ &= -2 - 2\sqrt{3}i\end{aligned}$

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Write $\sqrt{-14}\sqrt{-2}$ in standard form.

EXAMPLE 6 Complex Solutions of a Quadratic Equation

See LarsonPrecalculus.com for an interactive version of this type of example.

Solve (a) $x^2 + 4 = 0$ and (b) $3x^2 - 2x + 5 = 0$.

Solution

a. $x^2 + 4 = 0$ — Write original equation.

$x^2 = -4$ — Subtract 4 from each side.

$x = \pm 2i$ — Extract square roots.

b. $3x^2 - 2x + 5 = 0$ — Write original equation.

$x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$ — Quadratic Formula

$= \dfrac{2 \pm \sqrt{-56}}{6}$ — Simplify.

$= \dfrac{2 \pm 2\sqrt{14}i}{6}$ — Write $\sqrt{-56}$ in standard form.

$= \dfrac{1}{3} \pm \dfrac{\sqrt{14}}{3}i$ — Write in standard form.

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Solve $8x^2 + 14x + 9 = 0$.

2.4 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Match the type of complex number with its definition.

(a) real number — (i) $a + bi, a = 0, b \neq 0$

(b) imaginary number — (ii) $a + bi, b = 0$

(c) pure imaginary number — (iii) $a + bi, a \neq 0, b \neq 0$

In Exercises 2 and 3, fill in the blanks.

2. The imaginary unit i is defined as $i =$ ________ , where $i^2 =$ ________ .

3. When you add $(7 + 6i)$ and $(8 + 5i)$, the real part of the sum is ________ and the imaginary part of the sum is ________ .

4. What method for multiplying two polynomials can you use when multiplying two complex numbers?

5. What is the additive inverse of the complex number $2 - 4i$?

6. What is the complex conjugate of the complex number $2 - 4i$?

Procedures and Problem Solving

Equality of Complex Numbers **In Exercises 7–10, find real numbers a and b such that the equation is true.**

7. $a + bi = -9 + 4i$ **8.** $a + bi = 12 + 5i$

9. $3a + (b + 3)i = 9 + 8i$ **10.** $(a + 6) + 2bi = 6 - i$

Writing a Complex Number in Standard Form **In Exercises 11–20, write the complex number in standard form.**

11. $4 + \sqrt{-9}$ **12.** $7 - \sqrt{-25}$

13. 12 **14.** -3

15. $-8i - i^2$ **16.** $2i^2 - 6i$

17. $\left(\sqrt{-16}\right)^2 + 5$ **18.** $-i - \left(\sqrt{-23}\right)^2$

19. $\sqrt{-0.09}$ **20.** $\sqrt{-0.0004}$

Adding and Subtracting Complex Numbers **In Exercises 21–30, perform the addition or subtraction and write the result in standard form.**

21. $(4 + i) - (7 - 2i)$ **22.** $(11 - 2i) - (-3 + 6i)$

23. $(-1 + 8i) + (8 - 5i)$ **24.** $(7 + 6i) + (3 + 12i)$

25. $13i - (14 - 7i)$ **26.** $22 + (-5 + 8i) - 9i$

27. $\left(\frac{3}{2} + \frac{5}{2}i\right) + \left(\frac{5}{3} + \frac{11}{3}i\right)$ **28.** $\left(\frac{3}{4} + \frac{7}{5}i\right) - \left(\frac{5}{6} - \frac{1}{6}i\right)$

29. $(1.6 + 3.2i) + (-5.8 + 4.3i)$

30. $-(-3.7 - 12.8i) - (6.1 - 16.3i)$

Multiplying Complex Numbers **In Exercises 31–42, perform the operation(s) and write the result in standard form.**

31. $4(3 + 5i)$ **32.** $-6(5 - 3i)$

33. $(1 + i)(3 - 2i)$ **34.** $(6 - 2i)(2 - 3i)$

35. $4i(8 + 5i)$ **36.** $-3i(6 - i)$

37. $\left(\sqrt{14} + \sqrt{10}i\right)\left(\sqrt{14} - \sqrt{10}i\right)$

38. $\left(\sqrt{3} + \sqrt{15}i\right)\left(\sqrt{3} - \sqrt{15}i\right)$

39. $(6 + 7i)^2$ **40.** $(5 - 4i)^2$

41. $(4 + 5i)^2 - (4 - 5i)^2$ **42.** $(1 - 2i)^2 - (1 + 2i)^2$

Multiplying Conjugates **In Exercises 43–50, write the complex conjugate of the complex number. Then multiply the number by its complex conjugate.**

43. $6 - 2i$ **44.** $3 + 5i$

45. $-1 + \sqrt{7}i$ **46.** $-4 - \sqrt{3}i$

47. $\sqrt{-29}$ **48.** $\sqrt{-10}$

49. $9 - \sqrt{6}i$ **50.** $-8 + \sqrt{15}i$

Writing a Quotient of Complex Numbers in Standard Form **In Exercises 51–58, write the quotient in standard form.**

51. $\dfrac{6}{i}$ **52.** $-\dfrac{5}{2i}$

53. $\dfrac{2}{4 - 5i}$ **54.** $\dfrac{3}{1 - i}$

55. $\dfrac{2 + i}{2 - i}$ **56.** $\dfrac{8 - 7i}{1 - 2i}$

57. $i/(4 - 5i)^2$ **58.** $5i/(2 + 3i)^2$

Adding or Subtracting Quotients of Complex Numbers **In Exercises 59 and 60, perform the operation and write the result in standard form.**

