MacLaurin/Taylor/ Euler

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Question 4

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

(a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

(c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

(d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

(a) $f(1) = 15,\ f'(1) = 8$

An equation for the tangent line is $y = 15 + 8(x - 1)$.

$f(1.4) \approx 15 + 8(1.4 - 1) = 18.2$

2 : { 1 : tangent line; 1 : approximation }

(b) $\int_1^{1.4} f'(x)\,dx \approx (0.2)(10) + (0.2)(13) = 4.6$

$f(1.4) = f(1) + \int_1^{1.4} f'(x)\,dx$

$f(1.4) \approx 15 + 4.6 = 19.6$

3 : { 1 : midpoint Riemann sum; 1 : Fundamental Theorem of Calculus; 1 : answer }

(c) $f(1.2) \approx f(1) + (0.2)(8) = 16.6$

$f(1.4) \approx 16.6 + (0.2)(12) = 19.0$

2 : { 1 : Euler's method with two steps; 1 : answer }

(d) $T_2(x) = 15 + 8(x - 1) + \frac{20}{2!}(x - 1)^2$

$= 15 + 8(x - 1) + 10(x - 1)^2$

$f(1.4) \approx 15 + 8(1.4 - 1) + 10(1.4 - 1)^2 = 19.8$

2 : { 1 : Taylor polynomial; 1 : approximation }

Question 6

The function g has derivatives of all orders, and the Maclaurin series for g is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+3} = \frac{x}{3} - \frac{x^3}{5} + \frac{x^5}{7} - \cdots$.

(a) Using the ratio test, determine the interval of convergence of the Maclaurin series for g.

(b) The Maclaurin series for g evaluated at $x = \frac{1}{2}$ is an alternating series whose terms decrease in absolute value to 0. The approximation for $g\left(\frac{1}{2}\right)$ using the first two nonzero terms of this series is $\frac{17}{120}$. Show that this approximation differs from $g\left(\frac{1}{2}\right)$ by less than $\frac{1}{200}$.

(c) Write the first three nonzero terms and the general term of the Maclaurin series for $g'(x)$.

(a) $\left|\frac{x^{2n+3}}{2n+5} \cdot \frac{2n+3}{x^{2n+1}}\right| = \left(\frac{2n+3}{2n+5}\right) \cdot x^2$

$\lim_{n\to\infty}\left(\frac{2n+3}{2n+5}\right) \cdot x^2 = x^2$

$x^2 < 1 \;\Rightarrow\; -1 < x < 1$

The series converges when $-1 < x < 1$.

When $x = -1$, the series is $-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$

This series converges by the Alternating Series Test.

When $x = 1$, the series is $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots$

This series converges by the Alternating Series Test.

Therefore, the interval of convergence is $-1 \le x \le 1$.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies interior of interval of convergence
- 1 : considers both endpoints
- 1 : analysis and interval of convergence

(b) $\left|g\left(\frac{1}{2}\right) - \frac{17}{120}\right| < \frac{\left(\frac{1}{2}\right)^5}{7} = \frac{1}{224} < \frac{1}{200}$

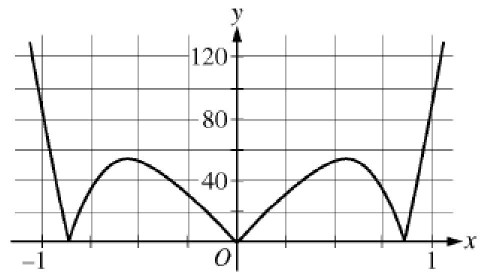
2 :
- 1 : uses the third term as an error bound
- 1 : error bound

(c) $g'(x) = \frac{1}{3} - \frac{3}{5}x^2 + \frac{5}{7}x^4 + \cdots + (-1)^n\left(\frac{2n+1}{2n+3}\right)x^{2n} + \cdots$

2 :
- 1 : first three terms
- 1 : general term

Question 6

Let $f(x) = \sin\left(x^2\right) + \cos x$. The graph of $y = \left|f^{(5)}(x)\right|$ is shown above.



Graph of $y = \left|f^{(5)}(x)\right|$

(a) Write the first four nonzero terms of the Taylor series for $\sin x$ about $x = 0$, and write the first four nonzero terms of the Taylor series for $\sin\left(x^2\right)$ about $x = 0$.

