1.4 Shifting, Reflecting, and Stretching Graphs

Summary of Graphs of Parent Functions

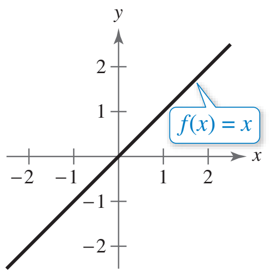
One of the goals of this text is to enable you to build your intuition for the basic shapes of the graphs of different types of functions. For instance, from your study of lines in Section 1.1, you can determine the basic shape of the graph of the parent linear function

$$f(x) = x.$$

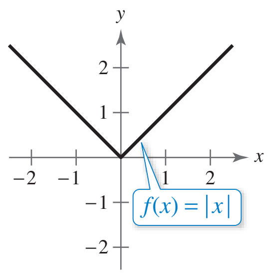
Specifically, you know that the graph of this function is a line whose slope is 1 and whose y-intercept is (0, 0).

The six graphs shown in Figure 1.24 represent the most commonly used types of functions in algebra. Familiarity with the basic characteristics of these parent graphs will help you analyze the shapes of more complicated graphs.

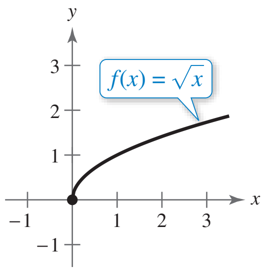
Library of Parent Functions: Commonly Used Functions



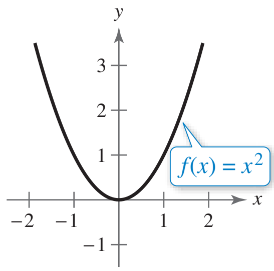
(a) *Linear Function*



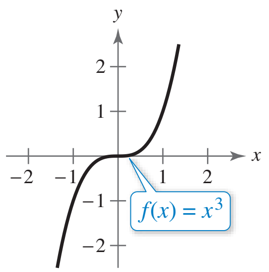
(b) *Absolute Value Function*



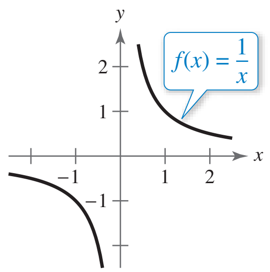
(c) *Square Root Function*



(d) *Quadratic Function*



(e) *Cubic Function*



(f) *Rational Function*

Figure 1.24

Throughout this section, you will discover how many complicated graphs are derived by shifting, stretching, shrinking, or reflecting the parent graphs shown above. Shifts, stretches, shrinks, and reflections are called *transformations*. Many graphs of functions can be created from combinations of these transformations.

What you should learn

- Recognize graphs of parent functions.
- Use vertical and horizontal shifts to sketch graphs of functions.
- Use reflections to sketch graphs of functions.
- Use nonrigid transformations to sketch graphs of functions.

Why you should learn it

Recognizing the graphs of parent functions and knowing how to shift, reflect, and stretch graphs of functions can help you sketch or describe the graphs of a wide variety of simple functions. For example, in Exercise 66 on page 49, you are asked to describe a transformation that produces the graph of a model for the sales of the WD-40 Company.

Emphasize that the graph of a function is related to a "family" of graphs, and when students learn these "families" of common graphs, graphing will be much easier. You can reinforce this concept with discovery methods such as graphing $f(x) = x^2$, $f(x) = x^2 + 2$, $f(x) = (x - 1)^2$, and $f(x) = (x - 1)^2 + 2$ and noting similarities and differences.

Vertical and Horizontal Shifts

Many functions have graphs that are simple transformations of the graphs of parent functions summarized in Figure 1.24. For example, you can obtain the graph of

$$h(x) = x^2 + 2$$

by shifting the graph of $f(x) = x^2$ two units *upward*, as shown in Figure 1.25. In function notation, h and f are related as follows.

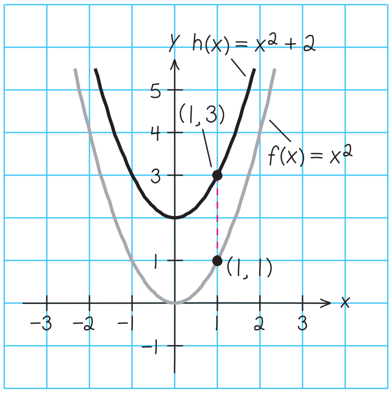
$$h(x) = x^2 + 2$$
$$= f(x) + 2 \quad \text{Upward shift of two units}$$

Similarly, you can obtain the graph of

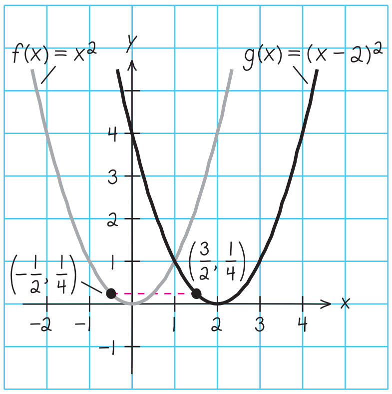
$$g(x) = (x - 2)^2$$

by shifting the graph of $f(x) = x^2$ two units to the *right*, as shown in Figure 1.26. In this case, the functions g and f have the following relationship.

$$g(x) = (x - 2)^2$$
$$= f(x - 2) \quad \text{Right shift of two units}$$



Vertical shift upward: two units
Figure 1.25



Horizontal shift to the right: two units
Figure 1.26

The following list summarizes vertical and horizontal shifts. In items 3 and 4, be sure you see that $h(x) = f(x - c)$ corresponds to a *right* shift and $h(x) = f(x + c)$ corresponds to a *left* shift for $c > 0$.

Vertical and Horizontal Shifts

Let c be a positive real number. **Vertical and horizontal shifts** in the graph of $y = f(x)$ are represented as follows.

1. Vertical shift c units *upward*: $h(x) = f(x) + c$
2. Vertical shift c units *downward*: $h(x) = f(x) - c$
3. Horizontal shift c units to the *right*: $h(x) = f(x - c)$
4. Horizontal shift c units to the *left*: $h(x) = f(x + c)$

Some graphs can be obtained from combinations of vertical and horizontal shifts, as shown in Example 1(c) on the next page. Vertical and horizontal shifts generate a *family of functions*, each with a graph that has the same shape but at a different location in the plane.

Explore the Concept

Use a graphing utility to display (in the same viewing window) the graphs of $y = x^2 + c$, where $c = -2, 0, 2$, and 4. Use the results to describe the effect that c has on the graph.

Use a graphing utility to display (in the same viewing window) the graphs of $y = (x + c)^2$, where $c = -2, 0, 2$, and 4. Use the results to describe the effect that c has on the graph.

You might also wish to illustrate simple transformations of functions numerically, using tables to emphasize what happens to individual ordered pairs. For instance, suppose you have

$$f(x) = x^2, h(x) = x^2 + 2 = f(x) + 2$$

and

$$g(x) = (x - 2)^2 = f(x - 2).$$

You can illustrate these transformations with the following tables.

x	$f(x)$	$h(x) = f(x) + 2$
-2	4	$4 + 2 = 6$
-1	1	$1 + 2 = 3$
0	0	$0 + 2 = 2$
1	1	$1 + 2 = 3$
2	4	$4 + 2 = 6$

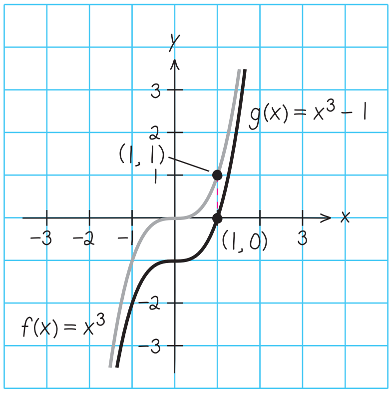
x	$x - 2$	$g(x) = f(x - 2)$
0	$0 - 2 = -2$	4
1	$1 - 2 = -1$	1
2	$2 - 2 = 0$	0
3	$3 - 2 = 1$	1
4	$4 - 2 = 2$	4

EXAMPLE 1 Shifts in the Graph of a Function

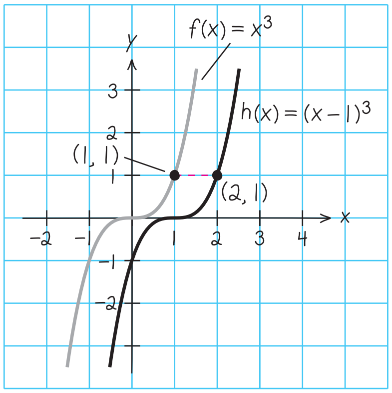
a. To use the graph of $f(x) = x^3$ to sketch the graph of $g(x) = x^3 - 1$, shift the graph of f one unit downward.

b. To use the graph of $f(x) = x^3$ to sketch the graph of $h(x) = (x - 1)^3$, shift the graph of f one unit to the right.

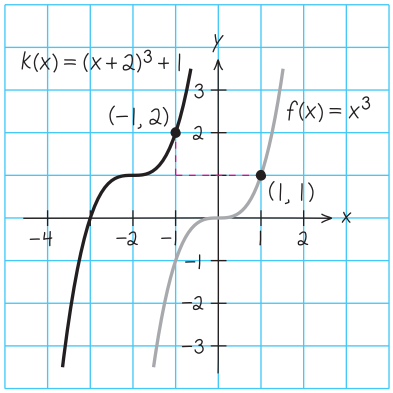
c. To use the graph of $f(x) = x^3$ to sketch the graph of $k(x) = (x + 2)^3 + 1$, shift the graph of f two units to the left and then one unit upward.



(a) *Vertical shift: one unit downward*



(b) *Horizontal shift: one unit right*



(c) *Two units left and one unit upward*

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

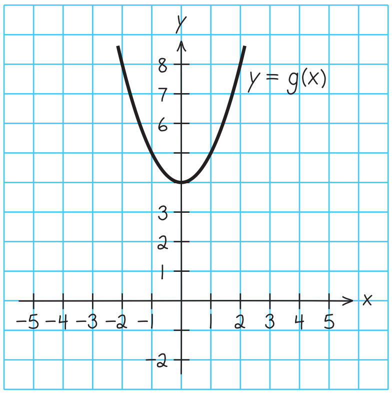
Use the graph of $f(x) = x^3$ to sketch the graph of each function.

a. $h(x) = x^3 + 5$ **b.** $g(x) = (x - 3)^3 + 2$

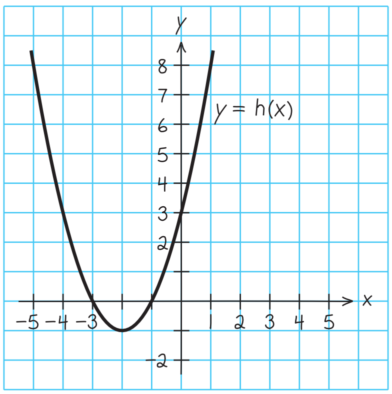
EXAMPLE 2 Writing Equations from Graphs

Each graph is a transformation of the graph of $f(x) = x^2$. Write an equation for each function.

a.



b.



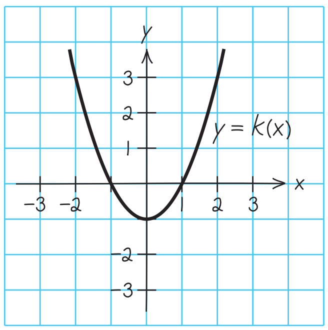
Solution

a. The graph of g is a vertical shift of four units upward of the graph of $f(x) = x^2$. So, the equation for g is $g(x) = x^2 + 4$.

b. The graph of h is a horizontal shift of two units to the left, and a vertical shift of one unit downward, of the graph of $f(x) = x^2$. So, the equation for h is $h(x) = (x + 2)^2 - 1$.

✓ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The graph at the right is a transformation of the graph of $f(x) = x^2$. Write an equation for the function.



Reflecting Graphs

Another common type of transformation is called a **reflection.** For instance, when you consider the x-axis to be a mirror, the graph of $h(x) = -x^2$ is the mirror image (or reflection) of the graph of $f(x) = x^2$, as shown in Figure 1.27.

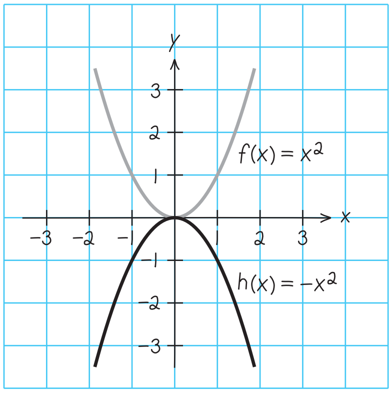


Figure 1.27

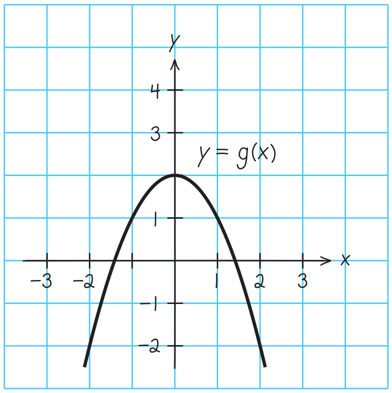
Reflections in the Coordinate Axes

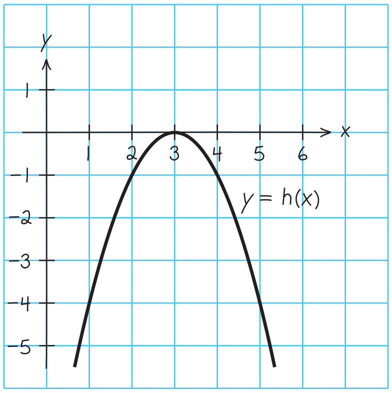
Reflections in the coordinate axes of the graph of $y = f(x)$ are represented as follows.

1. Reflection in the x-axis: $h(x) = -f(x)$
2. Reflection in the y-axis: $h(x) = f(-x)$

EXAMPLE 3 Writing Equations from Graphs

Each graph is a transformation of the graph of $f(x) = x^2$. Write an equation for each function.

a. 

b. 

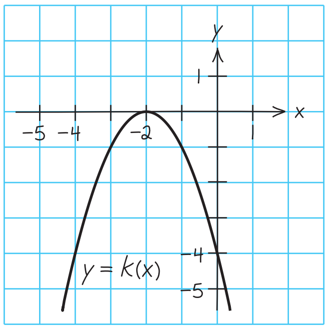
Solution

a. The graph of g is a reflection in the x-axis *followed by* an upward shift of two units of the graph of $f(x) = x^2$. So, the equation for g is $g(x) = -x^2 + 2$.

b. The graph of h is a horizontal shift of three units to the right *followed by* a reflection in the x-axis of the graph of $f(x) = x^2$. So, the equation for h is $h(x) = -(x - 3)^2$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The graph at the right is a transformation of the graph of $f(x) = x^2$. Write an equation for the function.

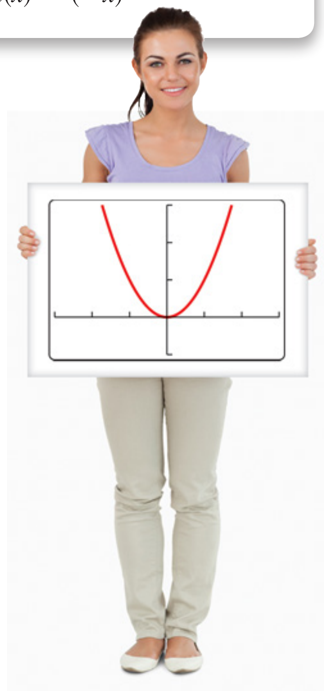


Explore the Concept

Compare the graph of each function with the graph of $f(x) = x^2$ by using a graphing utility to graph the function and f in the same viewing window. Describe the transformation.

a. $g(x) = -x^2$

b. $h(x) = (-x)^2$



Activity

Does the graph of $f(x) = -(x + 1)^3 + 4$ represent a horizontal shift of one unit to the left, followed by a vertical shift of four units upward, followed by a reflection in the x-axis?

Answer: No, it represents a horizontal shift of one unit to the left, followed by a reflection in the x-axis, followed by a vertical shift of four units upward.

EXAMPLE 4 Reflections and Shifts

Compare the graph of each function with the graph of

$$f(x) = \sqrt{x}.$$

a. $g(x) = -\sqrt{x}$

b. $h(x) = \sqrt{-x}$

c. $k(x) = -\sqrt{x+2}$

Algebraic Solution

a. Relative to the graph of $f(x) = \sqrt{x}$, the graph of g is a reflection in the x-axis because

$$g(x) = -\sqrt{x}$$
$$= -f(x).$$

b. The graph of h is a reflection of the graph of $f(x) = \sqrt{x}$ in the y-axis because

$$h(x) = \sqrt{-x}$$
$$= f(-x).$$

c. From the equation

$$k(x) = -\sqrt{x+2}$$
$$= -f(x+2)$$

you can conclude that the graph of k is a left shift of two units, followed by a reflection in the x-axis, of the graph of $f(x) = \sqrt{x}$.

Graphical Solution

a. From the graph in Figure 1.28, you can see that the graph of g is a reflection of the graph of f in the x-axis. Note that the domain of g is $x \geq 0$.

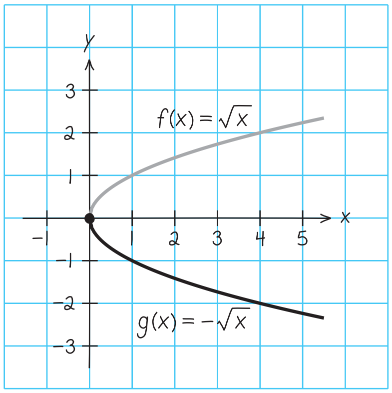


Figure 1.28

b. From the graph in Figure 1.29, you can see that the graph of h is a reflection of the graph of f in the y-axis. Note that the domain of h is $x \leq 0$.

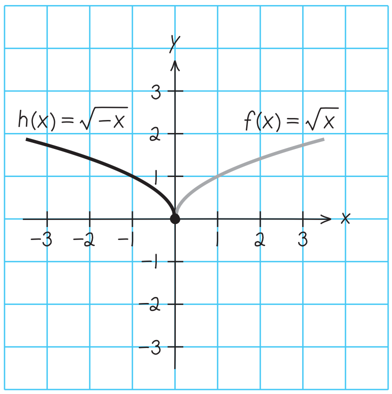


Figure 1.29

c. From the graph in Figure 1.30, you can see that the graph of k is a left shift of two units of the graph of f, followed by a reflection in the x-axis. Note that the domain of k is $x \geq -2$.

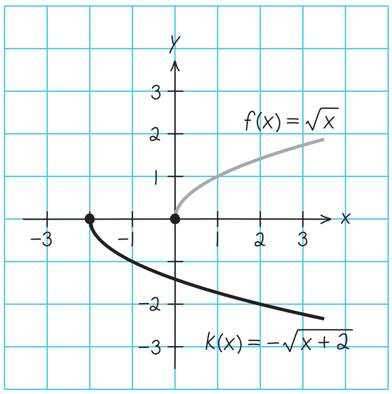


Figure 1.30

✔ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Compare the graph of $g(x) = -|x|$ with the graph of $f(x) = |x|$.

Nonrigid Transformations

Horizontal shifts, vertical shifts, and reflections are called **rigid transformations** because the basic shape of the graph is unchanged. These transformations change only the *position* of the graph in the coordinate plane. **Nonrigid transformations** are those that cause a *distortion*—a change in the shape of the original graph. For instance, a nonrigid transformation of the graph of $y = f(x)$ is represented by $g(x) = cf(x)$, where the transformation is a **vertical stretch** when $c > 1$ and a **vertical shrink** when $0 < c < 1$. Another nonrigid transformation of the graph of $y = f(x)$ is represented by $h(x) = f(cx)$, where the transformation is a **horizontal shrink** when $c > 1$ and a **horizontal stretch** when $0 < c < 1$.

EXAMPLE 5 Nonrigid Transformations

See LarsonPrecalculus.com for an interactive version of this type of example.

Compare the graphs of (a) $h(x) = 3|x|$ and (b) $g(x) = \frac{1}{3}|x|$ with the graph of $f(x) = |x|$.

Solution

a. Relative to the graph of $f(x) = |x|$, the graph of $h(x) = 3|x| = 3f(x)$ is a vertical stretch (each y-value is multiplied by 3) of the graph of f. (See Figure 1.31.)

b. Similarly, the graph of $g(x) = \frac{1}{3}|x| = \frac{1}{3}f(x)$ is a vertical shrink $\left(\text{each } y\text{-value is multiplied by } \frac{1}{3}\right)$ of the graph of f. (See Figure 1.32.)

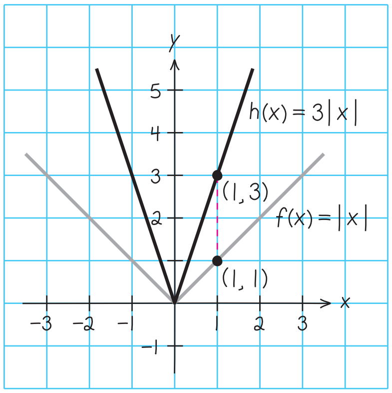


Figure 1.31

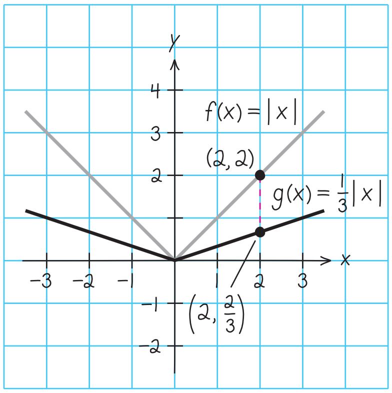


Figure 1.32

✔ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Compare the graphs of (a) $g(x) = 4x^2$ and (b) $h(x) = \frac{1}{4}x^2$ with the graph of $f(x) = x^2$.