59. $\dfrac{2}{1 + i} - \dfrac{3}{1 - i}$ **60.** $\dfrac{2i}{2 + i} + \dfrac{5}{2 - i}$

Writing Complex Numbers in Standard Form **In Exercises 61–70, perform the operation and write the result in standard form.**

61. $\sqrt{-18} - \sqrt{-54}$ **62.** $\sqrt{-50} + \sqrt{-275}$

63. $\left(-3 + \sqrt{-24}\right) + \left(7 - \sqrt{-44}\right)$

64. $\left(-12 - \sqrt{-72}\right) + \left(9 + \sqrt{-108}\right)$

65. $\sqrt{-6}\sqrt{-2}$ **66.** $\sqrt{-5}\sqrt{-10}$

67. $\left(\sqrt{-10}\right)^2$ **68.** $\left(\sqrt{-75}\right)^2$

69. $\left(2 - \sqrt{-6}\right)^2$ **70.** $\left(3 + \sqrt{-5}\right)\left(7 - \sqrt{-10}\right)$

Complex Solutions of a Quadratic Equation **In Exercises 71–82, solve the quadratic equation.**

71. $x^2 + 25 = 0$ **72.** $x^2 + 32 = 0$

73. $x^2 - 2x + 2 = 0$ **74.** $x^2 + 6x + 10 = 0$

75. $4x^2 + 16x + 17 = 0$ **76.** $9x^2 - 6x + 37 = 0$

77. $16t^2 - 4t + 3 = 0$ **78.** $4x^2 + 16x + 15 = 0$

79. $\frac{3}{2}x^2 - 6x + 9 = 0$ **80.** $\frac{7}{8}x^2 - \frac{3}{4}x + \frac{5}{16} = 0$

81. $1.4x^2 - 2x - 10 = 0$ **82.** $4.5x^2 - 3x + 12 = 0$

Expressions Involving Powers of *i* **In Exercises 83–88, simplify the complex number and write the result in standard form.**

83. $-6i^3 + i^2$ **84.** $4i^2 - 2i^3$

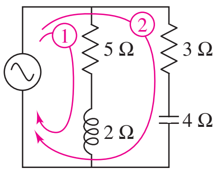
85. $\left(\sqrt{-75}\right)^3$ **86.** $\left(\sqrt{-2}\right)^6$

87. $\dfrac{1}{i^3}$ **88.** $\dfrac{1}{(2i)^3}$

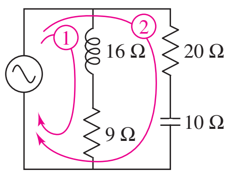
89. ***Why you should learn it*** *(p. 128)* The opposition to current in an electrical circuit is called its impedance. The impedance z in a parallel circuit with two pathways satisfies the equation $1/z = 1/z_1 + 1/z_2$, where z_1 is the impedance (in ohms) of pathway 1, and z_2 is the impedance (in ohms) of pathway 2. Use the table to determine the impedance of each parallel circuit. (*Hint:* You can find the impedance of each pathway in a parallel circuit by adding the impedances of all components in the pathway.)

	Resistor	Inductor	Capacitor
Symbol	$a\ \Omega$	$b\ \Omega$	$c\ \Omega$
Impedance	a	bi	$-ci$

(a)



(b)



90. Exploration Consider the functions

$$f(x) = 2(x - 3)^2 - 4 \quad \text{and} \quad g(x) = -2(x - 3)^2 - 4.$$

(a) Without graphing either function, determine whether the graph of f and the graph of g have x-intercepts. Explain your reasoning.

(b) Solve $f(x) = 0$ and $g(x) = 0$.

(c) Explain how the zeros of f and g are related to whether their graphs have x-intercepts.

(d) For the function $f(x) = a(x - h)^2 + k$, make a general statement about how a, h, and k affect whether the graph of f has x-intercepts, and whether the zeros of f are real or complex.

Conclusions

True or False? **In Exercises 91–94, determine whether the statement is true or false. Justify your answer.**

91. No complex number is equal to its complex conjugate.

92. $i^{44} + i^{150} - i^{74} - i^{109} + i^{61} = -1$

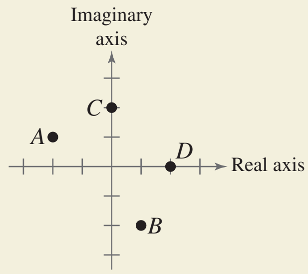
93. The conjugate of the product of two complex numbers is equal to the product of the conjugates of the two complex numbers.

94. The conjugate of the sum of two complex numbers is equal to the sum of the conjugates of the two complex numbers.

95. Error Analysis Describe the error.

~~$\sqrt{-6}\sqrt{-6} = \sqrt{(-6)(-6)} = \sqrt{36} = 6$~~

96. HOW DO YOU SEE IT? The coordinate system shown below is called the *complex plane*. In the complex plane, the point that corresponds to the complex number $a + bi$ is (a, b).



Match each complex number with its corresponding point.

(i) 2 (ii) $2i$ (iii) $-2 + i$ (iv) $1 - 2i$

Cumulative Mixed Review

Multiplying Polynomials **In Exercises 97–100, perform the operation and write the result in standard form.**

97. $(4x - 5)(4x + 5)$ **98.** $(x + 2)^3$

99. $\left(3x - \frac{1}{2}\right)(x + 4)$ **100.** $(2x - 5)^2$