(b) Write the first four nonzero terms of the Taylor series for $\cos x$ about $x = 0$. Use this series and the series for $\sin\left(x^2\right)$, found in part (a), to write the first four nonzero terms of the Taylor series for f about $x = 0$.

(c) Find the value of $f^{(6)}(0)$.

(d) Let $P_4(x)$ be the fourth-degree Taylor polynomial for f about $x = 0$. Using information from the graph of $y = \left|f^{(5)}(x)\right|$ shown above, show that $\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| < \frac{1}{3000}$.

(a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$

$\sin\left(x^2\right) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \sin x \\ 2 : \text{series for } \sin\left(x^2\right) \end{cases}$

(b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$f(x) = 1 + \frac{x^2}{2} + \frac{x^4}{4!} - \frac{121x^6}{6!} + \cdots$

$3 : \begin{cases} 1 : \text{series for } \cos x \\ 2 : \text{series for } f(x) \end{cases}$

(c) $\frac{f^{(6)}(0)}{6!}$ is the coefficient of x^6 in the Taylor series for f about $x = 0$. Therefore $f^{(6)}(0) = -121$.

1 : answer

(d) The graph of $y = \left|f^{(5)}(x)\right|$ indicates that $\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right| < 40$.

Therefore

$$\left|P_4\left(\frac{1}{4}\right) - f\left(\frac{1}{4}\right)\right| \le \frac{\max\limits_{0 \le x \le \frac{1}{4}} \left|f^{(5)}(x)\right|}{5!} \cdot \left(\frac{1}{4}\right)^5 < \frac{40}{120 \cdot 4^5} = \frac{1}{3072} < \frac{1}{3000}.$$

$2 : \begin{cases} 1 : \text{form of the error bound} \\ 1 : \text{analysis} \end{cases}$

Question 6

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x^2} & \text{for } x \neq 0 \\ -\dfrac{1}{2} & \text{for } x = 0 \end{cases}$$

The function f, defined above, has derivatives of all orders. Let g be the function defined by $g(x) = 1 + \int_0^x f(t)\,dt.$

(a) Write the first three nonzero terms and the general term of the Taylor series for $\cos x$ about $x = 0$. Use this series to write the first three nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use the Taylor series for f about $x = 0$ found in part (a) to determine whether f has a relative maximum, relative minimum, or neither at $x = 0$. Give a reason for your answer.

(c) Write the fifth-degree Taylor polynomial for g about $x = 0$.

(d) The Taylor series for g about $x = 0$, evaluated at $x = 1$, is an alternating series with individual terms that decrease in absolute value to 0. Use the third-degree Taylor polynomial for g about $x = 0$ to estimate the value of $g(1)$. Explain why this estimate differs from the actual value of $g(1)$ by less than $\frac{1}{6!}$.

Solution	Scoring
(a) $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots$ $f(x) = -\frac{1}{2} + \frac{x^2}{4!} - \frac{x^4}{6!} + \cdots + (-1)^{n+1} \frac{x^{2n}}{(2n+2)!} + \cdots$	3 : { 1 : terms for $\cos x$; 2 : terms for f { 1 : first three terms; 1 : general term } }
(b) $f'(0)$ is the coefficient of x in the Taylor series for f about $x = 0$, so $f'(0) = 0$. $\frac{f''(0)}{2!} = \frac{1}{4!}$ is the coefficient of x^2 in the Taylor series for f about $x = 0$, so $f''(0) = \frac{1}{12}$. Therefore, by the Second Derivative Test, f has a relative minimum at $x = 0$.	2 : { 1 : determines $f'(0)$; 1 : answer with reason }
(c) $P_5(x) = 1 - \frac{x}{2} + \frac{x^3}{3 \cdot 4!} - \frac{x^5}{5 \cdot 6!}$	2 : { 1 : two correct terms; 1 : remaining terms }
(d) $g(1) \approx 1 - \frac{1}{2} + \frac{1}{3 \cdot 4!} = \frac{37}{72}$ Since the Taylor series for g about $x = 0$ evaluated at $x = 1$ is alternating and the terms decrease in absolute value to 0, we know $\left\lvert g(1) - \frac{37}{72} \right\rvert < \frac{1}{5 \cdot 6!} < \frac{1}{6!}$.	2 : { 1 : estimate; 1 : explanation }

Question 4

Consider the differential equation $\frac{dy}{dx} = 6x^2 - x^2 y$. Let $y = f(x)$ be a particular solution to this differential equation with the initial condition $f(-1) = 2$.