EXAMPLE 6 Nonrigid Transformations

Compare the graph of $h(x) = f\left(\frac{1}{2}x\right)$ with the graph of $f(x) = 2 - x^3$.

Solution

Relative to the graph of $f(x) = 2 - x^3$, the graph of

$$h(x) = f\left(\tfrac{1}{2}x\right) = 2 - \left(\tfrac{1}{2}x\right)^3 = 2 - \tfrac{1}{8}x^3$$

is a horizontal stretch (each x-value is multiplied by 2) of the graph of f. (See Figure 1.33.)

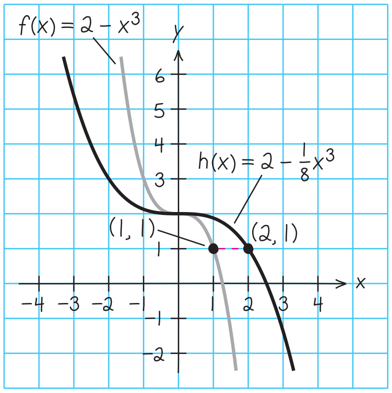


Figure 1.33

✔ **Checkpoint** Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Compare the graphs of (a) $g(x) = f(2x)$ and (b) $h(x) = f\left(\frac{1}{2}x\right)$ with the graph of $f(x) = x^2 + 3$.

1.4 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

1. Name three types of rigid transformations.

2. Match the rigid transformation of $y = f(x)$ with the correct representation, where $c > 0$.

(a) $h(x) = f(x) + c$ (i) horizontal shift c units to the left

(b) $h(x) = f(x) - c$ (ii) vertical shift c units upward

(c) $h(x) = f(x - c)$ (iii) horizontal shift c units to the right

(d) $h(x) = f(x + c)$ (iv) vertical shift c units downward

In Exercises 3 and 4, fill in the blanks.

3. A reflection in the x-axis of $y = f(x)$ is represented by $h(x) =$ ________ , while a reflection in the y-axis of $y = f(x)$ is represented by $h(x) =$ ________ .

4. A nonrigid transformation of $y = f(x)$ represented by $cf(x)$ is a vertical stretch when ________ and a vertical shrink when ________ .

Procedures and Problem Solving

Sketching Transformations **In Exercises 5–18, sketch the graphs of the three functions by hand on the same rectangular coordinate system. Verify your results with a graphing utility.**

5. $f(x) = x$
$g(x) = x - 4$
$h(x) = 3x$

6. $f(x) = \frac{1}{2}x$
$g(x) = \frac{1}{2}x + 2$
$h(x) = 4(x - 2)$

7. $f(x) = x^2$
$g(x) = x^2 + 2$
$h(x) = (x - 2)^2$

8. $f(x) = x^2$
$g(x) = 3x^2$
$h(x) = (x + 2)^2 + 1$

9. $f(x) = -x^2$
$g(x) = -x^2 + 1$
$h(x) = -(x + 3)^2$

10. $f(x) = (x - 2)^2$
$g(x) = (x + 2)^2 + 2$
$h(x) = -(x - 2)^2 - 1$

11. $f(x) = x^2$
$g(x) = \frac{1}{2}x^2$
$h(x) = (2x)^2$

12. $f(x) = x^2$
$g(x) = \frac{1}{4}x^2 + 2$
$h(x) = -\frac{1}{4}x^2$

13. $f(x) = |x|$
$g(x) = |x| - 1$
$h(x) = 3|x - 3|$

14. $f(x) = |x|$
$g(x) = |2x|$
$h(x) = -2|x + 2| - 1$

15. $f(x) = -\sqrt{x}$
$g(x) = \sqrt{x + 1}$
$h(x) = \sqrt{x - 2} + 1$

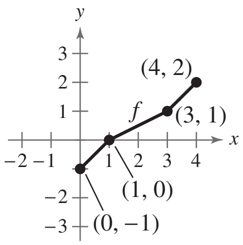
16. $f(x) = \sqrt{x}$
$g(x) = \frac{1}{2}\sqrt{x}$
$h(x) = -\sqrt{x - 4}$

17. $f(x) = \dfrac{1}{x}$
$g(x) = \dfrac{1}{x} - 2$
$h(x) = \dfrac{1}{x - 1} + 2$

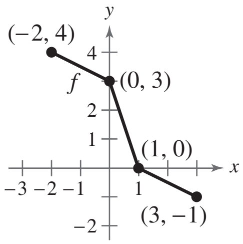
18. $f(x) = \dfrac{1}{x}$
$g(x) = \dfrac{1}{x} - 4$
$h(x) = \dfrac{1}{x + 3} - 1$

Sketching Transformations **In Exercises 19 and 20, use the graph of f to sketch each graph. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

19. (a) $y = f(x) + 2$
(b) $y = -f(x)$
(c) $y = f(x - 2)$
(d) $y = f(x + 3)$
(e) $y = 2f(x)$
(f) $y = f(-x)$
(g) $y = f\left(\frac{1}{2}x\right)$

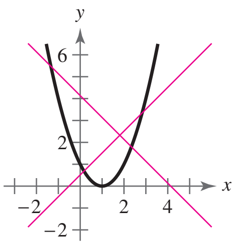


20. (a) $y = f(x) - 1$
(b) $y = f(x + 2)$
(c) $y = f(x - 1)$
(d) $y = -f(x - 2)$
(e) $y = f(-x)$
(f) $y = \frac{1}{2}f(x)$
(g) $y = f(2x)$

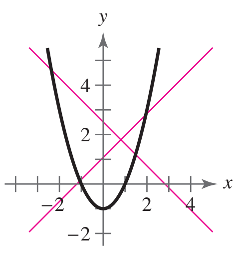


Error Analysis **In Exercises 21 and 22, describe the error in graphing the function.**

21. $f(x) = (x + 1)^2$



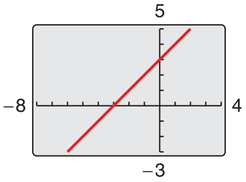
22. $f(x) = (x - 1)^2$



Library of Parent Functions **In Exercises 23–28, compare the graph of the function with the graph of its parent function.**

23. $y = \sqrt{x} + 2$
24. $y = \dfrac{1}{x} - 5$
25. $y = (x - 4)^3$
26. $y = |x + 5|$
27. $y = x^2 - 2$
28. $y = \sqrt{x - 2}$

Library of Parent Functions **In Exercises 29–34, identify the parent function and describe the transformation shown in the graph. Write an equation for the graphed function.**

29.


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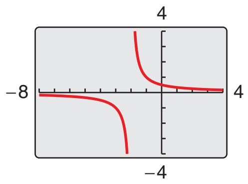

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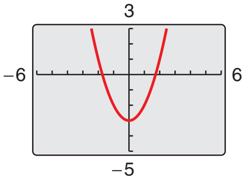

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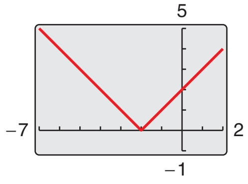
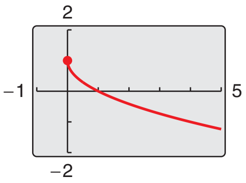


33.


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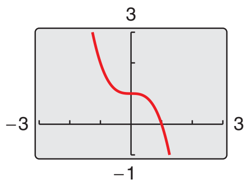


Rigid and Nonrigid Transformations **In Exercises 35–46, compare the graph of the function with the graph of its parent function.**

35. $y = -x$
36. $y = |-x|$
37. $y = (-x)^2$
38. $y = -x^3$
39. $y = \dfrac{1}{-x}$
40. $y = -\dfrac{1}{x}$
41. $h(x) = 4|x|$
42. $p(x) = \frac{1}{2}x^2$
43. $g(x) = \frac{1}{4}x^3$
44. $y = 2\sqrt{x}$
45. $f(x) = \sqrt{4x}$
46. $y = \left|\frac{1}{2}x\right|$

Rigid and Nonrigid Transformations **In Exercises 47–50, use a graphing utility to graph the three functions in the same viewing window. Describe the graphs of g and h relative to the graph of f.**

47. $f(x) = x^3 - 3x^2$
$g(x) = f(x + 2)$
$h(x) = \frac{1}{2}f(x)$

48. $f(x) = x^3 - 3x^2 + 2$
$g(x) = f(x - 1)$
$h(x) = f(3x)$

49. $f(x) = x^3 - 3x^2$
$g(x) = -\frac{1}{3}f(x)$
$h(x) = f(-x)$

50. $f(x) = x^3 - 3x^2 + 2$
$g(x) = -f(x)$
$h(x) = f(2x)$

Describing Transformations **In Exercises 51–64, g is related to one of the six parent functions on page 41. (a) Identify the parent function f. (b) Describe the sequence of transformations from f to g. (c) Sketch the graph of g by hand. (d) Use function notation to write g in terms of the parent function f.**

51. $g(x) = 2 - (x + 5)^2$
52. $g(x) = (x - 10)^2 + 5$
53. $g(x) = 3 + 2(x - 4)^2$
54. $g(x) = -\frac{1}{4}(x + 2)^2 - 2$
55. $g(x) = \frac{1}{3}(x - 2)^3$
56. $g(x) = \frac{1}{2}(x + 1)^3$
57. $g(x) = (x - 1)^3 + 2$
58. $g(x) = -(x + 3)^3 - 10$
59. $g(x) = \dfrac{1}{x + 8} - 9$
60. $g(x) = \dfrac{1}{x - 7} + 4$
61. $g(x) = -2|x - 1| - 4$
62. $g(x) = \frac{1}{2}|x - 2| - 3$
63. $g(x) = -\frac{1}{2}\sqrt{x + 3} - 1$
64. $g(x) = -3\sqrt{x + 1} - 6$

65. MODELING DATA

The numbers N (in millions) of households in the United States from 2000 through 2013 are given by the ordered pairs of the form $(t, N(t))$, where $t = 0$ represents 2000. A model for the data is

$$N(t) = -0.03(t - 26.17)^2 + 126.5.$$

(Source: U.S. Census Bureau)

(0, 104.7)
(1, 108.2)
(2, 109.3)
(3, 111.3)
(4, 112.0)
(5, 113.3)
(6, 114.4)
(7, 116.0)
(8, 116.8)
(9, 117.2)
(10, 117.5)
(11, 119.9)
(12, 121.1)
(13, 122.5)

(a) Describe the transformation of the parent function $f(t) = t^2$.

(b) Use a graphing utility to graph the model and the data in the same viewing window.

(c) Rewrite the function so that $t = 0$ represents 2009. Explain how you got your answer.

66. *Why you should learn it* (p. 41) The depreciation D (in millions of dollars) of the WD-40 Company assets from 2009 through 2013 can be approximated by the function

$$D(t) = 1.9\sqrt{t} + 3.7$$

where $t = 0$ represents 2009. (Source: WD-40 Company)

(a) Describe the transformation of the parent function $f(t) = \sqrt{t}$.

(b) Use a graphing utility to graph the model over the interval $0 \le t \le 4$.

(c) According to the model, in what year will the depreciation of WD-40 assets be approximately 6 million dollars?

(d) Rewrite the function so that $t = 0$ represents 2011. Explain how you got your answer.

Conclusions

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. Justify your answer.**

67. The graph of $y = f(-x)$ is a reflection of the graph of $y = f(x)$ in the x-axis.

68. The graphs of $f(x) = |x| + 6$ and $f(x) = |-x| + 6$ are identical.

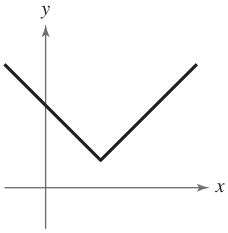
Exploration **In Exercises 69–72, use the fact that the graph of $y = f(x)$ has x-intercepts at $x = 2$ and $x = -3$ to find the x-intercepts of the given graph. If not possible, state the reason.**

69. $y = f(-x)$ **70.** $y = 2f(x)$

71. $y = f(x) + 2$ **72.** $y = f(x - 3)$

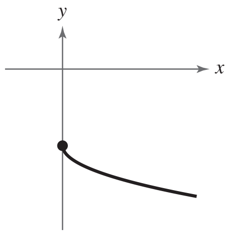
Library of Parent Functions **In Exercises 73–76, determine which equation(s) may be represented by the graph shown. (There may be more than one correct answer.)**

73.



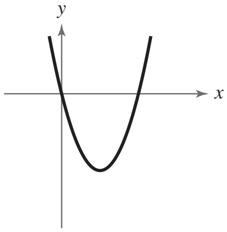
(a) $f(x) = |x + 2| + 1$
(b) $f(x) = |x - 1| + 2$
(c) $f(x) = |x - 2| + 1$
(d) $f(x) = 2 + |x - 2|$
(e) $f(x) = |(x - 2) + 1|$
(f) $f(x) = 1 - |x - 2|$

74.



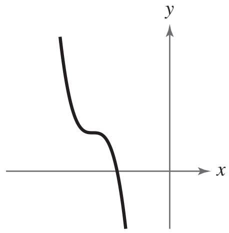
(a) $f(x) = -\sqrt{x} - 4$
(b) $f(x) = -4 - \sqrt{x}$
(c) $f(x) = -4 - \sqrt{-x}$
(d) $f(x) = \sqrt{-x} - 4$
(e) $f(x) = \sqrt{-x} + 4$
(f) $f(x) = \sqrt{x} - 4$

75.



(a) $f(x) = (x - 2)^2 - 2$
(b) $f(x) = (x + 4)^2 - 4$
(c) $f(x) = (x - 2)^2 - 4$
(d) $f(x) = (x + 2)^2 - 4$
(e) $f(x) = 4 - (x - 2)^2$
(f) $f(x) = 4 - (x + 2)^2$

76.



(a) $f(x) = -(x - 4)^3 + 2$
(b) $f(x) = -(x + 4)^3 + 2$
(c) $f(x) = -(x - 2)^3 + 4$
(d) $f(x) = (-x - 4)^3 + 2$
(e) $f(x) = (x + 4)^3 + 2$
(f) $f(x) = (-x + 4)^3 + 2$

77. **Think About It** You can use either of two methods to graph a function: plotting points, or translating a parent function as shown in this section. Which method do you prefer to use for each function? Explain.

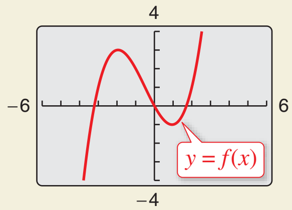
(a) $f(x) = 3x^2 - 4x + 1$ (b) $f(x) = 2(x - 1)^2 - 6$

78. **Think About It** The graph of $y = f(x)$ passes through the points $(0, 1)$, $(1, 2)$, and $(2, 3)$. Find the corresponding points on the graph of $y = f(x + 2) - 1$.

79. **Think About It** Compare the graph of $g(x) = ax^2$ with the graph of $f(x) = x^2$ when (a) $0 < a < 1$ and (b) $a > 1$.

80. **HOW DO YOU SEE IT?** Use the graph of $y = f(x)$ to find the intervals on which each of the graphs in (a)–(c) is increasing and decreasing. If not possible, then state the reason.

(a) $y = -f(x)$
(b) $y = f(x) - 3$
(c) $y = f(x - 1)$



Cumulative Mixed Review

Parallel and Perpendicular Lines **In Exercises 81 and 82, determine whether the lines L_1 and L_2 passing through the pairs of points are parallel, perpendicular, or neither.**

81. L_1: $(-2, -2)$, $(2, 10)$
L_2: $(-1, 3)$, $(3, 9)$

82. L_1: $(-1, -7)$, $(4, 3)$
L_2: $(1, 5)$, $(-2, -7)$

Finding the Domain of a Function **In Exercises 83–86, find the domain of the function.**

83. $f(x) = \dfrac{4}{9 - x}$ **84.** $f(x) = \dfrac{\sqrt{x - 5}}{x - 7}$

85. $f(x) = \sqrt{100 - x^2}$ **86.** $f(x) = \sqrt[3]{16 - x^2}$

1.5 Combinations of Functions

Arithmetic Combinations of Functions

Just as two real numbers can be combined by the operations of addition, subtraction, multiplication, and division to form other real numbers, two *functions* can be combined to create new functions. When $f(x) = 2x - 3$ and $g(x) = x^2 - 1$, you can form the sum, difference, product, and quotient of f and g as follows.

$$f(x) + g(x) = (2x - 3) + (x^2 - 1) = x^2 + 2x - 4 \qquad \text{Sum}$$

$$f(x) - g(x) = (2x - 3) - (x^2 - 1) = -x^2 + 2x - 2 \qquad \text{Difference}$$

$$f(x) \cdot g(x) = (2x - 3)(x^2 - 1) = 2x^3 - 3x^2 - 2x + 3 \qquad \text{Product}$$

$$\frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 - 1}, \quad x \neq \pm 1 \qquad \text{Quotient}$$

The domain of an **arithmetic combination** of functions f and g consists of all real numbers that are common to the domains of f and g. In the case of the quotient

$$\frac{f(x)}{g(x)}$$

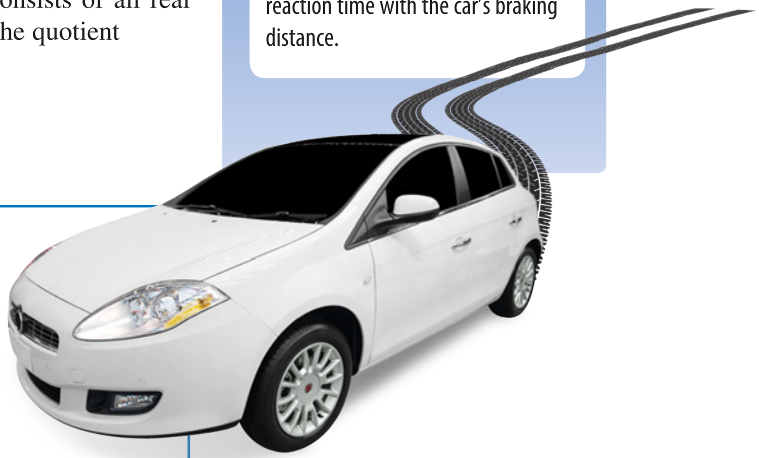
there is the further restriction that $g(x) \neq 0$.

What you should learn

- Add, subtract, multiply, and divide functions.
- Find compositions of one function with another function.
- Use combinations of functions to model and solve real-life problems.

Why you should learn it

You can model some situations by combining functions. For instance, in Exercise 79 on page 57, you will model the stopping distance of a car by combining the driver's reaction time with the car's braking distance.



Sum, Difference, Product, and Quotient of Functions

Let f and g be two functions with overlapping domains. Then, for all x common to both domains, the sum, difference, product, and quotient of f and g are defined as follows.

1. Sum: $(f + g)(x) = f(x) + g(x)$
2. Difference: $(f - g)(x) = f(x) - g(x)$
3. Product: $(fg)(x) = f(x) \cdot g(x)$
4. Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, \quad g(x) \neq 0$

EXAMPLE 1 Finding the Sum of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

Solution

$$\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ &= (2x + 1) + (x^2 + 2x - 1)\\ &= x^2 + 4x\end{aligned}$$

When $x = 2$, the value of this sum is $(f + g)(2) = 2^2 + 4(2) = 12$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Given $f(x) = x^2$ and $g(x) = 1 - x$, find $(f + g)(x)$. Then evaluate the sum when $x = 2$.

EXAMPLE 2 Finding the Difference of Two Functions

Given $f(x) = 2x + 1$ and $g(x) = x^2 + 2x - 1$, find $(f - g)(x)$. Then evaluate the difference when $x = 2$.

Algebraic Solution

The difference of the functions f and g is

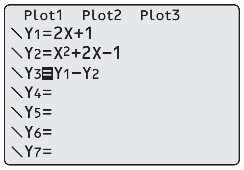
$$\begin{aligned}(f - g)(x) &= f(x) - g(x)\\ &= (2x + 1) - (x^2 + 2x - 1)\\ &= -x^2 + 2.\end{aligned}$$

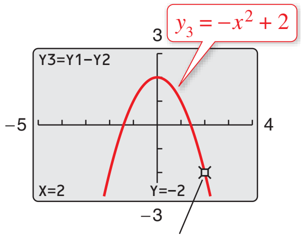
When $x = 2$, the value of this difference is

$$\begin{aligned}(f - g)(2) &= -(2)^2 + 2\\ &= -2.\end{aligned}$$

Graphical Solution

Enter $y_1 = f(x)$, $y_2 = g(x)$, and $y_3 = f(x) - g(x)$. Then graph the difference of the two functions, y_3.





The value of $(f - g)(2)$ is -2.

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Given $f(x) = x^2$ and $g(x) = 1 - x$, find $(f - g)(x)$. Then evaluate the difference when $x = 3$.

EXAMPLE 3 Finding the Product of Two Functions

Given $f(x) = x^2$ and $g(x) = x - 3$, find $(fg)(x)$. Then evaluate the product when $x = 4$.

Solution

$$(fg)(x) = f(x)g(x) = (x^2)(x - 3) = x^3 - 3x^2$$

When $x = 4$, the value of this product is $(fg)(4) = 4^3 - 3(4)^2 = 16$.

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Given $f(x) = x^2$ and $g(x) = 1 - x$, find $(fg)(x)$. Then evaluate the product when $x = 3$.

In Examples 1–3, both f and g have domains that consist of all real numbers. So, the domains of $(f + g)$ and $(f - g)$ are also the set of all real numbers. Remember to consider any restrictions on the domains of f or g when forming the sum, difference, product, or quotient of f and g. For instance, the domain of $f(x) = 1/x$ is all $x \neq 0$, and the domain of $g(x) = \sqrt{x}$ is $[0, \infty)$. This implies that the domain of $(f + g)$ is $(0, \infty)$.

