Calculus Stuff To Know Cold

Basic Derivatives

$$\frac{d}{dx}(x^n) = nx^{n-1}$$
$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$
$$\frac{d}{dx}(e^x) = e^x$$

The Intermediate Value Theorem

If the function f(x) is continuous on [a,b], then for any number c between f(a) and f(b), there exists a number k in the open interval (a,b) such that f(k)=c

The Mean-Value Theorem

If a function f(x) is continuous on [a,b] and the first derivative exists on the interval (a,b), then there exists a number c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Rolle's Theorem

If the function f(x) is continuous on [a,b], the first derivative exists on the interval (a,b) and f(a)=f(b); then there exists a number c on (a,b) such that $f'(c) = 0$

Curve Sketching and Analysis

1. Find all x-intercepts and y-intercepts
   a. **x-intercepts** can be found by letting y=0
   b. **y-intercepts** can be found by letting x=0
2. Find all asymptotes
   a. **Vertical Asymptotes** – can be found by letting the denominator of a rational expression equal zero
   b. **Horizontal Asymptotes**-can be found by finding $\lim_{x\to\pm\infty} f(x)$
   c. **Oblique Asymptotes**-can be found using long division
3. Find all relative extreme values where $f'(x) = 0$ or is undefined
4. Find all points of inflection where $f''(x) = 0$ or is undefined
5. Plot all critical points, points of inflection, and intercepts
6. Plot any additional points you want

Differentiation Rules

Chain Rule: $\frac{d}{dx}[f(u)] = f'(u)\frac{du}{dx}$

Product Rule: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

Quotient Rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v + \frac{dv}{dx}u}{v^2}$

Average Value of a Function

If the function f(x) is continuous on [a,b] and the first derivative exists on (a,b), then there exists a number c in (a,b) such that

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

This value f(c) is the "average value" of the function on the interval [a,b]

The Fundamental Theorem of Calculus

$$\int_a^b f(x)dx = F(b) - F(a)$$

The 2nd Fundamental Theorem of Calculus

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)dt = f[b(x)]b'(x) - f[a(x)]a'(x)$$

More Derivatives

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

Solids of Revolution

Disk Method:

$$V = \pi\int_a^b [R(x)]^2\,dx$$

Washer Method:

$$V = \pi\int_a^b [R(x)]^2 - [r(x)]^2\,dx$$

Shell Method:

$$V = 2\pi\int_a^b xf(x)dx$$

Cross Sections:

$$V = \int_a^b A(x)dx$$

Distance, Velocity, and Acceleration

Let s(t) be position

Velocity or Rate: $v(t) = s'(t)$

Speed = $|v|$

Acceleration: $a(t) = v'(t) = s''(t)$

Average Velocity over [a,b]= $\frac{f(b) - f(a)}{b - a}$

Total Distance = $\int_a^b |v|dt$

where a is initial time and b is final time.

The Trapezoidal Rule

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n)]$$

Integration by Parts

$$\int udv = uv - \int vdu$$

Simpson's Rule: Must use an even number of subdivisions!

$$\int_a^b f(x)dx \approx \frac{b-a}{3n}[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+...+4f(x_{n-1})+f(x_n)]$$

Integral of a Natural Log

$$\int \ln x dx = x\ln x - x + C$$

Distance, Velocity, Acceleration with Vectors

Velocity vector = $\left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$

Speed = $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

Total Distance = $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

where a is initial time and b is final time.

Lengths of Curves

Rectangular:

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Parametric:

$$L = \int_a^b \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

Polar:

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)}\, d\theta$$

L'Hopital's Rule

If $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Geometric Series:

$$\sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Average Value:

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Equation of Tangent Line at x=a

$$y - f(a) = f'(a)(x-a)$$

Taylor Series:

If the function f is smooth at x=a, then it can be approximated by the nth degree polynomial

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Maclaurin Series: A Taylor Series about x=0 is called a Maclaurin Series

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + ...$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - ...$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$$

The Ratio Test:

The series $\sum_{k=0}^{\infty} a_k$ converges if

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| < 1$$

If limit equals 1, you know nothing.

Polar Curves

For a polar curve $r(\theta)$, the area inside a "leaf" is

$$\int_{\theta_1}^{\theta_2} \frac{1}{2}[r(\theta)]^2 d\theta,$$

Where θ_1 and θ_2 are the "first" two times that r=0.

Convert Polar to Parametric

Given $r(\theta)$

$$x(\theta) = r\cdot\cos\theta$$
$$y(\theta) = r\cdot\sin\theta$$

Lagrange Error

If $P_n(x)$ is the nth degree Taylor polynomial of $f(x)$ about c and $\left|f^{(n+1)}(t)\right| \le M$ for all t between x and c, then

$$\left|f(x) - P(x)\right| \le \frac{M}{(n+1)!}\left|x-c\right|^{(n+1)}$$

Alternating Series Error Bound

If $S_N = \sum_{k=1}^{N}(-1)^n a_n$ is the Nth partial sum of a convergent alternating series, then $\left|S_\infty - S_N\right| \le \left|a_n + 1\right|$

Euler's Method

If given that $\frac{dy}{dx} = f(x,y)$ and that the solution passes through (x_0, y_0) where $y(x_0) = y_0$

$$y_{new} = y_{old} + dx\cdot\frac{dy}{dx}$$