(a) Use Euler's method with two steps of equal size, starting at $x = -1$, to approximate $f(0)$. Show the work that leads to your answer.

(b) At the point $(-1, 2)$, the value of $\frac{d^2y}{dx^2}$ is -12. Find the second-degree Taylor polynomial for f about $x = -1$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(-1) = 2$.

(a)

$$f\left(-\frac{1}{2}\right) \approx f(-1) + \left(\left.\frac{dy}{dx}\right|_{(-1,\,2)}\right) \cdot \Delta x$$

$$= 2 + 4 \cdot \frac{1}{2} = 4$$

$$f(0) \approx f\left(-\frac{1}{2}\right) + \left(\left.\frac{dy}{dx}\right|_{\left(-\frac{1}{2},\,4\right)}\right) \cdot \Delta x$$

$$\approx 4 + \frac{1}{2} \cdot \frac{1}{2} = \frac{17}{4}$$

2 : { 1 : Euler's method with two steps; 1 : answer }

(b) $P_2(x) = 2 + 4(x+1) - 6(x+1)^2$

1 : answer

(c)

$$\frac{dy}{dx} = x^2(6 - y)$$

$$\int \frac{1}{6-y}\, dy = \int x^2\, dx$$

$$-\ln|6 - y| = \frac{1}{3}x^3 + C$$

$$-\ln 4 = -\frac{1}{3} + C$$

$$C = \frac{1}{3} - \ln 4$$

$$\ln|6 - y| = -\frac{1}{3}x^3 - \left(\frac{1}{3} - \ln 4\right)$$

$$|6 - y| = 4e^{-\frac{1}{3}\left(x^3+1\right)}$$

$$y = 6 - 4e^{-\frac{1}{3}\left(x^3+1\right)}$$

6 : { 1 : separation of variables; 2 : antiderivatives; 1 : constant of integration; 1 : uses initial condition; 1 : solves for y }

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

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Question 6

The Maclaurin series for e^x is $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots + \frac{x^n}{n!} + \cdots$. The continuous function f is defined by $f(x) = \frac{e^{(x-1)^2} - 1}{(x-1)^2}$ for $x \neq 1$ and $f(1) = 1$. The function f has derivatives of all orders at $x = 1$.

(a) Write the first four nonzero terms and the general term of the Taylor series for $e^{(x-1)^2}$ about $x = 1$.

(b) Use the Taylor series found in part (a) to write the first four nonzero terms and the general term of the Taylor series for f about $x = 1$.

(c) Use the ratio test to find the interval of convergence for the Taylor series found in part (b).

(d) Use the Taylor series for f about $x = 1$ to determine whether the graph of f has any points of inflection.

(a) $1 + (x-1)^2 + \frac{(x-1)^4}{2} + \frac{(x-1)^6}{6} + \cdots + \frac{(x-1)^{2n}}{n!} + \cdots$	$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$
(b) $1 + \frac{(x-1)^2}{2} + \frac{(x-1)^4}{6} + \frac{(x-1)^6}{24} + \cdots + \frac{(x-1)^{2n}}{(n+1)!} + \cdots$	$2: \begin{cases} 1: \text{first four terms} \\ 1: \text{general term} \end{cases}$
(c) $\lim_{n\to\infty} \left\lvert \dfrac{\frac{(x-1)^{2n+2}}{(n+2)!}}{\frac{(x-1)^{2n}}{(n+1)!}} \right\rvert = \lim_{n\to\infty} \frac{(n+1)!}{(n+2)!}(x-1)^2 = \lim_{n\to\infty} \frac{(x-1)^2}{n+2} = 0$ Therefore, the interval of convergence is $(-\infty, \infty)$.	$3: \begin{cases} 1: \text{sets up ratio} \\ 1: \text{computes limit of ratio} \\ 1: \text{answer} \end{cases}$
(d) $f''(x) = 1 + \frac{4\cdot 3}{6}(x-1)^2 + \frac{6\cdot 5}{24}(x-1)^4 + \cdots + \frac{2n(2n-1)}{(n+1)!}(x-1)^{2n-2} + \cdots$ Since every term of this series is nonnegative, $f''(x) \geq 0$ for all x. Therefore, the graph of f has no points of inflection.	$2: \begin{cases} 1: f''(x) \\ 1: \text{answer} \end{cases}$