Additional Examples

a. Find $(fg)(x)$ given that $f(x) = x + 5$ and $g(x) = 3x$.

Solution

$$\begin{aligned}(fg)(x) &= f(x) \cdot g(x)\\ &= (x + 5)(3x)\\ &= 3x^2 + 15x\end{aligned}$$

b. Find $(gf)(x)$ given that $f(x) = \dfrac{1}{x}$ and $g(x) = \dfrac{x}{x + 1}$.

Solution

$$\begin{aligned}(gf)(x) &= g(x) \cdot f(x)\\ &= \left(\frac{x}{x + 1}\right)\left(\frac{1}{x}\right)\\ &= \frac{1}{x + 1}, \quad x \neq 0\end{aligned}$$

EXAMPLE 4 Finding the Quotient of Two Functions

Given $f(x) = \sqrt{x}$ and $g(x) = \sqrt{4 - x^2}$, find $(f/g)(x)$. Then find the domain of f/g.

Solution

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{\sqrt{x}}{\sqrt{4 - x^2}}$$

The domain of f is $[0, \infty)$ and the domain of g is $[-2, 2]$. The intersection of these domains is $[0, 2]$. So, the domain of f/g is $[0, 2)$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find $(f/g)(x)$ and $(g/f)(x)$ for the functions $f(x) = \sqrt{x - 3}$ and $g(x) = \sqrt{16 - x^2}$. Then find the domains of f/g and g/f.

Compositions of Functions

Another way of combining two functions is to form the **composition** of one with the other. For instance, when $f(x) = x^2$ and $g(x) = x + 1$, the composition of f with g is

$$\begin{aligned} f(g(x)) &= f(x + 1) \\ &= (x + 1)^2. \end{aligned}$$

This composition is denoted as $f \circ g$ and is read as "f composed with g."

Definition of Composition of Two Functions

The **composition** of the function f with the function g is

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. (See Figure 1.34.)

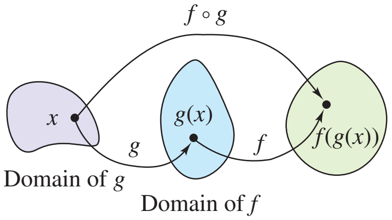


Figure 1.34

EXAMPLE 5 Forming the Composition of f with g

Find $(f \circ g)(x)$ for $f(x) = \sqrt{x}, x \geq 0$, and $g(x) = x - 1, x \geq 1$. If possible, find $(f \circ g)(2)$ and $(f \circ g)(0)$.

Solution

The composition of f with g is

$$\begin{aligned} (f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g \\ &= f(x - 1) && \text{Definition of } g(x) \\ &= \sqrt{x - 1}, \quad x \geq 1. && \text{Definition of } f(x) \end{aligned}$$

The domain of $(f \circ g)$ is $[1, \infty)$ (see Figure 1.35). So,

$$(f \circ g)(2) = \sqrt{2 - 1} = 1$$

is defined, but $(f \circ g)(0)$ is not defined because 0 is not in the domain of $f \circ g$.

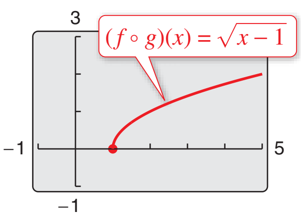


Figure 1.35

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Find $(f \circ g)(x)$ for $f(x) = x^2$ and $g(x) = x - 1$. If possible, find $(f \circ g)(0)$.

Explore the Concept

Let $f(x) = x + 2$ and $g(x) = 4 - x^2$. Are the compositions $f \circ g$ and $g \circ f$ equal? You can use your graphing utility to answer this question by entering and graphing the following functions.

$$y_1 = (4 - x^2) + 2$$

$$y_2 = 4 - (x + 2)^2$$

What do you observe? Which function represents $f \circ g$ and which represents $g \circ f$?

The composition of f with g is generally not the same as the composition of g with f. This is illustrated in Example 6.

EXAMPLE 6 Compositions of Functions

See LarsonPrecalculus.com for an interactive version of this type of example.

Given $f(x) = x + 2$ and $g(x) = 4 - x^2$, evaluate (a) $(f \circ g)(x)$ and (b) $(g \circ f)(x)$ when $x = 0$ and 1.

Algebraic Solution

a. $(f \circ g)(x) = f(g(x))$ — Definition of $f \circ g$

$= f(4 - x^2)$ — Definition of $g(x)$

$= (4 - x^2) + 2$ — Definition of $f(x)$

$= -x^2 + 6$

$(f \circ g)(0) = -0^2 + 6 = 6$

$(f \circ g)(1) = -1^2 + 6 = 5$

b. $(g \circ f)(x) = g(f(x))$ — Definition of $g \circ f$

$= g(x + 2)$ — Definition of $f(x)$

$= 4 - (x + 2)^2$ — Definition of $g(x)$

$= 4 - (x^2 + 4x + 4)$

$= -x^2 - 4x$

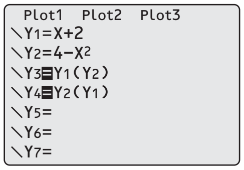
$(g \circ f)(0) = -0^2 - 4(0) = 0$

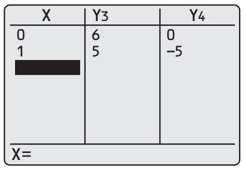
$(g \circ f)(1) = -1^2 - 4(1) = -5$

Note that $(f \circ g) \neq g \circ f$.

Numerical Solution

a. and b. Enter $y_1 = f(x)$, $y_2 = g(x)$, $y_3(f \circ g)(x)$, and $y_4 = (g \circ f)(x)$. Then use the *table* feature to find the desired function values.





From the table, you can see that $f \circ g \neq g \circ f$.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Given $f(x) = 2x + 5$ and $g(x) = 4x^2 + 1$, find the following.

a. $(f \circ g)(x)$ **b.** $(g \circ f)(x)$ **c.** $(f \circ g)\left(-\frac{1}{2}\right)$

EXAMPLE 7 Finding the Domain of a Composite Function

Find the domain of $f \circ g$ for the functions $f(x) = x^2 - 9$ and $g(x) = \sqrt{9 - x^2}$.

Algebraic Solution

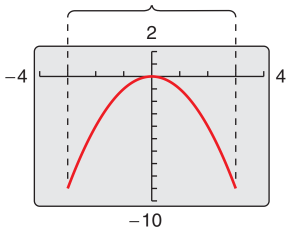
The composition of the functions is as follows.

$$
\begin{aligned}
(f \circ g)(x) &= f(g(x)) \\
&= f\left(\sqrt{9 - x^2}\right) \\
&= \left(\sqrt{9 - x^2}\right)^2 - 9 \\
&= 9 - x^2 - 9 \\
&= -x^2
\end{aligned}
$$

From this, it might appear that the domain of the composition is the set of all real numbers. This, however, is not true. Because the domain of f is the set of all real numbers and the domain of g is $[-3, 3]$, the domain of $f \circ g$ is $[-3, 3]$.

Graphical Solution

The x-coordinates of points on the graph extend from -3 to 3. So, the domain of $f \circ g$ is $[-3, 3]$.



Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the domain of $(f \circ g)(x)$ for the functions $f(x) = \sqrt{x}$ and $g(x) = x^2 + 4$.

EXAMPLE 8 A Case in Which $f \circ g = g \circ f$

Given $f(x) = 2x + 3$ and $g(x) = \frac{1}{2}(x - 3)$, find (a) $(f \circ g)(x)$ and (b) $(g \circ f)(x)$.

Solution

a.
$$\begin{aligned}(f \circ g)(x) &= f(g(x))\\ &= f\left(\frac{1}{2}(x-3)\right)\\ &= 2\left[\frac{1}{2}(x-3)\right] + 3\\ &= x - 3 + 3\\ &= x\end{aligned}$$

b.
$$\begin{aligned}(g \circ f)(x) &= g(f(x))\\ &= g(2x+3)\\ &= \frac{1}{2}[(2x+3) - 3]\\ &= \frac{1}{2}(2x)\\ &= x\end{aligned}$$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Given $f(x) = x^{1/3}$ and $g(x) = x^6$, find (a) $(f \circ g)(x)$ and (b) $(g \circ f)(x)$.

Remark

In Example 8, note that the two composite functions $f \circ g$ and $g \circ f$ are equal, and both represent the identity function. That is, $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. You will study this special case in the next section.

In Examples 5–8, you formed the composition of two given functions. In calculus, it is also important to be able to identify two functions that make up a given composite function. Basically, to "decompose" a composite function, look for an "inner" and an "outer" function.

EXAMPLE 9 Identifying a Composite Function

Write the function

$$h(x) = \frac{1}{(x-2)^2}$$

as a composition of two functions.

Solution

One way to write h as a composition of two functions is to take the inner function to be $g(x) = x - 2$ and the outer function to be

$$f(x) = \frac{1}{x^2} = x^{-2}.$$

Then you can write

$$h(x) = \frac{1}{(x-2)^2} = (x-2)^{-2} = f(x-2) = f(g(x)).$$

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Write the function

$$h(x) = \frac{\sqrt[3]{8-x}}{5}$$

as a composition of two functions.

Explore the Concept

Write each function as a composition of two functions.

a. $h(x) = |x^3 - 2|$

b. $r(x) = |x^3| - 2$

What do you notice about the inner and outer functions?

Activities

1. Find $(f + g)(-1)$ and $\left(\frac{f}{g}\right)(2)$ for $f(x) = 3x^2 + 2$, $g(x) = 2x$.

   Answer: $3; \frac{7}{2}$

2. Given $f(x) = 3x^2 + 2$ and $g(x) = 2x$, find $f \circ g$.

   Answer: $(f \circ g)(x) = 12x^2 + 2$

3. Find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)

   $h(x) = \frac{1}{\sqrt{3x+1}}$

   Answer:

   $f(x) = \frac{1}{\sqrt{x}}$ and $g(x) = 3x + 1$

Application

EXAMPLE 10 Bacteria Count

The number N of bacteria in a refrigerated petri dish is given by

$$N(T) = 20T^2 - 80T + 500, \quad 2 \le T \le 14$$

where T is the temperature of the petri dish (in degrees Celsius). When the petri dish is removed from refrigeration, the temperature of the petri dish is given by

$$T(t) = 4t + 2, \quad 0 \le t \le 3$$

where t is the time (in hours).

a. Find the composition $N(T(t))$ and interpret its meaning in context.

b. Find the number of bacteria in the petri dish when $t = 2$ hours.

c. Find the time when the bacteria count reaches 2000.

Microbiologist

Solution

a.
$$\begin{aligned} N(T(t)) &= 20(4t + 2)^2 - 80(4t + 2) + 500 \\ &= 20(16t^2 + 16t + 4) - 320t - 160 + 500 \\ &= 320t^2 + 320t + 80 - 320t - 160 + 500 \\ &= 320t^2 + 420 \end{aligned}$$

The composite function $N(T(t))$ represents the number of bacteria as a function of the amount of time the petri dish has been out of refrigeration.

b. When $t = 2$, the number of bacteria is

$$N = 320(2)^2 + 420 = 1280 + 420 = 1700.$$

c. The bacteria count will reach $N = 2000$ when $320t^2 + 420 = 2000$. You can solve this equation for t algebraically as follows.

$$320t^2 + 420 = 2000$$

$$320t^2 = 1580$$

$$t^2 = \frac{79}{16}$$

$$t = \frac{\sqrt{79}}{4} \quad \Rightarrow \quad t \approx 2.22 \text{ hours}$$

So, the count will reach 2000 when $t \approx 2.22$ hours. Note that the negative value is rejected because it is not in the domain of the composite function. To confirm your solution, graph the equation $N = 320t^2 + 420$, as shown in Figure 1.36. Then use the *zoom* and *trace* features to approximate $N = 2000$ when $t \approx 2.22$, as shown in Figure 1.37.

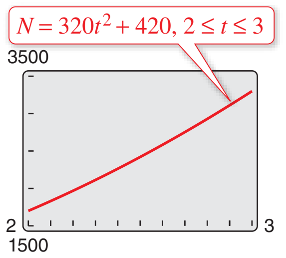


Figure 1.36

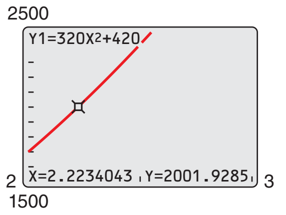


Figure 1.37

Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

The number N of bacteria in a refrigerated food is given by

$$N(T) = 8T^2 - 14T + 200, \quad 2 \le T \le 12$$

where T is the temperature of the food in degrees Celsius. When the food is removed from refrigeration, the temperature of the food is given by

$$T(t) = 2t + 2, \quad 0 \le t \le 5$$

where t is the time in hours. Find (a) $(N \circ T)(t)$ and (b) the time when the bacteria count reaches 1000.

1.5 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. Two functions f and g can be combined by the arithmetic operations of _______ , _______ , _______ , and _______ to create new functions.

2. The _______ of the function f with the function g is $(f \circ g)(x) = f(g(x))$.

3. The domain of $f \circ g$ is the set of all x in the domain of g such that _______ is in the domain of f.

4. To decompose a composite function, look for an _______ and an _______ function.

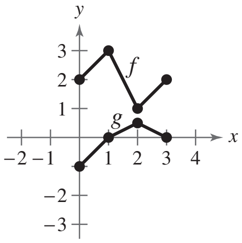
5. Given $f(x) = x^2 + 1$ and $(fg)(x) = 2x(x^2 + 1)$, what is $g(x)$?

6. Given $(f \circ g)(x) = f(x^2 + 1)$, what is $g(x)$?

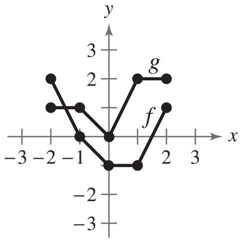
Procedures and Problem Solving

Graphing the Sum of Two Functions **In Exercises 7–10, use the graphs of f and g to graph $h(x) = (f + g)(x)$. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

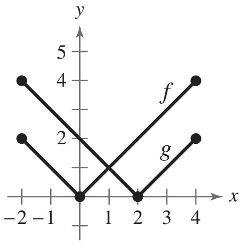
7.



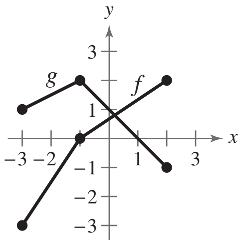
8.



9.



10.



Finding Arithmetic Combinations of Functions **In Exercises 11–18, find (a) $(f + g)(x)$, (b) $(f - g)(x)$, (c) $(fg)(x)$, and (d) $(f/g)(x)$. What is the domain of f/g?**

11. $f(x) = x + 3, \quad g(x) = x - 3$

12. $f(x) = 2x - 5, \quad g(x) = 1 - x$

13. $f(x) = 3x^2, \quad g(x) = 6 - 5x$

14. $f(x) = 2x + 5, \quad g(x) = x^2 - 9$

15. $f(x) = x^2 + 5, \quad g(x) = \sqrt{1 - x}$

16. $f(x) = \sqrt{x^2 - 4}, \quad g(x) = \dfrac{x^2}{x^2 + 1}$

17. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x^2}$

18. $f(x) = \dfrac{x}{x + 1}, \quad g(x) = \dfrac{1}{x^3}$

Evaluating an Arithmetic Combination of Functions **In Exercises 19–32, evaluate the indicated function for $f(x) = x^2 - 1$ and $g(x) = x - 2$ algebraically. If possible, use a graphing utility to verify your answer.**

19. $(f + g)(3)$ **20.** $(f - g)(-2)$

21. $(f - g)(0)$ **22.** $(f + g)(1)$

23. $(fg)(-6)$ **24.** $(fg)(4)$

25. $(f/g)(-5)$ **26.** $(f/g)(0)$

27. $(f - g)(t + 1)$ **28.** $(f + g)(t - 3)$

29. $(fg)(-5t)$ **30.** $(fg)(3t^2)$

31. $(f/g)(t - 4)$ **32.** $(f/g)(t + 2)$

Graphing an Arithmetic Combination of Functions **In Exercises 33–36, use a graphing utility to graph the functions f, g, and h in the same viewing window.**

33. $f(x) = \frac{1}{2}x, \quad g(x) = x - 1, \quad h(x) = f(x) + g(x)$

34. $f(x) = \frac{1}{3}x, \quad g(x) = -x + 4, \quad h(x) = f(x) - g(x)$

35. $f(x) = x^2, \quad g(x) = -2x + 5, \quad h(x) = f(x) \cdot g(x)$

36. $f(x) = 4 - x^2, \quad g(x) = x, \quad h(x) = f(x)/g(x)$

Graphing a Sum of Functions **In Exercises 37–40, use a graphing utility to graph f, g, and $f + g$ in the same viewing window. Which function contributes most to the magnitude of the sum when $0 \le x \le 2$? Which function contributes most to the magnitude of the sum when $x > 6$?**

37. $f(x) = 3x, \quad g(x) = -\dfrac{x^3}{10}$

38. $f(x) = \dfrac{x}{2}, \quad g(x) = \sqrt{x}$

39. $f(x) = 3x + 2, \quad g(x) = -\sqrt{x + 5}$

40. $f(x) = x^2 - \frac{1}{2}, \quad g(x) = -3x^2 - 1$

Compositions of Functions In Exercises 41–44, find (a) $f \circ g$, (b) $g \circ f$, and, if possible, (c) $(f \circ g)(0)$.

41. $f(x) = 2x^2, \quad g(x) = x + 4$
42. $f(x) = \sqrt[3]{x - 1}, \quad g(x) = x^3 + 1$
43. $f(x) = 3x + 5, \quad g(x) = 5 - x$
44. $f(x) = x^3, \quad g(x) = \dfrac{1}{x}$

Finding the Domain of a Composite Function In Exercises 45–54, determine the domains of (a) f, (b) g, and (c) $f \circ g$. Use a graphing utility to verify your results.

45. $f(x) = \sqrt{x - 7}, \quad g(x) = 4x^2$
46. $f(x) = \sqrt{x + 3}, \quad g(x) = \dfrac{x}{2}$
47. $f(x) = x^2 + 1, \quad g(x) = \sqrt{x}$
48. $f(x) = x^{1/4}, \quad g(x) = x^4$
49. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x + 3}$
50. $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{2x}$
51. $f(x) = |x - 4|, \quad g(x) = 3 - x$
52. $f(x) = \dfrac{2}{|x|} \quad g(x) = x - 5$
53. $f(x) = x + 2, \quad g(x) = \dfrac{1}{x^2 - 4}$
54. $f(x) = \dfrac{3}{x^2 - 1}, \quad g(x) = x + 1$

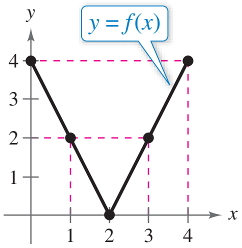
Determining Whether $f \circ g = g \circ f$ In Exercises 55–60, (a) find $f \circ g$, $g \circ f$, and the domain of $f \circ g$. (b) Use a graphing utility to graph $f \circ g$ and $g \circ f$. Determine whether $f \circ g = g \circ f$.

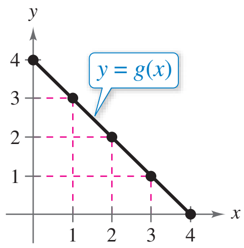
55. $f(x) = \sqrt{x + 4}, \quad g(x) = x^2$
56. $f(x) = \sqrt[3]{x + 1}, \quad g(x) = x^3 - 1$
57. $f(x) = \frac{1}{3}x - 3, \quad g(x) = 3x + 9$
58. $f(x) = \sqrt{x}, \quad g(x) = \sqrt{x}$
59. $f(x) = x^{2/3}, \quad g(x) = x^6$
60. $f(x) = |x|, \quad g(x) = -x^2 + 1$

Determining Whether $f \circ g = g \circ f$ In Exercises 61–66, (a) find $(f \circ g)(x)$ and $(g \circ f)(x)$, (b) determine algebraically whether $(f \circ g)(x) = (g \circ f)(x)$, and (c) use a graphing utility to complete a table of values for the two compositions to confirm your answer to part (b).

61. $f(x) = 5x + 4, \quad g(x) = \frac{1}{5}(x - 4)$
62. $f(x) = \frac{1}{4}(x - 1), \quad g(x) = 4x + 1$
63. $f(x) = \sqrt{x + 6}, \quad g(x) = x^2 - 5$
64. $f(x) = x^3 - 4, \quad g(x) = \sqrt[3]{x + 10}$
65. $f(x) = |x|, \quad g(x) = 2x^3$
66. $f(x) = \dfrac{6}{3x - 5}, \quad g(x) = -x$

Evaluating Combinations of Functions In Exercises 67–70, use the graphs of f and g to evaluate the functions.





67. (a) $(f + g)(3)$ (b) $(f/g)(2)$
68. (a) $(f - g)(1)$ (b) $(fg)(4)$
69. (a) $(f \circ g)(3)$ (b) $(g \circ f)(2)$
70. (a) $(f \circ g)(1)$ (b) $(g \circ f)(3)$

Identifying a Composite Function In Exercises 71–78, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)

71. $h(x) = (2x + 1)^2$
72. $h(x) = (1 - x)^3$
73. $h(x) = \sqrt[3]{x^2 - 4}$
74. $h(x) = \sqrt{9 - x}$
75. $h(x) = \dfrac{1}{x + 2}$
76. $h(x) = \dfrac{4}{(5x + 2)^2}$
77. $h(x) = (x + 4)^2 + 2(x + 4)$
78. $h(x) = (x + 3)^{3/2} + 4(x + 3)^{1/2}$

79. ***Why you should learn it*** *(p. 50)* The research and development department of an automobile manufacturer has determined that when required to stop quickly to avoid an accident, the distance (in feet) a car travels during the driver's reaction time is given by

$$R(x) = \tfrac{3}{4}x$$

where x is the speed of the car in miles per hour. The distance (in feet) traveled while the driver is braking is given by

$$B(x) = \tfrac{1}{15}x^2.$$

(a) Find the function that represents the total stopping distance T.