Question 3

x	$h(x)$	$h'(x)$	$h''(x)$	$h'''(x)$	$h^{(4)}(x)$
1	11	30	42	99	18
2	80	128	$\frac{488}{3}$	$\frac{448}{3}$	$\frac{584}{9}$
3	317	$\frac{753}{2}$	$\frac{1383}{4}$	$\frac{3483}{16}$	$\frac{1125}{16}$

Let h be a function having derivatives of all orders for $x > 0$. Selected values of h and its first four derivatives are indicated in the table above. The function h and these four derivatives are increasing on the interval $1 \le x \le 3$.

(a) Write the first-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$. Is this approximation greater than or less than $h(1.9)$? Explain your reasoning.

(b) Write the third-degree Taylor polynomial for h about $x = 2$ and use it to approximate $h(1.9)$.

(c) Use the Lagrange error bound to show that the third-degree Taylor polynomial for h about $x = 2$ approximates $h(1.9)$ with error less than 3×10^{-4}.

(a) $P_1(x) = 80 + 128(x-2)$, so $h(1.9) \approx P_1(1.9) = 67.2$

$P_1(1.9) < h(1.9)$ since h' is increasing on the interval $1 \le x \le 3$.

$4 : \begin{cases} 2 : P_1(x) \\ 1 : P_1(1.9) \\ 1 : P_1(1.9) < h(1.9) \text{ with reason} \end{cases}$

(b) $P_3(x) = 80 + 128(x-2) + \frac{488}{6}(x-2)^2 + \frac{448}{18}(x-2)^3$

$h(1.9) \approx P_3(1.9) = 67.988$

$3 : \begin{cases} 2 : P_3(x) \\ 1 : P_3(1.9) \end{cases}$

(c) The fourth derivative of h is increasing on the interval $1 \le x \le 3$, so $\max\limits_{1.9 \le x \le 2} \left|h^{(4)}(x)\right| = \frac{584}{9}$.

Therefore, $|h(1.9) - P_3(1.9)| \le \frac{584}{9} \frac{|1.9-2|^4}{4!}$

$= 2.7037 \times 10^{-4}$

$< 3 \times 10^{-4}$

$2 : \begin{cases} 1 : \text{form of Lagrange error estimate} \\ 1 : \text{reasoning} \end{cases}$

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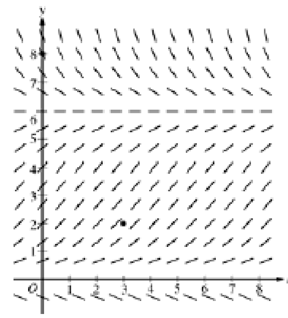
Question 6

Consider the logistic differential equation $\frac{dy}{dt} = \frac{y}{8}(6 - y)$. Let $y = f(t)$ be the particular solution to the differential equation with $f(0) = 8$.

(a) A slope field for this differential equation is given below. Sketch possible solution curves through the points $(3, 2)$ and $(0, 8)$.
(Note: Use the axes provided in the exam booklet.)

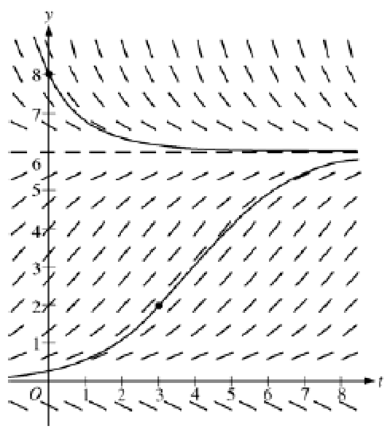


(b) Use Euler's method, starting at $t = 0$ with two steps of equal size, to approximate $f(1)$.

(c) Write the second-degree Taylor polynomial for f about $t = 0$, and use it to approximate $f(1)$.

(d) What is the range of f for $t \geq 0$?

(a)



$2 : \begin{cases} 1: \text{solution curve through } (0,8) \\ 1: \text{solution curve through } (3,2) \end{cases}$

(b) $f\left(\frac{1}{2}\right) \approx 8 + (-2)\left(\frac{1}{2}\right) = 7$

$f(1) \approx 7 + \left(-\frac{7}{8}\right)\left(\frac{1}{2}\right) = \frac{105}{16}$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{approximation of } f(1) \end{cases}$

(c) $\frac{d^2y}{dt^2} = \frac{1}{8}\frac{dy}{dt}(6 - y) + \frac{y}{8}\left(-\frac{dy}{dt}\right)$

$f(0) = 8;\ f'(0) = \left.\frac{dy}{dt}\right|_{t=0} = \frac{8}{8}(6 - 8) = -2;$ and

$f''(0) = \left.\frac{d^2y}{dt^2}\right|_{t=0} = \frac{1}{8}(-2)(-2) + \frac{8}{8}(2) = \frac{5}{2}$

The second-degree Taylor polynomial for f about $t = 0$ is $P_2(t) = 8 - 2t + \frac{5}{4}t^2$.

$f(1) \approx P_2(1) = \frac{29}{4}$

$4 : \begin{cases} 2 : \frac{d^2y}{dt^2} \\ 1 : \text{second-degree Taylor polynomial} \\ 1 : \text{approximation of } f(1) \end{cases}$

(d) The range of f for $t \geq 0$ is $6 < y \leq 8$.

1 : answer

Question 6

Let f be the function given by $f(x) = e^{-x^2}$.

(a) Write the first four nonzero terms and the general term of the Taylor series for f about $x = 0$.

(b) Use your answer to part (a) to find $\lim_{x\to 0} \frac{1 - x^2 - f(x)}{x^4}$.

(c) Write the first four nonzero terms of the Taylor series for $\int_0^x e^{-t^2}\,dt$ about $x = 0$. Use the first two terms of your answer to estimate $\int_0^{1/2} e^{-t^2}\,dt$.

(d) Explain why the estimate found in part (c) differs from the actual value of $\int_0^{1/2} e^{-t^2}\,dt$ by less than $\frac{1}{200}$.

(a) $$e^{-x^2} = 1 + \frac{(-x^2)}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots + \frac{(-x^2)^n}{n!} + \cdots$$
$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots$$

3 : { 1 : two of $1, -x^2, \frac{x^4}{2}, -\frac{x^6}{6}$; 1 : remaining terms; 1 : general term }

(b) $$\frac{1 - x^2 - f(x)}{x^4} = -\frac{1}{2} + \frac{x^2}{6} + \sum_{n=4}^{\infty} \frac{(-1)^{n+1} x^{2n-4}}{n!}$$

Thus, $\lim_{x\to 0}\left(\frac{1 - x^2 - f(x)}{x^4}\right) = -\frac{1}{2}$.

1 : answer

(c) $$\int_0^x e^{-t^2}\,dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2} - \frac{t^6}{6} + \cdots + \frac{(-1)^n t^{2n}}{n!} + \cdots\right) dt$$
$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots$$

Using the first two terms of this series, we estimate that
$$\int_0^{1/2} e^{-t^2}\,dt \approx \frac{1}{2} - \left(\frac{1}{3}\right)\left(\frac{1}{8}\right) = \frac{11}{24}.$$

3 : { 1 : two terms; 1 : remaining terms; 1 : estimate }

(d) $$\left|\int_0^{1/2} e^{-t^2}\,dt - \frac{11}{24}\right| < \left(\frac{1}{2}\right)^5 \cdot \frac{1}{10} = \frac{1}{320} < \frac{1}{200},$$ since

$$\int_0^{1/2} e^{-t^2}\,dt = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)^{2n+1}}{n!(2n+1)},$$ which is an alternating series with individual terms that decrease in absolute value to 0.

2 : { 1 : uses the third term as the error bound; 1 : explanation }

Question 5

Consider the differential equation $\frac{dy}{dx} = 5x^2 - \frac{6}{y-2}$ for $y \neq 2$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(-1) = -4$.