(b) Use a graphing utility to graph the functions R, B, and T in the same viewing window for $0 \le x \le 60$.

(c) Which function contributes most to the magnitude of the sum at higher speeds? Explain.

80. MODELING DATA

The table shows the total amounts (in billions of dollars) of health consumption expenditures in the United States (including Puerto Rico) for the years 2002 through 2012. The variables y_1, y_2, and y_3 represent out-of-pocket payments, insurance premiums, and other types of payments, respectively.
(Source: U.S. Centers for Medicare and Medicaid Services)

DATA

Spreadsheet at LarsonPrecalculus.com

Year	y_1	y_2	y_3
2002	222	1122	140
2003	238	1223	153
2004	252	1322	159
2005	267	1417	168
2006	277	1521	177
2007	294	1612	187
2008	301	1703	182
2009	301	1799	185
2010	306	1874	195
2011	316	1943	202
2012	328	2014	216

The data are approximated by the following models, where t represents the year, with $t = 2$ corresponding to 2002.

$$y_1 = -0.62t^2 + 18.7t + 188$$

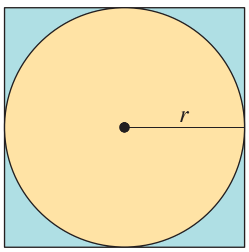
$$y_2 = -1.86t^2 + 116.4t + 890$$

$$y_3 = -0.12t^2 + 8.3t + 128$$

(a) Use the models and the *table* feature of a graphing utility to create a table showing the values of y_1, y_2, and y_3 for each year from 2002 through 2012. Compare these models with the original data. Are the models a good fit? Explain.

(b) Use the graphing utility to graph y_1, y_2, y_3, and $y_T = y_1 + y_2 + y_3$ in the same viewing window. What does the function y_T represent?

81. Geometry A square concrete foundation was prepared as a base for a large cylindrical gasoline tank (see figure).



x

(a) Write the radius r of the tank as a function of the length x of the sides of the square.

(b) Write the area A of the circular base of the tank as a function of the radius r.

(c) Find and interpret $(A \circ r)(x)$.

82. Geometry A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outermost ripple is given by $r(t) = 0.6t$, where t is the time (in seconds) after the pebble strikes the water. The area of the circle is given by $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

83. Business A company owns two retail stores. The annual sales (in thousands of dollars) of the stores each year from 2009 through 2015 can be approximated by the models

$$S_1 = 973 + 1.3t^2 \quad \text{and} \quad S_2 = 349 + 72.4t$$

where t is the year, with $t = 9$ corresponding to 2009.

(a) Write a function T that represents the total annual sales of the two stores.

(b) Use a graphing utility to graph S_1, S_2, and T in the same viewing window.

84. Business The annual cost C (in thousands of dollars) and revenue R (in thousands of dollars) for a company each year from 2009 through 2015 can be approximated by the models

$$C = 254 - 9t + 1.1t^2 \quad \text{and} \quad R = 341 + 3.2t$$

where t is the year, with $t = 9$ corresponding to 2009.

(a) Write a function P that represents the annual profits of the company.

(b) Use a graphing utility to graph C, R, and P in the same viewing window.

85. Biology The number of bacteria in a refrigerated food product is given by

$$N(T) = 10T^2 - 20T + 600, \quad 1 \le T \le 20$$

where T is the temperature of the food in degrees Celsius. When the food is removed from the refrigerator, the temperature of the food is given by

$$T(t) = 2t + 1$$

where t is the time in hours.

(a) Find the composite function $N(T(t))$ or $(N \circ T)(t)$ and interpret its meaning in the context of the situation.

(b) Find $(N \circ T)(12)$ and interpret its meaning.

(c) Find the time when the bacteria count reaches 1200.

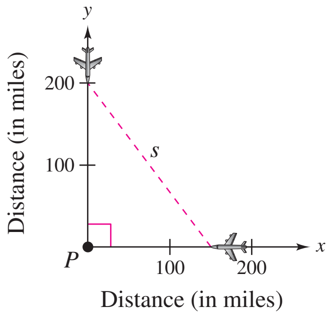
86. Environmental Science The spread of a contaminant is increasing in a circular pattern on the surface of a lake. The radius of the contaminant can be modeled by $r(t) = 5.25\sqrt{t}$, where r is the radius in meters and t is time in hours since contamination.

(a) Find a function that gives the area A of the circular leak in terms of the time t since the spread began.

(b) Find the size of the contaminated area after 36 hours.

(c) Find when the size of the contaminated area is 6250 square meters.

87. Air Traffic Control An air traffic controller spots two planes flying at the same altitude. Their flight paths form a right angle at point P. One plane is 150 miles from point P and is moving at 450 miles per hour. The other plane is 200 miles from point P and is moving at 450 miles per hour. Write the distance s between the planes as a function of time t.



88. Marketing The suggested retail price of a new car is p dollars. The dealership advertised a factory rebate of \$2000 and a 9% discount.

(a) Write a function R in terms of p giving the cost of the car after receiving the rebate from the factory.

(b) Write a function S in terms of p giving the cost of the car after receiving the dealership discount.

(c) Form the composite functions $(R \circ S)(p)$ and $(S \circ R)(p)$ and interpret each.

(d) Find $(R \circ S)(24{,}795)$ and $(S \circ R)(24{,}795)$. Which yields the lower cost for the car? Explain.

Conclusions

True or False? **In Exercises 89 and 90, determine whether the statement is true or false. Justify your answer.**

89. A function that represents the graph of $f(x) = x^2$ shifted three units to the right is $f(g(x))$, where $g(x) = x + 3$.

90. Given two functions f and g, you can calculate $(f \circ g)(x)$ if and only if the range of g is a subset of the domain of f.

91. Exploration The function in Example 9 can be decomposed in other ways. For which of the following pairs of functions is $h(x) = \dfrac{1}{(x-2)^2}$ equal to $f(g(x))$?

(a) $g(x) = \dfrac{1}{x-2}$ and $f(x) = x^2$

(b) $g(x) = x^2$ and $f(x) = \dfrac{1}{x-2}$

(c) $g(x) = (x-2)^2$ and $f(x) = \dfrac{1}{x}$

92. Proof Prove that the product of two odd functions is an even function, and that the product of two even functions is an even function.

93. Proof Use examples to hypothesize whether the product of an odd function and an even function is even or odd. Then prove your hypothesis.

Exploration **In Exercises 94 and 95, three siblings are of three different ages. The oldest is twice the age of the middle sibling, and the middle sibling is six years older than one-half the age of the youngest.**

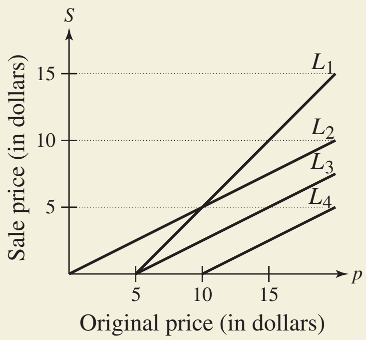
94. (a) Write a composite function that gives the oldest sibling's age in terms of the youngest. Explain how you arrived at your answer.

(b) The oldest sibling is 16 years old. Find the ages of the other two siblings.

95. (a) Write a composite function that gives the youngest sibling's age in terms of the oldest. Explain how you arrived at your answer.

(b) The youngest sibling is two years old. Find the ages of the other two siblings.

96. HOW DO YOU SEE IT? The graphs labeled L_1, L_2, L_3, and L_4 represent four different pricing discounts, where p is the original price (in dollars) and S is the sale price (in dollars). Match each function with its graph. Describe the situations in parts (c) and (d).



(a) $f(p)$: A 50% discount is applied.

(b) $g(p)$: A \$5 discount is applied.

(c) $(g \circ f)(p)$

(d) $(f \circ g)(p)$

Cumulative Mixed Review

Evaluating an Equation **In Exercises 97–100, find three points that lie on the graph of the equation. (There are many correct answers.)**

97. $y = -x^2 + x - 5$

98. $y = \frac{1}{5}x^3 - 4x^2 + 1$

99. $x^2 + y^2 = 49$

100. $y = \dfrac{x}{x^2 - 5}$

1.6 Inverse Functions

What you should learn

- Find inverse functions informally and verify that two functions are inverse functions of each other.
- Use graphs of functions to decide whether functions have inverse functions.
- Determine whether functions are one-to-one.
- Find inverse functions algebraically.

Why you should learn it

Inverse functions can be helpful in further exploring how two variables relate to each other. For example, in Exercise 115 on page 69, you will use inverse functions to find the European shoe sizes from the corresponding U.S. shoe sizes.

Inverse Functions

Recall from Section 1.2 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 4$ from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8\}$ can be written as follows.

$$f(x) = x + 4: \{(1, 5), (2, 6), (3, 7), (4, 8)\}$$

In this case, by interchanging the first and second coordinates of each of these ordered pairs, you can form the **inverse function** of f, which is denoted by f^{-1}. It is a function from the set B to the set A, and can be written as follows.

$$f^{-1}(x) = x - 4: \{(5, 1), (6, 2), (7, 3), (8, 4)\}$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 1.38. Also note that the functions f and f^{-1} have the effect of "undoing" each other. In other words, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = f(x - 4) = (x - 4) + 4 = x$$

$$f^{-1}(f(x)) = f^{-1}(x + 4) = (x + 4) - 4 = x$$

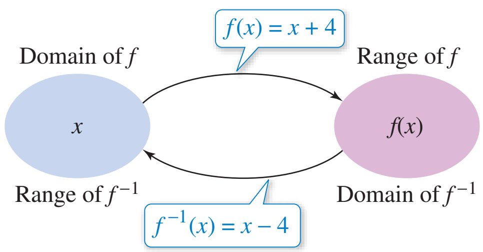


Figure 1.38

EXAMPLE 1 Finding Inverse Functions Informally

Find the inverse function of $f(x) = 4x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The function f *multiplies* each input by 4. To "undo" this function, you need to *divide* each input by 4. So, the inverse function of $f(x) = 4x$ is given by

$$f^{-1}(x) = \frac{x}{4}.$$

You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f\left(\frac{x}{4}\right) = 4\left(\frac{x}{4}\right) = x$$

$$f^{-1}(f(x)) = f^{-1}(4x) = \frac{4x}{4} = x$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the inverse function of $f(x) = \frac{1}{5}x$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Do not be confused by the use of the exponent -1 to denote the inverse function f^{-1}. In this text, whenever f^{-1} is written, it always refers to the inverse function of the function f and not to the reciprocal of $f(x)$, which is given by

$$\frac{1}{f(x)}.$$

EXAMPLE 2 Finding Inverse Functions Informally

Find the inverse function of $f(x) = x - 6$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

Solution

The function f *subtracts* 6 from each input. To "undo" this function, you need to *add* 6 to each input. So, the inverse function of $f(x) = x - 6$ is given by $f^{-1}(x) = x + 6$. You can verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function as follows.

$$f(f^{-1}(x)) = f(x + 6) = (x + 6) - 6 = x$$

$$f^{-1}(f(x)) = f^{-1}(x - 6) = (x - 6) + 6 = x$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the inverse function of $f(x) = x + 7$. Then verify that both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to the identity function.

A table of values can help you understand inverse functions. For instance, the first table below shows several values of the function in Example 2. Interchange the rows of this table to obtain values of the inverse function.

x	-2	-1	0	1	2
$f(x)$	-8	-7	-6	-5	-4

x	-8	-7	-6	-5	-4
$f^{-1}(x)$	-2	-1	0	1	2

In the table at the left, each output is 6 less than the input, and in the table at the right, each output is 6 more than the input.

The formal definition of an inverse function is as follows.

Definition of Inverse Function

Let f and g be two functions such that

$f(g(x)) = x$ for every x in the domain of g

and

$g(f(x)) = x$ for every x in the domain of f.

Under these conditions, the function g is the **inverse function** of the function f. The function g is denoted by f^{-1} (read "f-inverse"). So,

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

The domain of f must be equal to the range of f^{-1}, and the range of f must be equal to the domain of f^{-1}.

If the function g is the inverse function of the function f, then it must also be true that the function f is the inverse function of the function g. For this reason, you can say that the functions f and g are *inverse functions of each other*.

EXAMPLE 3 Verifying Inverse Functions Algebraically

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

Solution

$$f(g(x)) = f\left(\sqrt[3]{\frac{x+1}{2}}\right) = 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 = 2\left(\frac{x+1}{2}\right) - 1 = x + 1 - 1 = x$$

$$g(f(x)) = g(2x^3 - 1) = \sqrt[3]{\frac{(2x^3-1)+1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Show that $f(x) = x^5$ and $g(x) = \sqrt[5]{x}$ are inverse functions of each other.

EXAMPLE 4 Verifying Inverse Functions Algebraically

Which of the functions is the inverse function of $f(x) = \dfrac{5}{x-2}$?

$$g(x) = \frac{x-2}{5} \quad \text{or} \quad h(x) = \frac{5}{x} + 2$$

Solution

By forming the composition of f with g, you have

$$f(g(x)) = f\left(\frac{x-2}{5}\right) = \frac{5}{\left(\frac{x-2}{5}\right) - 2} = \frac{25}{x-12} \neq x.$$

Because this composition is not equal to the identity function x, it follows that g *is not* the inverse function of f. By forming the composition of f with h, you have

$$f(h(x)) = f\left(\frac{5}{x} + 2\right) = \frac{5}{\left(\frac{5}{x} + 2\right) - 2} = \frac{5}{\left(\frac{5}{x}\right)} = x.$$

So, it appears that h is the inverse function of f. You can confirm this by showing that the composition of h with f is also equal to the identity function.

$$h(f(x)) = h\left(\frac{5}{x-2}\right) = \frac{5}{\left(\frac{5}{x-2}\right)} + 2 = x - 2 + 2 = x$$

✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Which of the functions is the inverse function of $f(x) = \dfrac{x-4}{7}$?

$$g(x) = 7x + 4 \qquad h(x) = \frac{7}{x-4}$$

Technology Tip

Most graphing utilities can graph $y = x^{1/3}$ in two ways:

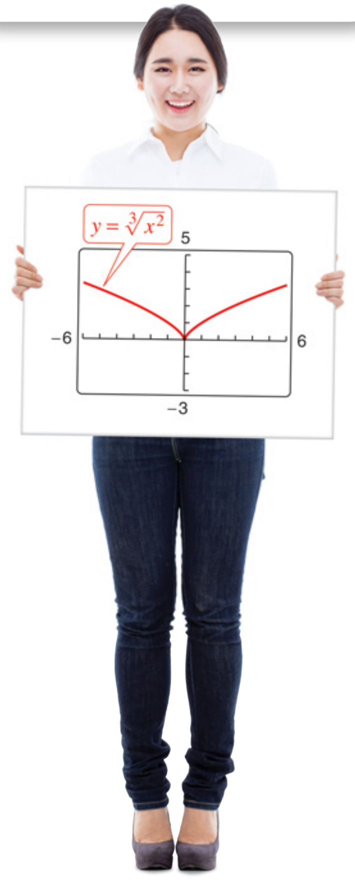
$y_1 = x \wedge (1/3)$ or

$y_1 = \sqrt[3]{x}$.

On some graphing utilities, you may not be able to obtain the complete graph of $y = x^{2/3}$ by entering $y_1 = x \wedge (2/3)$. If not, you should use

$y_1 = (x \wedge (1/3))^2$ or

$y_1 = \sqrt[3]{x^2}$.



Point out to students that when using a graphing utility, it is important to know a function's behavior because the graphing utility may show an incomplete function. For instance, it is important to know that the domain of $f(x) = x^{2/3}$ is all real numbers, because a graphing utility may show an incomplete graph of the function, depending on how the function was entered.

The Graph of an Inverse Function

The graphs of a function f and its inverse function f^{-1} are related to each other in the following way. If the point

(a, b)

lies on the graph of f, then the point

(b, a)

must lie on the graph of f^{-1}, and vice versa. This means that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$, as shown in Figure 1.39.

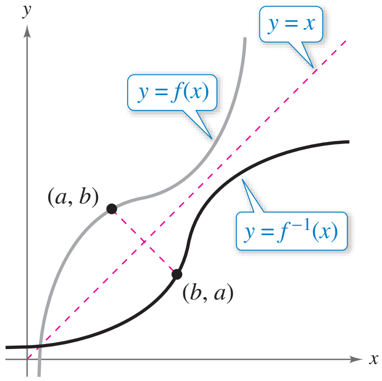


Figure 1.39

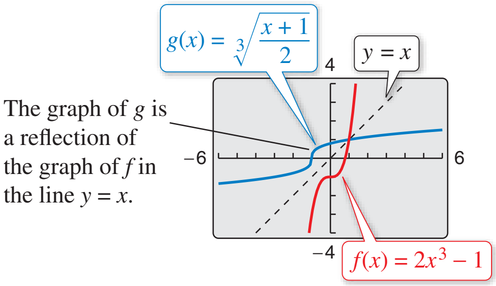
> **Technology Tip**
>
> Many graphing utilities have a built-in feature for drawing an inverse function. For instructions on how to use the *draw inverse* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website*.

EXAMPLE 5 Verifying Inverse Functions Graphically

Verify that the functions f and g from Example 3 are inverse functions of each other graphically.

Solution

From the figure, it appears that f and g are inverse functions of each other.



Checkpoint Audio-video solution in English & Spanish at LarsonPrecalculus.com.

Verify that $f(x) = x^5$ and $g(x) = \sqrt[5]{x}$ are inverse functions of each other graphically.

EXAMPLE 6 Verifying Inverse Functions Numerically

Verify that the functions $f(x) = \dfrac{x - 5}{2}$ and $g(x) = 2x + 5$ are inverse functions of each other numerically.

Solution

You can verify that f and g are inverse functions of each other *numerically* by using a graphing utility. Enter $y_1 = f(x)$, $y_2 = g(x)$, $y_3 = f(g(x))$, and $y_4 = g(f(x))$. Then use the *table* feature to create a table.

```
Plot1  Plot2  Plot3
\Y1=(X-5)/2
\Y2=2X+5
\Y3=Y1(Y2)
\Y4=Y2(Y1)
\Y5=
\Y6=
\Y7=
```

X	Y3	Y4
-2	-2	-2
-1	-1	-1
0	0	0
1	1	1
2	2	2
3	3	3
4	4	4

X=-2

Note that the entries for x, y_3, and y_4 are the same. So, it appears that $f(g(x)) = x$ and $g(f(x)) = x$, and that f and g are inverse functions of each other.

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Verify that $f(x) = x^5$ and $g(x) = \sqrt[5]{x}$ are inverse functions of each other numerically.

The Existence of an Inverse Function

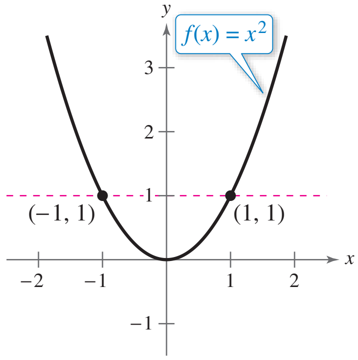
To have an inverse function, a function must be **one-to-one,** which means that no two elements in the domain of f correspond to the same element in the range of f.

Definition of a One-to-One Function

A function f is **one-to-one** when, for a and b in its domain, $f(a) = f(b)$ implies that $a = b$.

Existence of an Inverse Function

A function f has an inverse function f^{-1} if and only if f is one-to-one.



f*(*x*) = x^2 *is not one-to-one.
Figure 1.40

From its graph, it is easy to tell whether a function of x is one-to-one. Simply check to see that every horizontal line intersects the graph of the function at most once. This is called the **Horizontal Line Test.** For instance, Figure 1.40 shows the graph of $f(x) = x^2$. On the graph, you can find a horizontal line that intersects the graph twice. So, f is not one-to-one and does not have an inverse function.

Two special types of functions that pass the Horizontal Line Test are those that are increasing or decreasing on their entire domains. If f is *increasing* on its entire domain, then f is one-to-one. If f is *decreasing* on its entire domain, then f is one-to-one.

EXAMPLE 7 Testing Whether a Function Is One-to-One

Is the function $f(x) = \sqrt{x} + 1$ one-to-one?

Algebraic Solution

Let a and b be nonnegative real numbers with $f(a) = f(b)$.

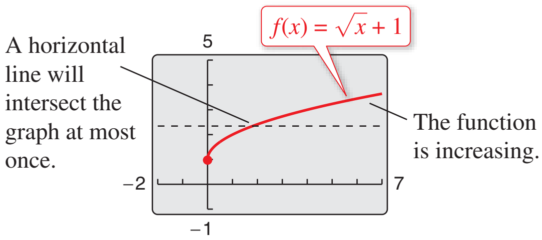
$$\sqrt{a} + 1 = \sqrt{b} + 1 \qquad \text{Set } f(a) = f(b).$$

$$\sqrt{a} = \sqrt{b}$$

$$a = b$$

So, $f(a) = f(b)$ implies that $a = b$. You can conclude that f *is* one-to-one and *does* have an inverse function.

Graphical Solution



From the figure, you can conclude that f *is* one-to-one and *does* have an inverse function.

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Is the function $f(x) = \sqrt[3]{x}$ one-to-one?

EXAMPLE 8 Testing Whether a Function Is One-to-One

See LarsonPrecalculus.com for an interactive version of this type of example.