(a) Evaluate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $(-1, -4)$.

(b) Is it possible for the x-axis to be tangent to the graph of f at some point? Explain why or why not.

(c) Find the second-degree Taylor polynomial for f about $x = -1$.

(d) Use Euler's method, starting at $x = -1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.

(a) $\left.\frac{dy}{dx}\right|_{(-1,\,-4)} = 6$

$$\frac{d^2y}{dx^2} = 10x + 6(y-2)^{-2}\frac{dy}{dx}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(-1,\,-4)} = -10 + 6\frac{1}{(-6)^2}6 = -9$$

$3 : \begin{cases} 1 : \left.\frac{dy}{dx}\right|_{(-1,\,-4)} \\ 1 : \frac{d^2y}{dx^2} \\ 1 : \left.\frac{d^2y}{dx^2}\right|_{(-1,\,-4)} \end{cases}$

(b) The x-axis will be tangent to the graph of f if $\left.\frac{dy}{dx}\right|_{(k,\,0)} = 0$.

The x-axis will never be tangent to the graph of f because $\left.\frac{dy}{dx}\right|_{(k,\,0)} = 5k^2 + 3 > 0$ for all k.

$2 : \begin{cases} 1 : \frac{dy}{dx} = 0 \text{ and } y = 0 \\ 1 : \text{answer and explanation} \end{cases}$

(c) $P(x) = -4 + 6(x+1) - \frac{9}{2}(x+1)^2$

$2 : \begin{cases} 1 : \text{quadratic and centered at } x = -1 \\ 1 : \text{coefficients} \end{cases}$

(d) $f(-1) = -4$

$$f\left(-\frac{1}{2}\right) \approx -4 + \frac{1}{2}(6) = -1$$

$$f(0) \approx -1 + \frac{1}{2}\left(\frac{5}{4} + 2\right) = \frac{5}{8}$$

$2 : \begin{cases} 1 : \text{Euler's method with 2 steps} \\ 1 : \text{Euler's approximation to } f(0) \end{cases}$

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Question 6

Let f be a function with derivatives of all orders and for which $f(2) = 7$. When n is odd, the nth derivative of f at $x = 2$ is 0. When n is even and $n \geq 2$, the nth derivative of f at $x = 2$ is given by $f^{(n)}(2) = \frac{(n-1)!}{3^n}$.

(a) Write the sixth-degree Taylor polynomial for f about $x = 2$.

(b) In the Taylor series for f about $x = 2$, what is the coefficient of $(x-2)^{2n}$ for $n \geq 1$?

(c) Find the interval of convergence of the Taylor series for f about $x = 2$. Show the work that leads to your answer.

(a) $P_6(x) = 7 + \frac{1!}{3^2} \cdot \frac{1}{2!}(x-2)^2 + \frac{3!}{3^4} \cdot \frac{1}{4!}(x-2)^4 + \frac{5!}{3^6} \cdot \frac{1}{6!}(x-2)^6$

3 : { 1 : polynomial about $x = 2$; 2 : $P_6(x)$; $\langle -1 \rangle$ each incorrect term; $\langle -1 \rangle$ max for all extra terms, $+ \cdots$, misuse of equality }

(b) $\frac{(2n-1)!}{3^{2n}} \cdot \frac{1}{(2n)!} = \frac{1}{3^{2n}(2n)}$

1 : coefficient

(c) The Taylor series for f about $x = 2$ is

$$f(x) = 7 + \sum_{n=1}^{\infty} \frac{1}{2n \cdot 3^{2n}}(x-2)^{2n}.$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{1}{2(n+1)} \cdot \frac{1}{3^{2(n+1)}}(x-2)^{2(n+1)}}{\frac{1}{2n} \cdot \frac{1}{3^{2n}}(x-2)^{2n}} \right|$$

$$= \lim_{n \to \infty} \left| \frac{2n}{2(n+1)} \cdot \frac{3^{2n}}{3^2 3^{2n}}(x-2)^2 \right| = \frac{(x-2)^2}{9}$$

$L < 1$ when $|x-2| < 3$.

Thus, the series converges when $-1 < x < 5$.