To determine whether $f(x) = x^2 - x$ is one-to-one, note that $f(-1) = (-1)^2 - (-1) = 2$ and $f(2) = 2^2 - 2 = 2$. Because you have two inputs matched with the same output, f *is not* one-to-one and *does not* have an inverse function. You can confirm this graphically by noticing that the horizontal line $y = 2$ intersects the graph of f twice, as shown in Figure 1.41.

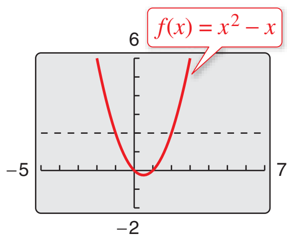


Figure 1.41

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Is the function $f(x) = |x|$ one-to-one?

Finding Inverse Functions Algebraically

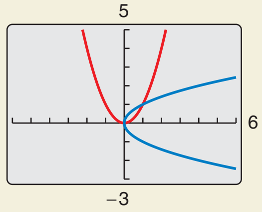
For relatively simple functions (such as the ones in Examples 1 and 2), you can find inverse functions by inspection. For more complicated functions, however, it is best to use the following guidelines.

Finding an Inverse Function

1. Use the Horizontal Line Test to decide whether f has an inverse function.
2. In the equation for $f(x)$, replace $f(x)$ with y.
3. Interchange the roles of x and y, and solve for y.
4. Replace y with $f^{-1}(x)$ in the new equation.
5. Verify that f and f^{-1} are inverse functions of each other by showing that the domain of f is equal to the range of f^{-1}, the range of f is equal to the domain of f^{-1}, and $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

What's Wrong?

You use a graphing utility to graph $y_1 = x^2$ and then use the *draw inverse* feature to conclude that $f(x) = x^2$ has an inverse function (see figure). What's wrong?



The key step in these guidelines is Step 3—interchanging the roles of x and y. This step corresponds to the fact that inverse functions have ordered pairs with the coordinates reversed.

EXAMPLE 9 Finding an Inverse Function Algebraically

Find the inverse function of

$$f(x) = \frac{5 - 3x}{2}.$$

Solution

The graph of f in Figure 1.42 passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

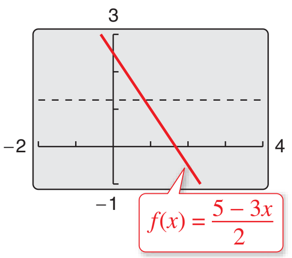


Figure 1.42

$$f(x) = \frac{5 - 3x}{2}$$ Write original function.

$$y = \frac{5 - 3x}{2}$$ Replace $f(x)$ with y.

$$x = \frac{5 - 3y}{2}$$ Interchange x and y.

$$2x = 5 - 3y$$ Multiply each side by 2.

$$3y = 5 - 2x$$ Isolate the y-term.

$$y = \frac{5 - 2x}{3}$$ Solve for y.

$$f^{-1}(x) = \frac{5 - 2x}{3}$$ Replace y with $f^{-1}(x)$.

The domains and ranges of f and f^{-1} consist of all real numbers. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

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Find the inverse function of $f(x) = 2x - 3$.

The *draw inverse* feature is particularly useful when you cannot find an expression for the inverse function of a given function. For example, it would be very difficult to determine the equation for the inverse function of the one-to-one function

$f(x) = \frac{1}{4}x^5 + \frac{1}{4}x^3 + \frac{1}{2}x - 1.$

However, it is easy to use the *draw inverse* feature to obtain the *graph* of the inverse function.

EXAMPLE 10 Finding an Inverse Function Algebraically

Find the inverse function of $f(x) = \sqrt{2x - 3}$.

Solution

The graph of f in Figure 1.43 passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

$f(x) = \sqrt{2x - 3}$ — Write original function.

$y = \sqrt{2x - 3}$ — Replace $f(x)$ with y.

$x = \sqrt{2y - 3}$ — Interchange x and y.

$x^2 = 2y - 3$ — Square each side.

$2y = x^2 + 3$ — Isolate y.

$y = \dfrac{x^2 + 3}{2}$ — Solve for y.

$f^{-1}(x) = \dfrac{x^2 + 3}{2}, \ x \geq 0$ — Replace y with $f^{-1}(x)$.

The range of f is the interval $[0, \infty)$, which implies that the domain of f^{-1} is the interval $[0, \infty)$. Moreover, the domain of f is the interval $\left[\frac{3}{2}, \infty\right)$, which implies that the range of f^{-1} is the interval $\left[\frac{3}{2}, \infty\right)$. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

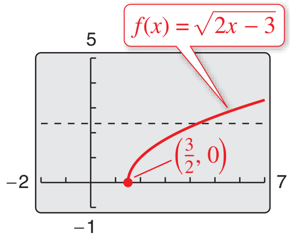


Figure 1.43

Activities

1. Given $f(x) = 5x - 7$, find $f^{-1}(x)$.

   Answer: $f^{-1}(x) = \dfrac{x + 7}{5}$

2. Show that f and g are inverse functions by showing that $f(g(x)) = x$ and $g(f(x)) = x$.

   $f(x) = 3x^3 + 1$

   $g(x) = \sqrt[3]{\dfrac{x - 1}{3}}$

3. Describe the graphs of functions that have inverse functions and show how the graph of a function and its inverse function are related.

✔ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the inverse function of $f(x) = \sqrt[3]{x + 10}$.

A function f with an implied domain of all real numbers may not pass the Horizontal Line Test. In this case, the domain of f may be restricted so that f does have an inverse function. Recall from Figure 1.40 that $f(x) = x^2$ is not one-to-one. By restricting the domain of f to $x \geq 0$, the function is one-to-one and does have an inverse function (see Figure 1.44).

EXAMPLE 11 Restricting the Domain of a Function

Find the inverse function of $f(x) = x^2, x \geq 0$.

Solution

The graph of f in Figure 1.44 passes the Horizontal Line Test. So, you know that f is one-to-one and has an inverse function.

$f(x) = x^2, \ x \geq 0$ — Write original function with restricted domain.

$y = x^2$ — Replace $f(x)$ with y.

$x = y^2, \ y \geq 0$ — Interchange x and y.

$y = \sqrt{x}$ — Solve for y. Extract positive square root $(y \geq 0)$.

$f^{-1}(x) = \sqrt{x}$ — Replace y with $f^{-1}(x)$.

The graph of f^{-1} in Figure 1.45 is the reflection of the graph of f in the line $y = x$. The domains and ranges of f and f^{-1} consist of all nonnegative real numbers. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

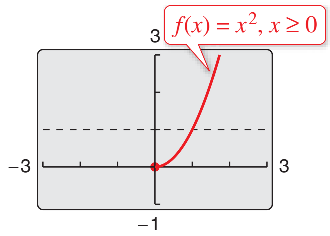


Figure 1.44

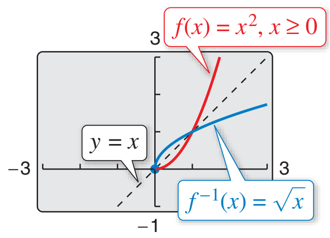


Figure 1.45

✔ ***Checkpoint*** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find the inverse function of $f(x) = (x - 1)^2, x \geq 1$.

1.6 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1–4, fill in the blank(s).

1. If f and g are functions such that $f(g(x)) = x$ and $g(f(x)) = x$, then the function g is the ________ function of f, and is denoted by ________ .

2. The domain of f is the ________ of f^{-1}, and the ________ of f^{-1} is the range of f.

3. The graphs of f and f^{-1} are reflections of each other in the line ________ .

4. To have an inverse function, a function f must be ________ ; that is, $f(a) = f(b)$ implies $a = b$.

5. How many times can a horizontal line intersect the graph of a function that is one-to-one?

6. Can $(1, 4)$ and $(2, 4)$ be two ordered pairs of a one-to-one function?

Procedures and Problem Solving

Finding Inverse Functions Informally **In Exercises 7–14, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.**

7. $f(x) = 6x$ **8.** $f(x) = \frac{1}{3}x$

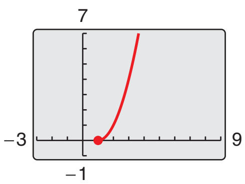
9. $f(x) = x + 11$ **10.** $f(x) = x + 3$

11. $f(x) = (x - 1)/2$ **12.** $f(x) = 4(x - 1)$

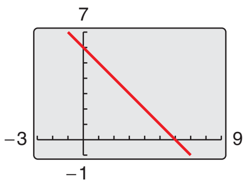
13. $f(x) = \sqrt[3]{x}$ **14.** $f(x) = x^7$

Identifying Graphs of Inverse Functions **In Exercises 15–18, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]**

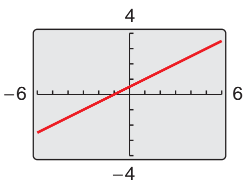
(a)



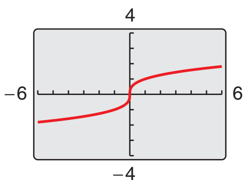
(b)



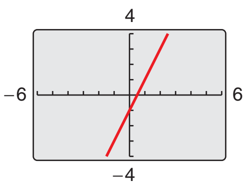
(c)



(d)



15.



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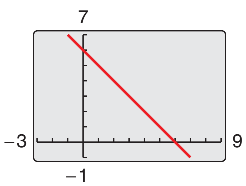
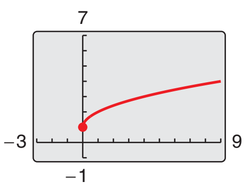
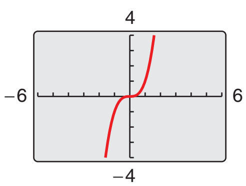


17.



18.



Verifying Inverse Functions Algebraically **In Exercises 19–24, show that f and g are inverse functions algebraically. Use a graphing utility to graph f and g in the same viewing window. Describe the relationship between the graphs.**

19. $f(x) = x^3, \quad g(x) = \sqrt[3]{x}$ **20.** $f(x) = \dfrac{1}{x}, \quad g(x) = \dfrac{1}{x}$

21. $f(x) = \sqrt{x - 4}; \quad g(x) = x^2 + 4, \quad x \ge 0$

22. $f(x) = 9 - x^2, x \ge 0; \quad g(x) = \sqrt{9 - x}$

23. $f(x) = 1 - x^3, \quad g(x) = \sqrt[3]{1 - x}$

24. $f(x) = \dfrac{1}{1 + x}, x \ge 0; \quad g(x) = \dfrac{1 - x}{x}, 0 < x \le 1$

Algebraic-Graphical-Numerical **In Exercises 25–34, show that f and g are inverse functions (a) algebraically, (b) graphically, and (c) numerically.**

25. $f(x) = -\dfrac{7}{2}x - 3, \quad g(x) = -\dfrac{2x + 6}{7}$

26. $f(x) = \dfrac{x - 9}{4}, \quad g(x) = 4x + 9$

27. $f(x) = x^3 + 5, \quad g(x) = \sqrt[3]{x - 5}$

28. $f(x) = \dfrac{x^3}{2}, \quad g(x) = \sqrt[3]{2x}$

29. $f(x) = -\sqrt{x - 8}; \quad g(x) = 8 + x^2, \quad x \le 0$

30. $f(x) = \sqrt[4]{3x - 10}; \quad g(x) = \dfrac{x^4 + 10}{3}, \quad x \ge 0$

31. $f(x) = 2x, \quad g(x) = \dfrac{x}{2}$

32. $f(x) = -3x + 5, \quad g(x) = -\dfrac{x - 5}{3}$

33. $f(x) = \dfrac{x - 1}{x + 5}, \quad g(x) = -\dfrac{5x + 1}{x - 1}$

34. $f(x) = \dfrac{x + 3}{x - 2}, \quad g(x) = \dfrac{2x + 3}{x - 1}$

Identifying Whether Functions Have Inverses **In Exercises 35–38, does the function have an inverse? Explain.**

35. *Domain* → *Range*

- 1 can → \$1
- 6 cans → \$5
- 12 cans → \$9
- 24 cans → \$16

36. *Domain* → *Range*

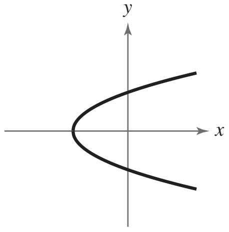
- 1/2 hour → \$40
- 1 hour → \$40
- 2 hours → \$70
- 4 hours → \$120

37. $\{(-3, 6), (-1, 5), (0, 6)\}$

38. $\{(2, 4), (3, 7), (7, 2)\}$

Recognizing One-to-One Functions **In Exercises 39–44, determine whether the graph is that of a function. If so, determine whether the function is one-to-one.**

39.



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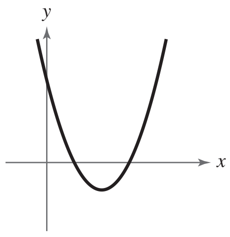

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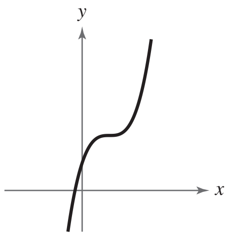

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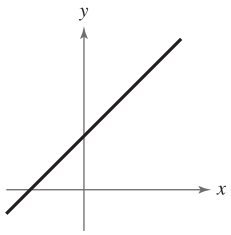

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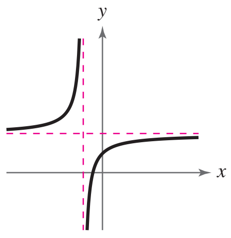
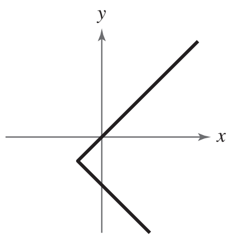


44.



Using the Horizontal Line Test **In Exercises 45–56, use a graphing utility to graph the function and use the Horizontal Line Test to determine whether the function is one-to-one and so has an inverse function.**

45. $f(x) = 3 - \frac{1}{2}x$

46. $f(x) = 2x^{1/3} + 5$

47. $h(x) = \dfrac{x^2}{x^2 + 1}$

48. $g(x) = \dfrac{4 - x}{6x^2}$

49. $h(x) = \sqrt{16 - x^2}$

50. $f(x) = -2x\sqrt{16 - x^2}$

51. $f(x) = 10$

52. $f(x) = -0.65$

53. $g(x) = (x + 5)^3$

54. $f(x) = x^5 - 7$

55. $h(x) = |x| - |x - 4|$

56. $f(x) = -\dfrac{|x^2 - 9|}{|x^2 + 7|}$

Analyzing a Piecewise-Defined Function **In Exercises 57 and 58, sketch the graph of the piecewise-defined function by hand and use the graph to determine whether an inverse function exists.**

57. $f(x) = \begin{cases} x^2, & 0 \le x \le 1 \\ x, & x > 1 \end{cases}$

58. $f(x) = \begin{cases} (x - 2)^3, & x < 3 \\ (x - 4)^2, & x \ge 3 \end{cases}$

Testing Whether a Function Is One-to-One **In Exercises 59–70, determine algebraically whether the function is one-to-one. Verify your answer graphically. If the function is one-to-one, find its inverse.**

59. $f(x) = x^4$

60. $g(x) = x^2 - x^4$

61. $f(x) = \dfrac{3x + 4}{5}$

62. $f(x) = 3x + 5$

63. $f(x) = \dfrac{1}{x^2}$

64. $h(x) = \dfrac{4}{x^2}$

65. $f(x) = (x + 3)^2, \quad x \ge -3$

66. $q(x) = (x - 5)^2, \quad x \le 5$

67. $f(x) = \sqrt{2x + 3}$

68. $f(x) = \sqrt{x - 2}$

69. $f(x) = |x - 2|, \quad x \le 2$

70. $f(x) = \dfrac{x^2}{x^2 + 1}, \quad x \ge 0$

Finding an Inverse Function Algebraically **In Exercises 71–80, find the inverse function of f algebraically. Use a graphing utility to graph both f and f^{-1} in the same viewing window. Describe the relationship between the graphs.**

71. $f(x) = 4x - 9$

72. $f(x) = 3x$

73. $f(x) = x^5$

74. $f(x) = x^3 + 1$

75. $f(x) = x^4, \quad x \le 0$

76. $f(x) = x^2, \quad x \ge 0$

77. $f(x) = \sqrt{4 - x^2}, \quad 0 \le x \le 2$

78. $f(x) = \sqrt{16 - x^2}, \quad -4 \le x \le 0$

79. $f(x) = \dfrac{4}{x^3}$

80. $f(x) = \dfrac{6}{\sqrt{x}}$

Think About It In Exercises 81–90, restrict the domain of the function f so that the function is one-to-one and has an inverse function. Then find the inverse function f^{-1}. State the domains and ranges of f and f^{-1}. Explain your results. (There are many correct answers.)

81. $f(x) = (x - 2)^2$

82. $f(x) = x^4 + 1$

83. $f(x) = |x + 2|$

84. $f(x) = |x - 2|$

85. $f(x) = (x + 3)^2$

86. $f(x) = (x - 4)^2$

87. $f(x) = -2x^2 - 5$

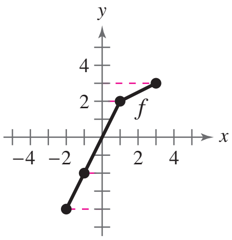
88. $f(x) = \frac{1}{2}x^2 + 1$

89. $f(x) = |x - 4| + 1$

90. $f(x) = -|x - 1| - 2$

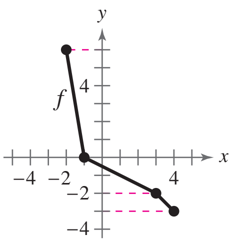
Using the Properties of Inverse Functions In Exercises 91 and 92, use the graph of the function f to complete the table and sketch the graph of f^{-1}.

91.



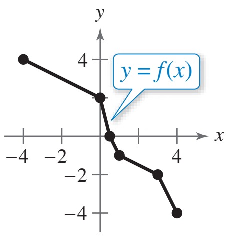
x	$f^{-1}(x)$
−4	
−2	
2	
3	

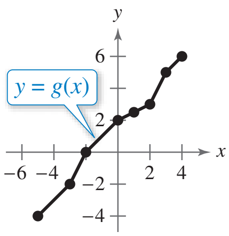
92.



x	$f^{-1}(x)$
−3	
−2	
0	
6	

Using Graphs to Evaluate a Function In Exercises 93–100, use the graphs of $y = f(x)$ and $y = g(x)$ to evaluate the function.





93. $f^{-1}(-4)$

94. $g^{-1}(0)$

95. $(f \circ g)(2)$

96. $g(f(-4))$

97. $f^{-1}(g(0))$

98. $(g^{-1} \circ f)(3)$

99. $(g \circ f^{-1})(2)$

100. $(f^{-1} \circ g^{-1})(6)$

Using the *Draw Inverse* Feature In Exercises 101–104, (a) use a graphing utility to graph the function f, (b) use the *draw inverse* feature of the graphing utility to draw the inverse relation of the function, and (c) determine whether the inverse relation is an inverse function. Explain your reasoning.

101. $f(x) = x^3 + x + 1$

102. $f(x) = x\sqrt{4 - x^2}$

103. $f(x) = \dfrac{3x^2}{x^2 + 1}$

104. $f(x) = \dfrac{4x}{\sqrt{x^2 + 15}}$

Evaluating a Composition of Functions In Exercises 105–110, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value or function.

105. $(f^{-1} \circ g^{-1})(1)$

106. $(g^{-1} \circ f^{-1})(-3)$

107. $(f^{-1} \circ f^{-1})(-6)$

108. $(g^{-1} \circ g^{-1})(4)$

109. $f^{-1} \circ g^{-1}$

110. $g^{-1} \circ f^{-1}$

Finding a Composition of Functions In Exercises 111–114, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the specified function.

111. $g^{-1} \circ f^{-1}$

112. $f^{-1} \circ g^{-1}$

113. $(f \circ g)^{-1}$

114. $(g \circ f)^{-1}$

115. ***Why you should learn it*** *(p. 60)* The table shows men's shoe sizes in the United States and the corresponding European shoe sizes. Let $y = f(x)$ represent the function that gives the men's European shoe size in terms of x, the men's U.S. size.

Men's U.S. shoe size	Men's European shoe size
8	41
9	42
10	43
11	44
12	45
13	46

(a) Is f one-to-one? Explain.

(b) Find $f(11)$.

(c) Find $f^{-1}(43)$, if possible.

(d) Find $f(f^{-1}(41))$.

(e) Find $f^{-1}(f(12))$.

116. Fashion Design Let $y = g(x)$ represent the function that gives the women's European shoe size in terms of x, the women's U.S. size. A women's U.S. size 6 shoe corresponds to a European size 37. Find $g^{-1}(g(6))$.

117. **Military Science** You can encode and decode messages using functions and their inverses. To code a message, first translate the letters to numbers using 1 for "A," 2 for "B," and so on. Use 0 for a space. So, "A ball" becomes

1 0 2 1 12 12.

Then, use a one-to-one function to convert to coded numbers. Using $f(x) = 2x - 1$, "A ball" becomes

1 −1 3 1 23 23.

(a) Encode "Call me later" using the function $f(x) = 5x + 4$.

(b) Find the inverse function of $f(x) = 5x + 4$ and use it to decode 119 44 9 104 4 104 49 69 29.

118. **Production Management** Your wage is \$12.00 per hour plus \$0.55 for each unit produced per hour. So, your hourly wage y in terms of the number of units produced x is $y = 12 + 0.55x$.

(a) Find the inverse function. What does each variable in the inverse function represent?

(b) Use a graphing utility to graph the function and its inverse function.

(c) Use the *trace* feature of the graphing utility to find the hourly wage when 9 units are produced per hour.

(d) Use the *trace* feature of the graphing utility to find the number of units produced per hour when your hourly wage is \$21.35.

Conclusions

True or False? **In Exercises 119 and 120, determine whether the statement is true or false. Justify your answer.**

119. If f is an even function, then f^{-1} exists.

120. If the inverse function of f exists, and the graph of f has a y-intercept, then the y-intercept of f is an x-intercept of f^{-1}.