When $x = 5$, the series is $7 + \sum_{n=1}^{\infty} \frac{3^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges.

When $x = -1$, the series is $7 + \sum_{n=1}^{\infty} \frac{(-3)^{2n}}{2n \cdot 3^{2n}} = 7 + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$,

which diverges, because $\sum_{n=1}^{\infty} \frac{1}{n}$, the harmonic series, diverges.

The interval of convergence is $(-1, 5)$.

5 : { 1 : sets up ratio; 1: computes limit of ratio; 1: identifies interior of interval of convergence; 1 : considers both endpoints; 1 : analysis/conclusion for both endpoints }

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Question 6

Let f be the function given by $f(x) = \sin\left(5x + \frac{\pi}{4}\right)$, and let $P(x)$ be the third-degree Taylor polynomial for f about $x = 0$.

(a) Find $P(x)$.

(b) Find the coefficient of x^{22} in the Taylor series for f about $x = 0$.

(c) Use the Lagrange error bound to show that $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| < \frac{1}{100}$.

(d) Let G be the function given by $G(x) = \int_0^x f(t)\,dt$. Write the third-degree Taylor polynomial for G about $x = 0$.

(a) $f(0) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$

$f'(0) = 5\cos\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{2}$

$f''(0) = -25\sin\left(\frac{\pi}{4}\right) = -\frac{25\sqrt{2}}{2}$

$f'''(0) = -125\cos\left(\frac{\pi}{4}\right) = -\frac{125\sqrt{2}}{2}$

$P(x) = \frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}x - \frac{25\sqrt{2}}{2(2!)}x^2 - \frac{125\sqrt{2}}{2(3!)}x^3$

4 : $P(x)$

⟨−1⟩ each error or missing term

deduct only once for $\sin\left(\frac{\pi}{4}\right)$ evaluation error

deduct only once for $\cos\left(\frac{\pi}{4}\right)$ evaluation error

⟨−1⟩ max for all extra terms, $+\cdots$, misuse of equality

(b) $\frac{-5^{22}\sqrt{2}}{2(22!)}$

2 : { 1 : magnitude; 1 : sign }

(c) $\left|f\left(\frac{1}{10}\right) - P\left(\frac{1}{10}\right)\right| \le \max_{0 \le c \le \frac{1}{10}} \left|f^{(4)}(c)\right| \left(\frac{1}{4!}\right)\left(\frac{1}{10}\right)^4$

$\le \frac{625}{4!}\left(\frac{1}{10}\right)^4 = \frac{1}{384} < \frac{1}{100}$

1 : error bound in an appropriate inequality

(d) The third-degree Taylor polynomial for G about $x = 0$ is $\int_0^x \left(\frac{\sqrt{2}}{2} + \frac{5\sqrt{2}}{2}t - \frac{25\sqrt{2}}{4}t^2\right) dt$

$= \frac{\sqrt{2}}{2}x + \frac{5\sqrt{2}}{4}x^2 - \frac{25\sqrt{2}}{12}x^3$

2 : third-degree Taylor polynomial for G about $x = 0$

⟨−1⟩ each incorrect or missing term

⟨−1⟩ max for all extra terms, $+\cdots$, misuse of equality

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Question 6

The Maclaurin series for the function f is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{(2x)^{n+1}}{n+1} = 2x + \frac{4x^2}{2} + \frac{8x^3}{3} + \frac{16x^4}{4} + \cdots + \frac{(2x)^{n+1}}{n+1} + \cdots$$

on its interval of convergence.

(a) Find the interval of convergence of the Maclaurin series for f. Justify your answer.

(b) Find the first four terms and the general term for the Maclaurin series for $f'(x)$.

(c) Use the Maclaurin series you found in part (b) to find the value of $f'\left(-\frac{1}{3}\right)$.

(a) $$\lim_{n\to\infty} \left| \frac{\dfrac{(2x)^{n+2}}{n+2}}{\dfrac{(2x)^{n+1}}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)}{(n+2)} 2x \right| = \left| 2x \right|$$

$\left| 2x \right| < 1$ for $-\frac{1}{2} < x < \frac{1}{2}$

At $x = \frac{1}{2}$, the series is $\sum_{n=0}^{\infty} \frac{1}{n+1}$ which diverges since this is the harmonic series.

At $x = -\frac{1}{2}$, the series is $\sum_{n=0}^{\infty} (-1)^{n+1} \frac{1}{n+1}$ which converges by the Alternating Series Test.

Hence, the interval of convergence is $-\frac{1}{2} \le x < \frac{1}{2}$.

5 {
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies interior of interval of convergence
- 2 : analysis/conclusion at endpoints
  - 1 : right endpoint
  - 1 : left endpoint

$< -1 >$ if endpoints not $x = \pm\frac{1}{2}$

$< -1 >$ if multiple intervals

(b) $f'(x) = 2 + 4x + 8x^2 + 16x^3 + \ldots + 2(2x)^n + \ldots$

2 {
- 1 : first 4 terms
- 1 : general term

(c) The series in (b) is a geometric series.

$$f'\left(-\frac{1}{3}\right) = 2 + 4\left(-\frac{1}{3}\right) + 8\left(-\frac{1}{3}\right)^2 + \ldots + 2\left(2\cdot\left(-\frac{1}{3}\right)\right)^n + \ldots$$

$$= 2 - \frac{4}{3} + \frac{8}{9} - \frac{16}{27} + \ldots + 2\left(-\frac{2}{3}\right)^n + \ldots$$

$$= \frac{2}{1+\frac{2}{3}} = \frac{6}{5}$$

OR

$f'(x) = \frac{2}{1-2x}$ for $-\frac{1}{2} < x < \frac{1}{2}$. Therefore,

$$f'\left(-\frac{1}{3}\right) = \frac{2}{1+\frac{2}{3}} = \frac{6}{5}$$

2 {
- 1 : substitutes $x = -\frac{1}{3}$ into infinite series from (b) or expresses series from (b) in closed form
- 1 : answer for student's series

The Taylor series about $x = 5$ for a certain function f converges to $f(x)$ for all x in the interval of convergence. The nth derivative of f at $x = 5$ is given by $f^{(n)}(5) = \dfrac{(-1)^n n!}{2^n (n+2)}$, and $f(5) = \dfrac{1}{2}$.

(a) Write the third-degree Taylor polynomial for f about $x = 5$.

(b) Find the radius of convergence of the Taylor series for f about $x = 5$.

(c) Show that the sixth-degree Taylor polynomial for f about $x = 5$ approximates $f(6)$ with error less than $\dfrac{1}{1000}$.

(a) $f'(5) = \dfrac{-1!}{2(3)}$, $f''(5) = \dfrac{2!}{4(4)}$, $f'''(5) = \dfrac{-3!}{8(5)}$

$$P_3(f,5)(x) = \frac{1}{2} - \frac{1}{6}(x-5) + \frac{1}{16}(x-5)^2 - \frac{1}{40}(x-5)^3$$

3 : $P_3(f,5)(x)$

<−1> each error or missing term

Note: <−1> max for improper use of extra terms, equality or $+\ldots$

(b) $a_n = \dfrac{f^{(n)}(5)}{n!} = \dfrac{(-1)^n}{2^n (n+2)}$

$$\lim_{n\to\infty}\left|\frac{\dfrac{(-1)^{n+1}(x-5)^{n+1}}{2^{n+1}(n+3)}}{\dfrac{(-1)^n(x-5)^n}{2^n(n+2)}}\right| = \lim_{n\to\infty}\frac{1}{2}\left(\frac{n+2}{n+3}\right)|x-5|$$

$$= \frac{|x-5|}{2} < 1$$

The radius of convergence is 2.

4 {
1 : general term
1 : sets up ratio test
1 : computes the limit
1 : applies ratio test to get radius of convergence

(c) The Taylor series about $x = 5$ for the function f, when evaluated at $x = 6$, is an alternating series with absolute value of terms decreasing to 0. The error in approximating $f(6)$ with the 6th degree Taylor polynomial at $x = 6$ is less than the first omitted term in the series.

$$\left|f(6) - P_6(f,5)(6)\right| \le \frac{1}{2^7(9)} = \frac{1}{1152} < \frac{1}{1000}$$

2 {
1 : error bound $< \dfrac{1}{1000}$
1 : refers to an alternating series and indicates the error bound is found from the next term