Think About It **In Exercises 121–124, determine whether the situation could be represented by a one-to-one function. If so, write a statement that describes the inverse function.**

121. The number of miles n a marathon runner has completed in terms of the time t in hours

122. The population p of a town in terms of the year t from 2005 through 2015 given that the population was greatest in 2012

123. The depth of the tide d at a beach in terms of the time t over a 24-hour period

124. The height h in inches of a human child from age 2 to age 14 in terms of his or her age n in years

125. **Writing** Describe the relationship between the graph of a function f and the graph of its inverse function f^{-1}.

126. **Think About It** The domain of a one-to-one function f is $[0, 9]$ and the range is $[-3, 3]$. Find the domain and range of f^{-1}.

127. **Think About It** The function $f(x) = \frac{9}{5}x + 32$ can be used to convert a temperature of x degrees Celsius to its corresponding temperature in degrees Fahrenheit.

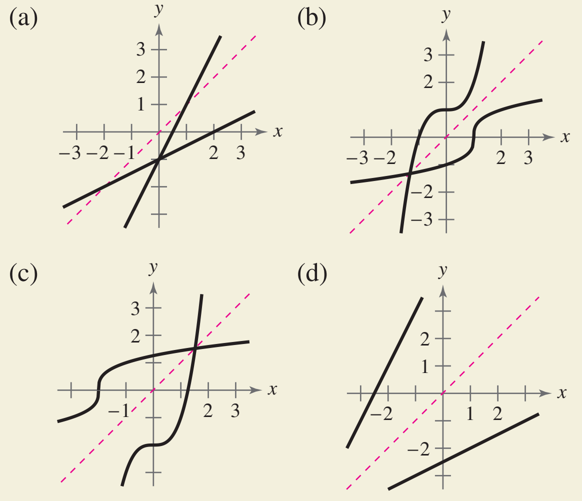
(a) Using the expression for f, make a conceptual argument to show that f has an inverse function.

(b) What does $f^{-1}(50)$ represent?

128. **Think About It** A function f is increasing over its entire domain. Does f have an inverse function? Explain.

129. **Think About It** Describe a type of function that is *not* one-to-one on any interval of its domain.

130. **HOW DO YOU SEE IT?** Decide whether the two functions shown in each graph appear to be inverse functions of each other. Explain your reasoning.



131. **Proof** Prove that if f and g are one-to-one functions, then $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

132. **Proof** Prove that if f is a one-to-one odd function, then f^{-1} is an odd function.

Cumulative Mixed Review

Simplifying a Rational Expression **In Exercises 133–136, write the rational expression in simplest form.**

133. $\dfrac{27x^3}{3x^2}$

134. $\dfrac{5x^2y^2 + 25x^2y}{xy + 5x}$

135. $\dfrac{x^2 - 36}{6 - x}$

136. $\dfrac{x^2 + 3x - 40}{x^2 - 3x - 10}$

Testing for Functions **In Exercises 137–140, determine whether the equation represents y as a function of x.**

137. $x = 5$

138. $y - 7 = -3$

139. $x^2 + y = 5$

140. $x - y^2 = 0$

1.7 Linear Models and Scatter Plots

Scatter Plots and Correlation

Many real-life situations involve finding relationships between two variables, such as the year and the population of the United States. In a typical situation, data are collected and written as a set of ordered pairs. The graph of such a set is called a *scatter plot*. (For a brief discussion of scatter plots, see Appendix B.1.)

EXAMPLE 1 Constructing a Scatter Plot

The populations P (in millions) of the United States from 2008 through 2013 are shown in the table. Construct a scatter plot of the data. (Source: U.S. Census Bureau)

DATA

Year	Population, P (in millions)
2008	304.1
2009	306.8
2010	309.3
2011	311.6
2012	313.9
2013	316.1

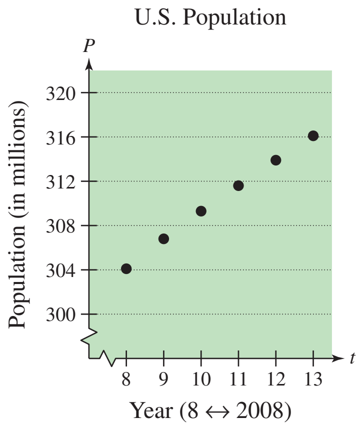
Spreadsheet at LarsonPrecalculus.com

Solution

Begin by representing the data with a set of ordered pairs. Let t represent the year, with $t = 8$ corresponding to 2008.

(8, 304.1), (9, 306.8), (10, 309.3), (11, 311.6), (12, 313.9), (13, 316.1)

Then plot each point in a coordinate plane, as shown in the figure.



✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The median sales prices (in thousands of dollars) of new homes sold in a neighborhood from 2006 through 2013 are given by the following ordered pairs. *(Spreadsheet at LarsonPrecalculus.com)* Construct a scatter plot of the data.

DATA (2006, 179.4), (2007, 185.4), (2008, 191.0), (2009, 196.7), (2010, 202.6), (2011, 208.7), (2012, 214.9), (2013, 221.4)

From the scatter plot in Example 1, it appears that the points describe a relationship that is nearly linear. The relationship is not *exactly* linear because the population did not increase by precisely the same amount each year.

A mathematical equation that approximates the relationship between t and P is a *mathematical model*. When developing a mathematical model to describe a set of data, you strive for two (often conflicting) goals—accuracy and simplicity. For the data above, a linear model of the form $P = at + b$ (where a and b are constants) appears to be best. It is simple and relatively accurate.

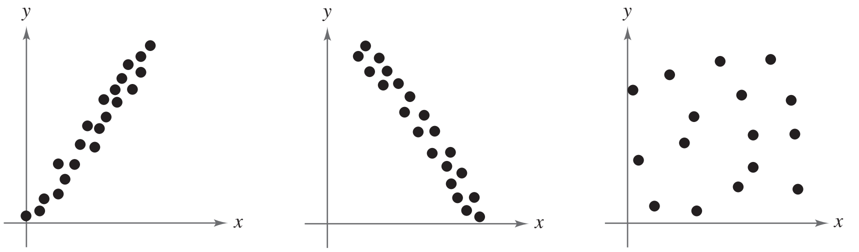
What you should learn

- Construct scatter plots and interpret correlation.
- Use scatter plots and a graphing utility to find linear models for data.

Why you should learn it

Real-life data often follow a linear pattern. For instance, in Exercise 27 on page 79, you will find a linear model for the winning times in the women's 400-meter freestyle Olympic swimming event.

Consider a collection of ordered pairs of the form (x, y). If y tends to increase as x increases, then the collection is said to have a **positive correlation.** If y tends to decrease as x increases, then the collection is said to have a **negative correlation.** Figure 1.46 shows three examples: one with a positive correlation, one with a negative correlation, and one with no (discernible) correlation.



Positive Correlation *Negative Correlation* *No Correlation*

Figure 1.46

EXAMPLE 2 Interpreting Correlation

On a Friday, 22 students in a class were asked to record the numbers of hours they spent studying for a test on Monday. The results are shown below. The first coordinate is the number of hours and the second coordinate is the score obtained on the test. *(Spreadsheet at LarsonPrecalculus.com)*

DATA *Study Hours:* (0, 40), (1, 41), (2, 51), (3, 58), (3, 49), (4, 48), (4, 64), (5, 55), (5, 69), (5, 58), (5, 75), (6, 68), (6, 63), (6, 93), (7, 84), (7, 67), (8, 90), (8, 76), (9, 95), (9, 72), (9, 85), (10, 98)

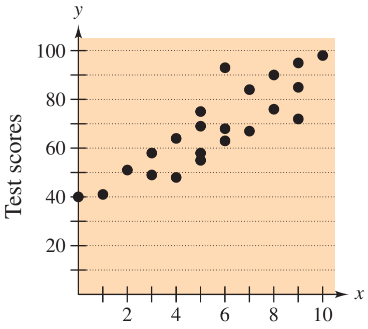
a. Construct a scatter plot of the data.

b. Determine whether the points are positively correlated, are negatively correlated, or have no discernible correlation. What can you conclude?

Solution

a. The scatter plot is shown in the figure.

b. The scatter plot relating study hours and test scores has a positive correlation. This means that the more a student studied, the higher his or her score tended to be.



✓ **Checkpoint** *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The students in Example 2 also recorded the numbers of hours they spent watching television. The results are shown below. The first coordinate is the number of hours and the second coordinate is the score obtained on the test. *(Spreadsheet at LarsonPrecalculus.com)*

DATA *TV Hours:* (0, 98), (1, 85), (2, 72), (2, 90), (3, 67), (3, 93), (3, 95), (4, 68), (4, 84), (5, 76), (7, 75), (7, 58), (9, 63), (9, 69), (11, 55), (12, 58), (14, 64), (16, 48), (17, 51), (18, 41), (19, 49), (20, 40)

a. Construct a scatter plot of the data.

b. Determine whether the points are positively correlated, are negatively correlated, or have no discernible correlation. What can you conclude?

Fitting a Line to Data

Finding a linear model to represent the relationship described by a scatter plot is called **fitting a line to data.** You can do this graphically by simply sketching the line that appears to fit the points, finding two points on the line, and then finding the equation of the line that passes through the two points.

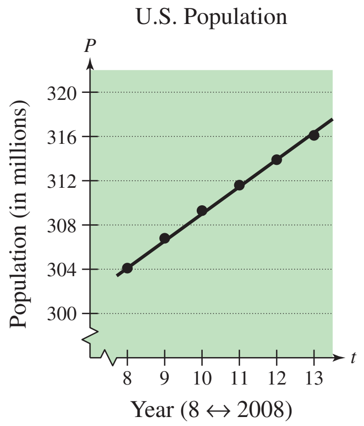
EXAMPLE 3 Fitting a Line to Data

See LarsonPrecalculus.com for an interactive version of this type of example.

Find a linear model that relates the year to the population of the United States. (See Example 1.)

Year	U.S. Population, P (in millions)
2008	304.1
2009	306.8
2010	309.3
2011	311.6
2012	313.9
2013	316.1

Spreadsheet at LarsonPrecalculus.com



Solution

Let t represent the year, with $t = 8$ corresponding to 2008. After plotting the data in the table, draw the line that you think best represents the data, as shown in the figure. Two points that lie on this line are (8, 304.1) and (12, 313.9). Using the point-slope form, you can find the equation of the line to be

$$P = 2.45(t - 8) + 304.1$$

$$= 2.45t + 284.5. \quad \text{Linear model}$$

Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

Find a linear model that relates the year to the median sales prices (in thousands of dollars) of new homes sold in a neighborhood from 2006 through 2013. (See Checkpoint for Example 1.) *(Spreadsheet at LarsonPrecalculus.com)*

(2006, 179.4), (2007, 185.4), (2008, 191.0), (2009, 196.7), (2010, 202.6), (2011, 208.7), (2012, 214.9), (2013, 221.4)

After finding a model, you can measure how well the model fits the data by comparing the actual values with the values given by the model, as shown in the table.

	t	8	9	10	11	12	13
Actual	P	304.1	306.8	309.3	311.6	313.9	316.1
Model	P	304.1	306.6	309.0	311.5	313.9	316.4

The sum of the squares of the differences between the actual values and the model values is called the **sum of the squared differences.** The model that has the least sum is called the **least squares regression line** for the data. For the model in Example 3, the sum of the squared differences is 0.23. The least squares regression line for the data is

$$P = 2.39t + 285.2. \quad \text{Best-fitting linear model}$$

Its sum of squared differences is 0.14. For more on the least squares regression line, see Appendix C.2 at this textbook's *Companion Website.*

Remark

The model in Example 3 is based on the two data points chosen. When different points are chosen, the model may change somewhat. For instance, when you choose (9, 306.8) and (12, 313.9), the new model is

$$P = 2.37(t - 9) + 306.8$$

$$= 2.37t + 285.5.$$

Another way to find a linear model to represent the relationship described by a scatter plot is to enter the data points into a graphing utility and use the *linear regression* feature. This method is demonstrated in Example 4.

Technology Tip

For instructions on how to use the *linear regression* feature, see Appendix A; for specific keystrokes, go to this textbook's *Companion Website.*

EXAMPLE 4 A Mathematical Model

The table shows the numbers N (in millions) of tax returns made through e-file from 2006 through 2013. (Source: Internal Revenue Service)

Year	Number of tax returns made through e-file, N (in millions)
2006	73.3
2007	80.0
2008	89.9
2009	95.0
2010	98.7
2011	112.2
2012	119.6
2013	122.5

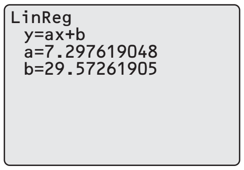
Spreadsheet at LarsonPrecalculus.com

a. Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 6$ corresponding to 2006.

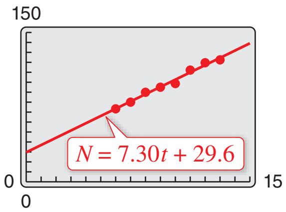
b. How closely does the model represent the data?

Graphical Solution

a. Use the *linear regression* feature of a graphing utility to obtain the model shown in the figure. You can approximate the model to be $N = 7.30t + 29.6$.



b. Graph the actual data and the model. From the figure, it appears that the model is a good fit for the actual data.



Numerical Solution

a. Using the *linear regression* feature of a graphing utility, you can find that a linear model for the data is $N = 7.30t + 29.6$.

b. You can see how well the model fits the data by comparing the actual values of N with the values of N given by the model, which are labeled N^* in the table below. From the table, you can see that the model appears to be a good fit for the actual data.

Year	N	N^*
2006	73.3	73.4
2007	80.0	80.7
2008	89.9	88.0
2009	95.0	95.3
2010	98.7	102.6
2011	112.2	109.9
2012	119.6	117.2
2013	122.5	124.5

✓ Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

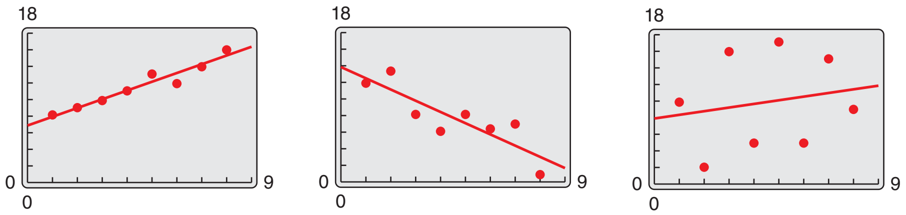
The numbers N (in millions) of people unemployed in the U.S. from 2007 through 2013 are given by the following ordered pairs. *(Spreadsheet at LarsonPrecalculus.com)* (Source: U.S. Bureau of Labor Statistics)

(2007, 7.078), (2008, 8.924), (2009, 14.265), (2010, 14.825),
(2011, 13.747), (2012, 12.506), (2013, 11.460)

a. Use the *regression* feature of a graphing utility to find a linear model for the data. Let t represent the year, with $t = 7$ corresponding to 2007.

b. How closely does the model represent the data?

When you use the *regression* feature of a graphing calculator or computer program to find a linear model for data, you will notice that the program may also output an "r-value." For instance, the r-value from Example 4 was $r \approx 0.992$. This r-value is the **correlation coefficient** of the data and gives a measure of how well the model fits the data. The correlation coefficient r varies between -1 and 1. Basically, the closer $|r|$ is to 1, the better the points can be described by a line. Three examples are shown in Figure 1.47.



$r = 0.972$ $r = -0.856$ $r = 0.190$

Figure 1.47

Technology Tip

For some calculators, the *diagnostics on* feature must be selected before the *regression* feature is used in order to see the value of the correlation coefficient r. To learn how to use this feature, consult your user's manual.

Emphasize to your students that the correlation coefficient measures the strength of the linear relationship between two variables. Point out that a correlation value near $r = 0$ indicates that there is no linear relationship between the variables, but this does not rule out the possibility of there being some other type of relationship—for instance, a quadratic relationship.

EXAMPLE 5 Finding a Least Squares Regression Line

The ordered pairs (w, h) represent the shoe sizes w and the heights h (in inches) of 25 men. Use the *regression* feature of a graphing utility to find the least squares regression line for the data.

(10.0, 70.5)	(10.5, 71.0)	(9.5, 69.0)	(11.0, 72.0)	(12.0, 74.0)
(8.5, 67.0)	(9.0, 68.5)	(13.0, 76.0)	(10.5, 71.5)	(10.5, 70.5)
(10.0, 71.0)	(9.5, 70.0)	(10.0, 71.0)	(10.5, 71.0)	(11.0, 71.5)
(12.0, 73.5)	(12.5, 75.0)	(11.0, 72.0)	(9.0, 68.0)	(10.0, 70.0)
(13.0, 75.5)	(10.5, 72.0)	(10.5, 71.0)	(11.0, 73.0)	(8.5, 67.5)

Solution

After entering the data into a graphing utility (see Figure 1.48), you obtain the model shown in Figure 1.49. So, the least squares regression line for the data is

$$h = 1.84w + 51.9.$$

In Figure 1.50, this line is plotted with the data. Note that the plot does not have 25 points because some of the ordered pairs graph as the same point. The correlation coefficient for this model is $r \approx 0.981$, which implies that the model is a good fit for the data.

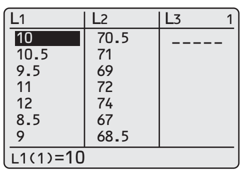


Figure 1.48

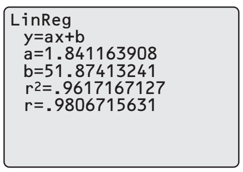


Figure 1.49

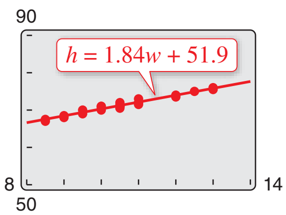
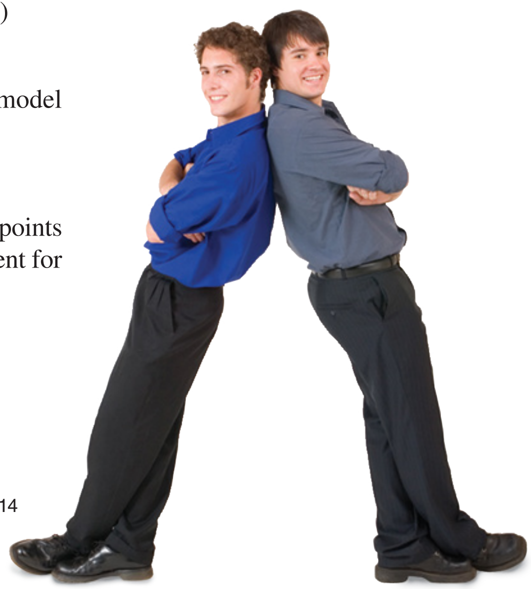


Figure 1.50



Checkpoint *Audio-video solution in English & Spanish at LarsonPrecalculus.com.*

The ordered pairs (g, e) represent the gross domestic products g (in millions of U.S. dollars) and the carbon dioxide emissions e (in millions of metric tons) for seven countries. *(Spreadsheet at LarsonPrecalculus.com)* Use the *regression* feature of a graphing utility to find the least squares regression line for the data. (Sources: World Bank and U.S. Energy Information Administration)

DATA (14.99, 5491), (5.87, 1181), (3.6, 748), (2.77, 374), (2.48, 475), (2.45, 497), (2.19, 401)

1.7 Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Vocabulary and Concept Check

In Exercises 1 and 2, fill in the blank.

1. Consider a collection of ordered pairs of the form (x, y). If y tends to increase as x increases, then the collection is said to have a _______ correlation.

2. To find the least squares regression line for data, you can use the _______ feature of a graphing utility.

3. In a collection of ordered pairs (x, y), y tends to decrease as x increases. Does the collection have a positive correlation or a negative correlation?

4. You find the least squares regression line for a set of data. The correlation coefficient is 0.114. Is the model a good fit?

Procedures and Problem Solving

5. **Constructing a Scatter Plot** The following ordered pairs give the years of experience x for 15 sales representatives and the monthly sales y (in thousands of dollars).

(1.5, 41.7), (1.0, 32.4), (0.3, 19.2), (3.0, 48.4), (4.0, 51.2), (0.5, 28.5), (2.5, 50.4), (1.8, 35.5), (2.0, 36.0), (1.5, 40.0), (3.5, 50.3), (4.0, 55.2), (0.5, 29.1), (2.2, 43.2), (2.0, 41.6)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? Explain.

6. **Constructing a Scatter Plot** The following ordered pairs give the scores on two consecutive 15-point quizzes for a class of 18 students.

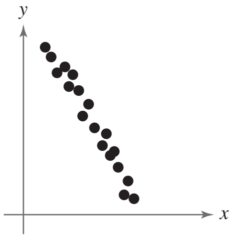
(7, 13), (9, 7), (14, 14), (15, 15), (10, 15), (9, 7), (14, 11), (14, 15), (8, 10), (9, 10), (15, 9), (10, 11), (11, 14), (7, 14), (11, 10), (14, 11), (10, 15), (9, 6)

(a) Create a scatter plot of the data.

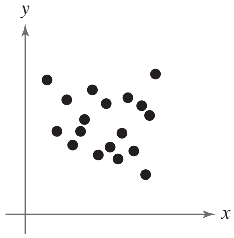
(b) Does the relationship between consecutive quiz scores appear to be approximately linear? If not, give some possible explanations.

Interpreting Correlation **In Exercises 7–10, the scatter plot of a set of data is shown. Determine whether the points are positively correlated, are negatively correlated, or have no discernible correlation.**

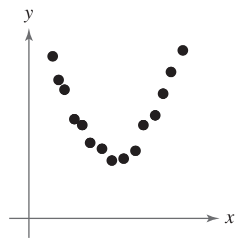
7.



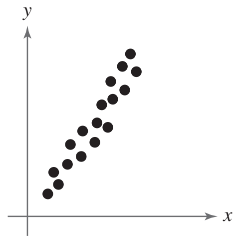
8.



9.



10.



Fitting a Line to Data **In Exercises 11–16, (a) create a scatter plot of the data, (b) draw a line of fit that passes through two of the points, and (c) use the two points to find an equation of the line.**

11. $(-3, -3), (3, 4), (1, 1), (3, 2), (4, 4), (-1, -1)$

12. $(-2, 3), (-2, 4), (-1, 2), (1, -2), (0, 0), (0, 1)$

13. $(0, 2), (-2, 1), (3, 3), (1, 3), (4, 4)$

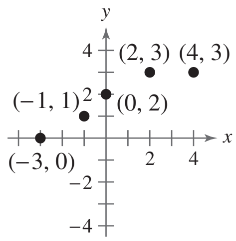
14. $(3, 2), (2, 3), (1, 5), (4, 0), (5, 0)$

15. $(0, 7), (3, 2), (6, 0), (4, 3), (2, 5)$

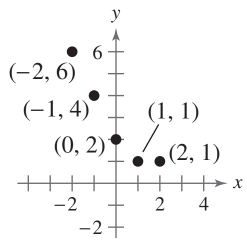
16. $(3, 4), (2, 2), (5, 6), (1, 1), (0, 2)$

A Mathematical Model **In Exercises 17 and 18, use the *regression* feature of a graphing utility to find a linear model for the data. Then use the graphing utility to decide how closely the model fits the data (a) graphically and (b) numerically. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

17.



18.



19. MODELING DATA

Hooke's Law states that the force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is, $F = kd$, where k is the measure of the stiffness of the spring and is called the *spring constant*. The table shows the elongation d in centimeters of a spring when a force of F kilograms is applied.

Force, F	Elongation, d
20	1.4
40	2.5
60	4.0
80	5.3
100	6.6

(a) Sketch a scatter plot of the data.

(b) Find the equation of the line that seems to best fit the data.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data.

(d) Use the model from part (c) to estimate the elongation of the spring when a force of 55 kilograms is applied.

20. MODELING DATA

The numbers of subscribers S (in millions) to wireless networks from 2006 through 2012 are shown in the table. (Source: CTIA–The Wireless Association)

Spreadsheet at LarsonPrecalculus.com

Year	Subscribers, S (in millions)
2006	233.0
2007	255.4
2008	270.3
2009	285.6
2010	296.3
2011	316.0
2012	326.5

(a) Use a graphing utility to create a scatter plot of the data, with $t = 6$ corresponding to 2006.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the number of subscribers in 2020. Is your answer reasonable? Explain.

21. MODELING DATA

The total enterprise values V (in millions of dollars) for the Pittsburgh Penguins from 2008 through 2012 are shown in the table. (Source: Forbes)

Year	Total enterprise value, V (in millions of dollars)
2008	195
2009	222
2010	235
2011	264
2012	288

(a) Use a graphing utility to create a scatter plot of the data, with $t = 8$ corresponding to 2008.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the values in 2015 and 2020. Do the results seem reasonable? Explain.

(e) What is the slope of your model? What does it tell you about the enterprise value?

22. MODELING DATA

The mean salaries S (in thousands of dollars) of public school teachers in the United States from 2005 through 2011 are shown in the table. (Source: National Education Association)

Spreadsheet at LarsonPrecalculus.com

Year	Mean salary, S (in thousands of dollars)
2005	47.5
2006	49.1
2007	51.1
2008	52.8
2009	54.3
2010	55.5
2011	57.2

(a) Use a graphing utility to create a scatter plot of the data, with $t = 5$ corresponding to 2005.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. Is the model a good fit? Explain.

(d) Use the model to predict the mean salaries in 2020 and 2022. Do the results seem reasonable? Explain.

23. MODELING DATA

The populations P (in thousands) of New Jersey from 2008 through 2013 are shown in the table. (Source: U.S. Census Bureau)

Year	Population, P (in thousands)
2008	8711
2009	8756
2010	8792
2011	8821
2012	8868
2013	8899

DATA Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to create a scatter plot of the data, with $t = 8$ corresponding to 2008.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window.

(d) Create a table showing the actual values of P and the values of P given by the model. How closely does the model fit the data?

(e) Use the model to predict the population of New Jersey in 2050. Does the result seem reasonable? Explain.

24. MODELING DATA

The populations P (in thousands) of Wyoming from 2009 through 2013 are shown in the table. (Source: U.S. Census Bureau)

Year	Population, P (in thousands)
2009	560
2010	564
2011	568
2012	577
2013	583

(a) Use a graphing utility to create a scatter plot of the data, with $t = 9$ corresponding to 2009.

(b) Use the *regression* feature of the graphing utility to find a linear model for the data.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window.

(d) Create a table showing the actual values of P and the values of P given by the model. How closely does the model fit the data?

(e) Use the model to predict the population of Wyoming in 2050. Does the result seem reasonable? Explain.

25. MODELING DATA

The table shows the advertising expenditures x and sales volumes y for a company for nine randomly selected months. Both are measured in thousands of dollars.

Month	Advertising expenditures, x	Sales volume, y
1	2.4	215
2	1.6	182
3	1.4	172
4	2.3	214
5	2.0	201
6	2.6	230
7	1.4	176
8	1.6	177
9	2.0	196

DATA Spreadsheet at LarsonPrecalculus.com

(a) Use the *regression* feature of a graphing utility to find a linear model for the data.

(b) Use the graphing utility to plot the data and graph the model in the same viewing window.

(c) Interpret the slope of the model in the context of the problem.

(d) Use the model to estimate sales for advertising expenditures of $1500.

26. MODELING DATA

The table shows the numbers T of stores owned by the Target Corporation from 2006 through 2012. (Source: Target Corp.)

Year	Number of stores, T
2006	1488
2007	1591
2008	1682
2009	1740
2010	1750
2011	1763
2012	1778

DATA Spreadsheet at LarsonPrecalculus.com

(a) Use a graphing utility to make a scatter plot of the data, with $t = 6$ corresponding to 2006. Identify two sets of points in the scatter plot that are approximately linear.

(b) Use the *regression* feature of the graphing utility to find a linear model for each set of points.

(c) Write a piecewise-defined model for the data. Use the graphing utility to graph the piecewise-defined model.

(d) Describe a scenario that could be the cause of the break in the data.

27. ***Why you should learn it*** *(p. 71)* The following ordered pairs (t, T) represent the Olympic year t and the winning time T (in minutes) in the women's 400-meter freestyle swimming event. *(Spreadsheet at LarsonPrecalculus.com)* (Source: International Olympic Committee)

DATA

(1956, 4.91)	(1988, 4.06)
(1960, 4.84)	(1992, 4.12)
(1964, 4.72)	(1996, 4.12)
(1968, 4.53)	(2000, 4.10)
(1972, 4.32)	(2004, 4.09)
(1976, 4.16)	(2008, 4.05)
(1980, 4.15)	(2012, 4.02)
(1984, 4.12)	

(a) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the correlation coefficient. Let t represent the year, with $t = 0$ corresponding to 1950.

(b) What information is given by the sign of the slope of the model?

(c) Use the graphing utility to plot the data and graph the model in the same viewing window.

(d) Create a table showing the actual values of y and the values of y given by the model. How closely does the model fit the data?

(e) How can you use the value of the correlation coefficient to help answer the question in part (d)?

(f) Would you use the model to predict the winning times in the future? Explain.

28. **MODELING DATA**

In a study, 60 colts were measured every 14 days from birth. The ordered pairs (d, l) represent the average length l (in centimeters) of the 60 colts d days after birth. *(Spreadsheet at LarsonPrecalculus.com)* (Source: American Society of Animal Science)

DATA

(14, 81.2)	(28, 87.1)	(42, 93.7)
(56, 98.3)	(70, 102.4)	(84, 106.2)
(98, 110.0)		

(a) Use the *regression* feature of a graphing utility to find a linear model for the data and to identify the correlation coefficient.

(b) According to the correlation coefficient, does the model represent the data well? Explain.

(c) Use the graphing utility to plot the data and graph the model in the same viewing window. How closely does the model fit the data?

(d) Use the model to predict the average length of a colt 112 days after birth.

Conclusions

True or False? **In Exercises 29 and 30, determine whether the statement is true or false. Justify your answer.**

29. A linear regression model with a positive correlation will have a slope that is greater than 0.

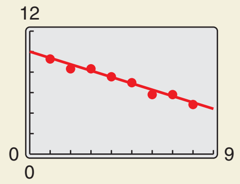
30. When the correlation coefficient for a linear regression model is close to -1, the regression line is a poor fit for the data.

31. **Writing** Use your school's library, the Internet, or some other reference source to locate data that you think describe a linear relationship. Create a scatter plot of the data and find the least squares regression line that represents the points. Interpret the slope and y-intercept in the context of the data. Write a summary of your findings.

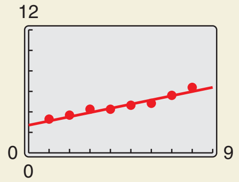
32. **HOW DO YOU SEE IT?** Each graphing utility screen below shows the least squares regression line for a set of data. The equations and r-values for the models are given.

$y = 0.68x + 2.7$
$y = 0.41x + 2.7$
$y = -0.62x + 10.0$
$r = 0.973$
$r = -0.986$
$r = 0.624$

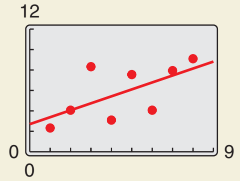
(i)



(ii)



(iii)



(a) Determine the equation and correlation coefficient (r-value) that represent each graph. Explain how you found your answers.

(b) According to the correlation coefficients, which model is the best fit for its data? Explain.

Cumulative Mixed Review

Evaluating a Function **In Exercises 33 and 34, evaluate the function at each value of the independent variable and simplify.**

33. $f(x) = 2x^2 - 3x + 5$
 (a) $f(-1)$
 (b) $f(w + 2)$

34. $g(x) = 5x^2 - 6x + 1$
 (a) $g(-2)$
 (b) $g(z - 2)$

Solving Equations **In Exercises 35–38, solve the equation algebraically. Check your solution graphically.**

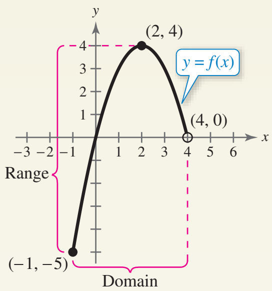
35. $6x + 1 = -9x - 8$

36. $3(x - 3) = 7x + 2$

37. $8x^2 - 10x - 3 = 0$

38. $10x^2 - 23x - 5 = 0$

1 Chapter Summary

	What did you learn?	Explanation and Examples	Review Exercises
1.1	**Find the slopes of lines *(p. 3)*.**	The slope m of the nonvertical line through (x_1, y_1) and (x_2, y_2) is $m = \dfrac{y_2 - y_1}{x_2 - x_1}$, $x_1 \neq x_2$.	1–8
	Write linear equations given points on lines and their slopes *(p. 5)*.	The point-slope form of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is $y - y_1 = m(x - x_1)$.	9–16
	Use slope-intercept forms of linear equations to sketch lines *(p. 7)*.	The graph of the equation $y = mx + b$ is a line whose slope is m and whose y-intercept is $(0, b)$.	17–30
	Use slope to identify parallel and perpendicular lines *(p. 9)*.	**Parallel lines:** Slopes are equal. **Perpendicular lines:** Slopes are negative reciprocals of each other.	31, 32
1.2	**Decide whether a relation between two variables represents a function *(p. 16)*.**	A function f from a set A to a set B is a relation that assigns to each element x in the set A exactly one element y in the set B. The set A is the domain (or set of inputs) of the function f, and the set B contains the range (or set of outputs).	33–42
	Use function notation and evaluate functions *(p. 18)*, and find the domains of functions *(p. 20)*.	**Equation:** $f(x) = 5 - x^2$ **$f(2)$:** $f(2) = 5 - 2^2 = 1$ **Domain of $f(x) = 5 - x^2$:** All real numbers x	43–50
	Use functions to model and solve real-life problems *(p. 22)*.	A function can be used to model the number of interior design services employees in the United States. (See Example 8.)	51, 52
	Evaluate difference quotients *(p. 23)*.	**Difference quotient:** $\dfrac{f(x + h) - f(x)}{h}$, $h \neq 0$	53, 54
1.3	**Find the domains and ranges of functions *(p. 29)*.**		55–62
	Use the Vertical Line Test for functions *(p. 30)*.	A set of points in a coordinate plane is the graph of y as a function of x if and only if no *vertical* line intersects the graph at more than one point.	63–66
	Determine intervals on which functions are increasing, decreasing, or constant *(p. 31)*.	A function f is increasing on an interval when, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) < f(x_2)$. A function f is decreasing on an interval when, for any x_1 and x_2 in the interval, $x_1 < x_2$ implies $f(x_1) > f(x_2)$. A function f is constant on an interval when, for any x_1 and x_2 in the interval, $f(x_1) = f(x_2)$.	67–70

	What did you learn?	Explanation and Examples	Review Exercises
1.3	**Determine relative maximum and relative minimum values of functions *(p. 32)*.**	A function value $f(a)$ is called a relative minimum of f when there exists an interval (x_1, x_2) that contains a such that $x_1 < x < x_2$ implies $f(a) \le f(x)$. A function value $f(a)$ is called a relative maximum of f when there exists an interval (x_1, x_2) that contains a such that $x_1 < x < x_2$ implies $f(a) \ge f(x)$.	71–74
	Identify and graph step functions and other piecewise-defined functions *(p. 34)*.	**Greatest integer:** $f(x) = [\![x]\!]$	75–78
	Identify even and odd functions *(p. 35)*.	**Even:** For each x in the domain of f, $f(-x) = f(x)$. **Odd:** For each x in the domain of f, $f(-x) = -f(x)$.	79–86
1.4	**Recognize graphs of parent functions *(p. 41)*.**	**Linear:** $f(x) = x$; **Quadratic:** $f(x) = x^2$; **Cubic:** $f(x) = x^3$; **Absolute value:** $f(x) = \|x\|$; **Square root:** $f(x) = \sqrt{x}$; **Rational:** $f(x) = 1/x$ (See Figure 1.24, page 41.)	87–92
	Use vertical and horizontal shifts *(p. 42)*, reflections *(p. 44)*, and nonrigid transformations *(p. 46)* to graph functions.	**Vertical shifts:** $h(x) = f(x) + c$ or $h(x) = f(x) - c$ **Horizontal shifts:** $h(x) = f(x - c)$ or $h(x) = f(x + c)$ **Reflection in the *x*-axis:** $h(x) = -f(x)$ **Reflection in the *y*-axis:** $h(x) = f(-x)$ **Nonrigid transformations:** $h(x) = cf(x)$ or $h(x) = f(cx)$	93–106
1.5	**Add, subtract, multiply, and divide functions *(p. 50)*, find the compositions of functions *(p. 52)*, and write a function as a composition of two functions *(p. 54)*.**	$(f + g)(x) = f(x) + g(x)$ $(f - g)(x) = f(x) - g(x)$ $(fg)(x) = f(x) \cdot g(x)$ $(f/g)(x) = f(x)/g(x),\ g(x) \ne 0$ **Composition of functions:** $(f \circ g)(x) = f(g(x))$	107–122
	Use combinations of functions to model and solve real-life problems *(p. 55)*.	A composite function can be used to represent the number of bacteria in a petri dish as a function of the amount of time the petri dish has been out of refrigeration. (See Example 10.)	123, 124
1.6	**Find inverse functions informally and verify that two functions are inverse functions of each other *(p. 60)*.**	Let f and g be two functions such that $f(g(x)) = x$ for every x in the domain of g and $g(f(x)) = x$ for every x in the domain of f. Under these conditions, the function g is the inverse function of the function f.	125–128
	Use graphs of functions to decide whether functions have inverse functions *(p. 63)*.	If the point (a, b) lies on the graph of f, then the point (b, a) must lie on the graph of f^{-1}, and vice versa. In short, f^{-1} is a reflection of f in the line $y = x$.	129, 130
	Determine whether functions are one-to-one *(p. 64)*.	A function f is one-to-one when, for a and b in its domain, $f(a) = f(b)$ implies $a = b$.	131–134
	Find inverse functions algebraically *(p. 65)*.	To find inverse functions, replace $f(x)$ with y, interchange the roles of x and y, and solve for y. Replace y with $f^{-1}(x)$.	135–142
1.7	**Construct scatter plots *(p. 71)* and interpret correlation *(p. 72)*.**	A scatter plot is a graphical representation of data written as a set of ordered pairs.	143–146
	Use scatter plots *(p. 73)* and a graphing utility *(p. 74)* to find linear models for data.	The best-fitting linear model can be found using the *linear regression* feature of a graphing utility or a computer program.	147, 148

1 Review Exercises

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

1.1

Finding the Slope of a Line **In Exercises 1–8, plot the two points and find the slope of the line passing through the points.**

1. $(-3, 2), (8, 2)$ **2.** $(3, -1), (-3, -1)$

3. $(-5, -1), (-5, 9)$ **4.** $(8, -1), (8, 2)$

5. $\left(\frac{3}{2}, 1\right), \left(5, \frac{5}{2}\right)$ **6.** $\left(-\frac{3}{4}, \frac{5}{6}\right), \left(\frac{1}{2}, -\frac{5}{2}\right)$

7. $(-4.5, 6), (2.1, 3)$ **8.** $(-2.7, -6.3), (0, 1.8)$

The Point-Slope Form of the Equation of a Line **In Exercises 9–16, (a) use the point on the line and the slope of the line to find an equation of the line, and (b) find three additional points through which the line passes. (There are many correct answers.)**

	Point	*Slope*
9.	$(2, -1)$	$m = \frac{1}{4}$
10.	$(-3, 5)$	$m = -\frac{3}{2}$
11.	$(0, -5)$	$m = -\frac{3}{2}$
12.	$(0, 1)$	$m = \frac{4}{5}$
13.	$(-2, 6)$	$m = 0$
14.	$(-8, 8)$	$m = 0$
15.	$(10, -6)$	m is undefined.
16.	$(-5, 4)$	m is undefined.

Finding the Slope-Intercept Form **In Exercises 17–24, write an equation of the line that passes through the points. Use the slope-intercept form, if possible. If not possible, explain why. Use a graphing utility to graph the line (if possible).**

17. $(2, -1), (4, -1)$ **18.** $(0, 0), (0, 10)$

19. $\left(\frac{5}{6}, -1\right), \left(\frac{5}{6}, 3\right)$ **20.** $\left(7, \frac{4}{3}\right), \left(9, \frac{4}{3}\right)$

21. $(-1, 0), (6, 2)$ **22.** $(1, 6), (4, 2)$

23. $(3, -1), (-3, 2)$ **24.** $\left(-\frac{5}{2}, 1\right), \left(-4, \frac{2}{9}\right)$

Using a Rate of Change to Write an Equation **In Exercises 25–28, you are given the dollar value of a product in 2015 *and* the rate at which the value of the item is expected to change during the next 5 years. Use this information to write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 5$ represent 2015.)**

	2015 Value	*Rate*
25.	\$12,500	\$850 increase per year
26.	\$4025	\$275 decrease per year
27.	\$625.50	\$42.70 increase per year
28.	\$72.95	\$5.15 decrease per year

29. Business During the second and third quarters of the year, an e-commerce business had sales of \$170,000 and \$195,000, respectively. The growth of sales follows a linear pattern. Estimate sales during the fourth quarter.

30. Accounting The dollar value of an MP3 player in 2015 is \$285. The product will decrease in value at an expected rate of \$47.50 per year.

(a) Write a linear equation that gives the dollar value V of the MP3 player in terms of the year t. (Let $t = 0$ represent 2015.)

(b) Use a graphing utility to graph the equation found in part (a). Be sure to choose an appropriate viewing window. State the dimensions of your viewing window and explain why you chose the values that you did.

(c) Use the *value* or *trace* feature of the graphing utility to estimate the dollar value of the MP3 player in 2019. Confirm your answer algebraically.

(d) According to the model, when will the MP3 player have no value?

Equations of Parallel and Perpendicular Lines **In Exercises 31 and 32, write the slope-intercept forms of the equations of the lines through the given point (a) parallel to the given line and (b) perpendicular to the given line. Verify your results with a graphing utility (use a *square setting*).**

	Point	*Line*
31.	$(3, -2)$	$5x - 4y = 8$
32.	$(-8, 3)$	$2x + 3y = 5$

1.2

Testing for Functions **In Exercises 33 and 34, which set of ordered pairs represents a function from A to B? Explain.**

33. $A = \{10, 20, 30, 40\}$ and $B = \{0, 2, 4, 6\}$

(a) $\{(20, 4), (40, 0), (20, 6), (30, 2)\}$

(b) $\{(10, 4), (20, 4), (30, 4), (40, 4)\}$

34. $A = \{u, v, w\}$ and $B = \{-2, -1, 0, 1, 2\}$

(a) $\{(u, -2), (v, 2), (w, 1)\}$

(b) $\{(w, -2), (v, 0), (w, 2)\}$

Testing for Functions Represented Algebraically **In Exercises 35–42, determine whether the equation represents y as a function of x.**

35. $16x^2 - y^2 = 0$ **36.** $x^3 + y^2 = 64$

37. $2x - y - 3 = 0$ **38.** $2x + y = 10$

39. $y = \sqrt{1 - x}$ **40.** $y = \sqrt{x^2 + 4}$

41. $|y| = x + 2$ **42.** $16 - |y| - 4x = 0$

Evaluating a Function **In Exercises 43–46, evaluate the function at each specified value of the independent variable, and simplify.**

43. $f(x) = x^2 + 1$

(a) $f(1)$ (b) $f(-3)$

(c) $f(b^3)$ (d) $f(x - 1)$

44. $g(x) = \sqrt{x^2 + 1}$

(a) $g(-1)$ (b) $g(3)$

(c) $g(3x)$ (d) $g(x + 2)$

45. $h(x) = \begin{cases} 2x + 1, & x \le -1 \\ x^2 + 2, & x > -1 \end{cases}$

(a) $h(-2)$ (b) $h(-1)$

(c) $h(0)$ (d) $h(2)$

46. $f(x) = \dfrac{3}{2x - 5}$

(a) $f(1)$ (b) $f(-2)$

(c) $f(t)$ (d) $f(10)$

Finding the Domain of a Function **In Exercises 47–50, find the domain of the function.**

47. $f(x) = \dfrac{x - 1}{x + 2}$ **48.** $f(x) = \dfrac{x^2}{x^2 + 1}$

49. $f(x) = \sqrt{25 - x^2}$ **50.** $f(x) = \sqrt{x^2 - 16}$

51. Industrial Engineering A hand tool manufacturer produces a product for which the variable cost is \$5.25 per unit and the fixed costs are \$17,500. The company sells the product for \$8.43 and can sell all that it produces.

(a) Write the total cost C as a function of x, the number of units produced.

(b) Write the profit P as a function of x.

52. Education The numbers n (in millions) of students enrolled in public schools in the United States from 2005 through 2012 can be approximated by

$$n(t) = \begin{cases} -0.35t^3 + 6.95t^2 - 44.8t + 158, & 5 \le t \le 8 \\ -0.35t^2 + 7.7t + 26, & 8 < t \le 12 \end{cases}$$

where t is the year, with $t = 5$ corresponding to 2005. (Source: U.S. Census Bureau)

(a) Use the *table* feature of a graphing utility to approximate the enrollment from 2005 through 2012.

(b) Use the graphing utility to graph the model and estimate the enrollment for the years 2013 through 2017. Do the values seem reasonable? Explain.

Evaluating a Difference Quotient **In Exercises 53 and 54, find the difference quotient** $\dfrac{f(x + h) - f(x)}{h}$ **for the given function and simplify your answer.**

53. $f(x) = 2x^2 + 3x - 1$ **54.** $f(x) = x^2 - 3x + 5$

1.3

Finding the Domain and Range of a Function **In Exercises 55–62, use a graphing utility to graph the function and estimate its domain and range. Then find the domain and range algebraically.**

55. $f(x) = 3 - 2x^2$ **56.** $f(x) = 2x^2 + 5$

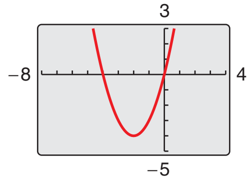
57. $f(x) = \sqrt{x + 3} + 4$ **58.** $f(x) = 2 - \sqrt{x - 5}$

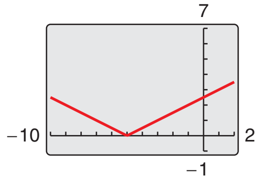
59. $h(x) = \sqrt{36 - x^2}$ **60.** $f(x) = \sqrt{x^2 - 9}$

61. $f(x) = |x - 5| + 2$ **62.** $f(x) = |x + 1| - 3$

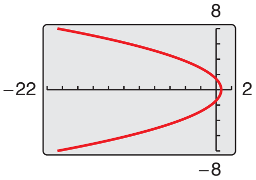
Vertical Line Test for Functions **In Exercises 63–66, use the Vertical Line Test to determine whether y is a function of x. Describe how to enter the equation into a graphing utility to produce the given graph.**

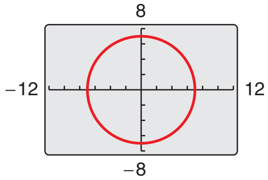
63. $y - 4x = x^2$ **64.** $|x + 5| - 2y = 0$





65. $3x + y^2 - 2 = 0$ **66.** $x^2 + y^2 - 49 = 0$





Increasing and Decreasing Functions **In Exercises 67–70, (a) use a graphing utility to graph the function and (b) determine the open intervals on which the function is increasing, decreasing, or constant.**

67. $f(x) = x^3 - 3x$ **68.** $f(x) = \sqrt{x^2 - 9}$

69. $f(x) = x\sqrt{x - 6}$ **70.** $f(x) = \dfrac{|x + 8|}{2}$

Approximating Relative Minima and Maxima **In Exercises 71–74, use a graphing utility to approximate (to two decimal places) any relative minimum or relative maximum values of the function.**

71. $f(x) = (x^2 - 4)^2$ **72.** $f(x) = x^2 + x$

73. $h(x) = x^3 + 4x^2 + 3$ **74.** $f(x) = x^3 - 4x^2 - 1$

Sketching Graphs **In Exercises 75–78, sketch the graph of the function by hand.**

75. $f(x) = \begin{cases} 3x + 5, & x < 0 \\ x - 4, & x \ge 0 \end{cases}$ **76.** $f(x) = \begin{cases} \frac{1}{2}x + 3, & x < 0 \\ 4 - x^2, & x \ge 0 \end{cases}$

77. $f(x) = [\![x]\!] - 3$ **78.** $f(x) = [\![x + 2]\!]$

Even and Odd Functions **In Exercises 79–86, determine algebraically whether the function is even, odd, or neither. Verify your answer using a graphing utility.**

79. $f(x) = x^2 + 6$ **80.** $f(x) = x^2 - x - 1$

81. $f(x) = (x^2 - 8)^2$ **82.** $f(x) = 2x^3 - x$

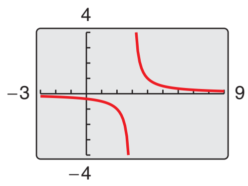
83. $f(x) = 3x^{5/2}$ **84.** $f(x) = 3x^{2/5}$

85. $f(x) = 2x\sqrt{4 - x^2}$ **86.** $f(x) = x\sqrt{x^2 - 1}$

1.4

Library of Parent Functions **In Exercises 87–92, identify the parent function and describe the transformation shown in the graph. Write an equation for the graphed function.**

87.



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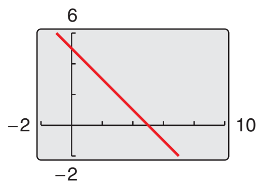

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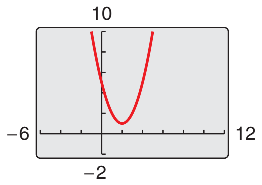

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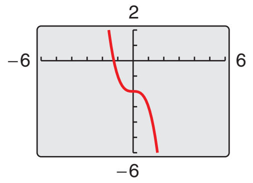

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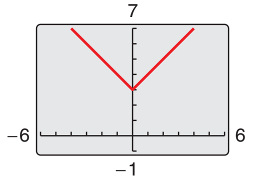
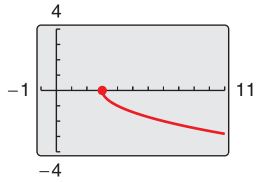
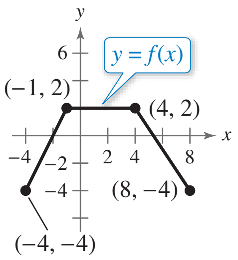


92.



Sketching Transformations **In Exercises 93–96, use the graph of $y = f(x)$ to graph the function.**



93. $y = f(-x)$

94. $y = -f(x)$

95. $y = f(x) + 2$

96. $y = f(x - 1)$

Describing Transformations **In Exercises 97–106, h is related to one of the six parent functions on page 41. (a) Identify the parent function f. (b) Describe the sequence of transformations from f to h. (c) Sketch the graph of h by hand. (d) Use function notation to write h in terms of the parent function f.**

97. $h(x) = \dfrac{1}{x} - 6$ **98.** $h(x) = -\dfrac{1}{x} + 3$

99. $h(x) = (x - 2)^3 + 5$ **100.** $h(x) = (-x)^2 - 8$

101. $h(x) = -\sqrt{x} - 6$ **102.** $h(x) = \sqrt{x - 1} + 4$

103. $h(x) = |3x| + 9$ **104.** $h(x) = |2x + 8| - 1$

105. $h(x) = \dfrac{-2}{x + 1} - 3$ **106.** $h(x) = \dfrac{5}{x + 2} - 4$

1.5

Evaluating a Combination of Functions **In Exercises 107–116, let $f(x) = 3 - 2x$, $g(x) = \sqrt{x}$, and $h(x) = 3x^2 + 2$, and find the indicated values.**

107. $(f - g)(4)$ **108.** $(f + h)(5)$

109. $(f + g)(25)$ **110.** $(g - h)(1)$

111. $(fh)(1)$ **112.** $\left(\dfrac{g}{h}\right)(1)$

113. $(h \circ g)(5)$ **114.** $(g \circ f)(-3)$

115. $(f \circ h)(-4)$ **116.** $(g \circ h)(6)$

Identifying a Composite Function **In Exercises 117–122, find two functions f and g such that $(f \circ g)(x) = h(x)$. (There are many correct answers.)**

117. $h(x) = (x + 3)^2$ **118.** $h(x) = (1 - 2x)^3$

119. $h(x) = \sqrt{4x + 2}$ **120.** $h(x) = \sqrt[3]{(x + 2)^2}$

121. $h(x) = \dfrac{4}{x + 2}$ **122.** $h(x) = \dfrac{6}{(3x + 1)^3}$

Education **In Exercises 123 and 124, the numbers (in thousands) of students taking the SAT (y_1) and the ACT (y_2) for the years 2008 through 2013 can be modeled by**

$$y_1 = -5.824t^3 + 182.05t^2 - 1832.8t + 7515 \text{ and}$$

$$y_2 = 3.398t^3 - 103.63t^2 + 1106.5t - 2543$$

where t represents the year, with $t = 8$ corresponding to 2008. (Sources: College Entrance Examination Board and ACT, Inc.)

123. Use a graphing utility to graph y_1, y_2, and $y_1 + y_2$ in the same viewing window.

124. Use the model $y_1 + y_2$ to estimate the total number of students taking the SAT and the ACT in 2017.

1.6

Finding Inverse Functions Informally **In Exercises 125–128, find the inverse function of f informally. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.**

125. $f(x) = 6x$ **126.** $f(x) = x + 5$

127. $f(x) = \dfrac{1}{2}x + 3$ **128.** $f(x) = \dfrac{x - 4}{5}$

Algebraic-Graphical-Numerical **In Exercises 129 and 130, show that f and g are inverse functions (a) algebraically, (b) graphically, and (c) numerically.**

129. $f(x) = 3 - 4x$, $g(x) = \dfrac{3 - x}{4}$

130. $f(x) = \sqrt{x + 1}$; $g(x) = x^2 - 1, x \geq 0$

Using the Horizontal Line Test **In Exercises 131–134, use a graphing utility to graph the function and use the Horizontal Line Test to determine whether the function is one-to-one and an inverse function exists.**

131. $f(x) = \frac{1}{2}x - 3$

132. $f(x) = (x - 1)^2$

133. $h(t) = (t + 5)^{2/3}$

134. $g(x) = \dfrac{5}{x - 4}$

Finding an Inverse Function Algebraically **In Exercises 135–142, find the inverse function of f algebraically.**

135. $f(x) = \dfrac{1}{2}x - 5$

136. $f(x) = \dfrac{7x + 3}{8}$

137. $f(x) = -5x^3 - 3$

138. $f(x) = 5x^3 + 2$

139. $f(x) = \sqrt{x + 10}$

140. $f(x) = 4\sqrt{6 - x}$

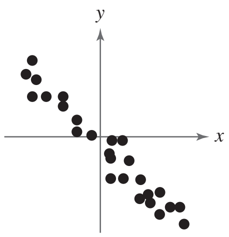
141. $f(x) = \frac{1}{4}x^2 + 1, \quad x \ge 0$

142. $f(x) = 5 - \frac{1}{9}x^2, \quad x \le 0$

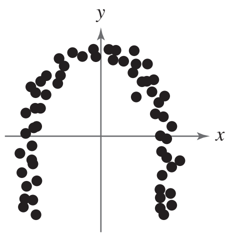
1.7

Interpreting Correlation **In Exercises 143 and 144, the scatter plot of a set of data is shown. Determine whether the points are positively correlated, are negatively correlated, or have no discernible correlation.**

143.



144.



145. Education The following ordered pairs give the entrance exam scores x and the grade-point averages y after 1 year of college for 10 students. *(Spreadsheet at LarsonPrecalculus.com)*

DATA (75, 2.3), (82, 3.0), (90, 3.6), (65, 2.0), (70, 2.1), (88, 3.5), (93, 3.9), (69, 2.0), (80, 2.8), (85, 3.3)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? Explain.

146. Industrial Engineering A machine part was tested by bending it x centimeters 10 times per minute until it failed (y equals the time to failure in hours). The results are given as the following ordered pairs. *(Spreadsheet at LarsonPrecalculus.com)*

DATA (3, 61), (6, 56), (9, 53), (12, 55), (15, 48), (18, 35), (21, 36), (24, 33), (27, 44), (30, 23)

(a) Create a scatter plot of the data.

(b) Does the relationship between x and y appear to be approximately linear? If not, give some possible explanations.

147. MODELING DATA

In an experiment, students measured the speed s (in meters per second) of a ball t seconds after it was released. The results are shown in the table.

Time, t	Speed, s
0	0
1	11.0
2	19.4
3	29.2
4	39.4

(a) Sketch a scatter plot of the data.

(b) Find an equation of a line that seems to fit the data best.

(c) Use the *regression* feature of a graphing utility to find a linear model for the data and identify the correlation coefficient.

(d) Use the model from part (c) to estimate the speed of the ball after 2.5 seconds.

148. MODELING DATA

The following ordered pairs (x, y) represent the Olympic year x and the winning time y (in minutes) in the men's 400-meter freestyle swimming event. *(Spreadsheet at LarsonPrecalculus.com)* (Source: International Olympic Committee)

(1968, 4.150) (1984, 3.854) (2000, 3.677)
(1972, 4.005) (1988, 3.783) (2004, 3.718)
(1976, 3.866) (1992, 3.750) (2008, 3.698)
(1980, 3.855) (1996, 3.800) (2012, 3.669)

(a) Use the *regression* feature of a graphing utility to find a linear model for the data. Let x represent the year, with $x = 8$ corresponding to 1968.

(b) Use the graphing utility to create a scatter plot of the data. Graph the model in the same viewing window.

(c) Is the model a good fit for the data? Explain.

(d) Is this model appropriate for predicting the winning times in future Olympics? Explain.

Conclusions

True or False? **In Exercises 149–151, determine whether the statement is true or false. Justify your answer.**

149. If the graph of the parent function $f(x) = x^2$ is moved six units to the right, moved three units upward, and reflected in the x-axis, then the point $(-1, 28)$ will lie on the graph of the transformation.

150. If $f(x) = x^n$ where n is odd, then f^{-1} exists.

151. There exists no function f such that $f = f^{-1}$.

1 Chapter Test

See *CalcChat.com* for tutorial help and worked-out solutions to odd-numbered exercises.
For instructions on how to use a graphing utility, see Appendix A.

Take this test as you would take a test in class. After you are finished, check your work against the answers given in the back of the book.

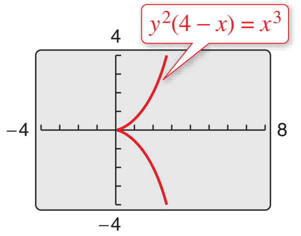


Figure for 3

1. Find the equations of the lines that pass through the point $(0, 4)$ and are (a) parallel to and (b) perpendicular to the line $5x + 2y = 3$.
2. Find the slope-intercept form of the equation of the line that passes through the points $(2, -1)$ and $(-3, 4)$.
3. Does the graph at the right represent y as a function of x? Explain.
4. Evaluate $f(x) = |x + 2| - 15$ at each value of the independent variable and simplify.
   (a) $f(-8)$ (b) $f(14)$ (c) $f(t - 6)$
5. Find the domain of $f(x) = 10 - \sqrt{3 - x}$.
6. An electronics company produces a car stereo for which the variable cost is \$24.60 per unit and the fixed costs are \$25,000. The product sells for \$101.50. Write the total cost C as a function of the number of units produced and sold, x. Write the profit P as a function of the number of units produced and sold, x.

In Exercises 7 and 8, determine algebraically whether the function is even, odd, or neither.

7. $f(x) = 2x^3 - 3x$
8. $f(x) = 3x^4 + 5x^2$

In Exercises 9 and 10, determine the open intervals on which the function is increasing, decreasing, or constant.

9. $h(x) = \frac{1}{4}x^4 - 2x^2$
10. $g(t) = |t + 2| - |t - 2|$

In Exercises 11 and 12, use a graphing utility to graph the functions and to approximate (to two decimal places) any relative minimum or relative maximum values of the function.

11. $f(x) = -x^3 - 5x^2 + 12$
12. $f(x) = x^5 - 3x^3 + 4$

In Exercises 13–15, (a) identify the parent function f, (b) describe the sequence of transformations from f to g, and (c) sketch the graph of g.

13. $g(x) = -2(x - 5)^3 + 3$
14. $g(x) = \sqrt{-7x - 14}$
15. $g(x) = 4|-x| - 7$

16. Use the functions $f(x) = x^2$ and $g(x) = \sqrt{2 - x}$ to find the specified function and its domain.
   (a) $(f - g)(x)$ (b) $\left(\frac{f}{g}\right)(x)$ (c) $(f \circ g)(x)$ (d) $(g \circ f)(x)$

In Exercises 17–19, determine whether the function has an inverse function, and if so, find the inverse function.

17. $f(x) = x^3 + 8$
18. $f(x) = x^2 + 6$
19. $f(x) = \dfrac{3x\sqrt{x}}{8}$

20. The table shows the average monthly costs C of expanded basic cable television from 2004 through 2012, where t represents the year, with $t = 4$ corresponding to 2004. Use the *regression* feature of a graphing utility to find a linear model for the data. Use the model to estimate the year in which the average monthly cost reached \$85. (Source: Federal Communications Commission)

DATA

Year, t	Average monthly cost, C (in dollars)
4	41.04
5	43.04
6	45.26
7	47.27
8	49.65
9	52.37
10	54.44
11	57.46
12	61.63

Spreadsheet at LarsonPrecalculus.com

Table for 20

Proofs in Mathematics

Conditional Statements

Many theorems are written in the **if-then form** "if p, then q," which is denoted by

$p \rightarrow q$ Conditional statement

where p is the **hypothesis** and q is the **conclusion.** Here are some other ways to express the conditional statement $p \rightarrow q$.

p implies q. p, only if q. p is sufficient for q.

Conditional statements can be either true or false. The conditional statement $p \rightarrow q$ is false only when p is true and q is false. To show that a conditional statement is true, you must prove that the conclusion follows for all cases that fulfill the hypothesis. To show that a conditional statement is false, you need to describe only a single **counterexample** that shows that the statement is not always true.

For instance, $x = -4$ is a counterexample that shows that the following statement is false.

If $x^2 = 16$, then $x = 4$.

The hypothesis "$x^2 = 16$" is true because $(-4)^2 = 16$. However, the conclusion "$x = 4$" is false. This implies that the given conditional statement is false.

For the conditional statement $p \rightarrow q$, there are three important associated conditional statements.

1. The **converse** of $p \rightarrow q$: $q \rightarrow p$
2. The **inverse** of $p \rightarrow q$: $\sim p \rightarrow \sim q$
3. The **contrapositive** of $p \rightarrow q$: $\sim q \rightarrow \sim p$

The symbol ~ means the **negation** of a statement. For instance, the negation of "The engine is running" is "The engine is not running."

EXAMPLE 1 Writing the Converse, Inverse, and Contrapositive

Write the converse, inverse, and contrapositive of the conditional statement "If I get a B on my test, then I will pass the course."

Solution

a. *Converse:* If I pass the course, then I got a B on my test.

b. *Inverse:* If I do not get a B on my test, then I will not pass the course.

c. *Contrapositive:* If I do not pass the course, then I did not get a B on my test.

In the example above, notice that neither the converse nor the inverse is logically equivalent to the original conditional statement. On the other hand, the contrapositive is logically equivalent to the original conditional statement.

Biconditional Statements

Recall that a conditional statement is a statement of the form "if p, then q." A statement of the form "p if and only if q" is called a **biconditional statement.** A biconditional statement, denoted by

$p \leftrightarrow q$ Biconditional statement

is the conjunction of the conditional statement $p \rightarrow q$ and its converse $q \rightarrow p$.

A biconditional statement can be either true or false. To be true, *both* the conditional statement and its converse must be true.

EXAMPLE 2 Analyzing a Biconditional Statement

Consider the statement $x = 3$ if and only if $x^2 = 9$.

a. Is the statement a biconditional statement?

b. Is the statement true?

Solution

a. The statement is a biconditional statement because it is of the form "p if and only if q."

b. The statement can be rewritten as the following conditional statement and its converse.

Conditional statement: If $x = 3$, then $x^2 = 9$.

Converse: If $x^2 = 9$, then $x = 3$.

The first of these statements is true, but the second is false because x could also equal -3. So, the biconditional statement is false.

Knowing how to use biconditional statements is an important tool for reasoning in mathematics.

EXAMPLE 3 Analyzing a Biconditional Statement

Determine whether the biconditional statement is true or false. If it is false, provide a counterexample.

A number is divisible by 5 if and only if it ends in 0.

Solution

The biconditional statement can be rewritten as the following conditional statement and its converse.

Conditional statement: If a number is divisible by 5, then it ends in 0.

Converse: If a number ends in 0, then it is divisible by 5.

The conditional statement is false. A counterexample is the number 15, which is divisible by 5 but does not end in 0. So, the biconditional statement is false.