

Separation of Variables

AP[®] CALCULUS BC 2012 SCORING GUIDELINES

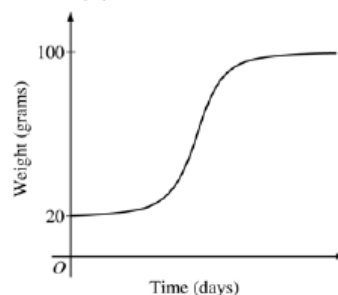
Question 5

The rate at which a baby bird gains weight is proportional to the difference between its adult weight and its current weight. At time $t = 0$, when the bird is first weighed, its weight is 20 grams. If $B(t)$ is the weight of the bird, in grams, at time t days after it is first weighed, then

$$\frac{dB}{dt} = \frac{1}{5}(100 - B).$$

Let $y = B(t)$ be the solution to the differential equation above with initial condition $B(0) = 20$.

- (a) Is the bird gaining weight faster when it weighs 40 grams or when it weighs 70 grams? Explain your reasoning.
- (b) Find $\frac{d^2B}{dt^2}$ in terms of B . Use $\frac{d^2B}{dt^2}$ to explain why the graph of B cannot resemble the following graph.
- (c) Use separation of variables to find $y = B(t)$, the particular solution to the differential equation with initial condition $B(0) = 20$.



(a) $\left. \frac{dB}{dt} \right|_{B=40} = \frac{1}{5}(60) = 12$

$$\left. \frac{dB}{dt} \right|_{B=70} = \frac{1}{5}(30) = 6$$

Because $\left. \frac{dB}{dt} \right|_{B=40} > \left. \frac{dB}{dt} \right|_{B=70}$, the bird is gaining weight faster when it weighs 40 grams.

(b) $\frac{d^2B}{dt^2} = -\frac{1}{5} \frac{dB}{dt} = -\frac{1}{5} \cdot \frac{1}{5}(100 - B) = -\frac{1}{25}(100 - B)$

Therefore, the graph of B is concave down for $20 \leq B < 100$. A portion of the given graph is concave up.

(c) $\frac{dB}{dt} = \frac{1}{5}(100 - B)$

$$\int \frac{1}{100 - B} dB = \int \frac{1}{5} dt$$

$$-\ln|100 - B| = \frac{1}{5}t + C$$

Because $20 \leq B < 100$, $|100 - B| = 100 - B$.

$$-\ln(100 - 20) = \frac{1}{5}(0) + C \Rightarrow -\ln(80) = C$$

$$100 - B = 80e^{-t/5}$$

$$B(t) = 100 - 80e^{-t/5}, \quad t \geq 0$$

2 : $\left\{ \begin{array}{l} 1 : \text{uses } \frac{dB}{dt} \\ 1 : \text{answer with reason} \end{array} \right.$

2 : $\left\{ \begin{array}{l} 1 : \frac{d^2B}{dt^2} \text{ in terms of } B \\ 1 : \text{explanation} \end{array} \right.$

5 : $\left\{ \begin{array}{l} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } B \end{array} \right.$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

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2011 SCORING GUIDELINES

Question 5

At the beginning of 2010, a landfill contained 1400 tons of solid waste. The increasing function W models the total amount of solid waste stored at the landfill. Planners estimate that W will satisfy the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ for the next 20 years. W is measured in tons, and t is measured in years from the start of 2010.

- (a) Use the line tangent to the graph of W at $t = 0$ to approximate the amount of solid waste that the landfill contains at the end of the first 3 months of 2010 (time $t = \frac{1}{4}$).
- (b) Find $\frac{d^2W}{dt^2}$ in terms of W . Use $\frac{d^2W}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the amount of solid waste that the landfill contains at time $t = \frac{1}{4}$.
- (c) Find the particular solution $W = W(t)$ to the differential equation $\frac{dW}{dt} = \frac{1}{25}(W - 300)$ with initial condition $W(0) = 1400$.

(a) $\left. \frac{dW}{dt} \right|_{t=0} = \frac{1}{25}(W(0) - 300) = \frac{1}{25}(1400 - 300) = 44$

The tangent line is $y = 1400 + 44t$.

$$W\left(\frac{1}{4}\right) \approx 1400 + 44\left(\frac{1}{4}\right) = 1411 \text{ tons}$$

(b) $\frac{d^2W}{dt^2} = \frac{1}{25} \frac{dW}{dt} = \frac{1}{625}(W - 300)$ and $W \geq 1400$

Therefore $\frac{d^2W}{dt^2} > 0$ on the interval $0 \leq t \leq \frac{1}{4}$.

The answer in part (a) is an underestimate.

(c) $\frac{dW}{dt} = \frac{1}{25}(W - 300)$

$$\int \frac{1}{W - 300} dW = \int \frac{1}{25} dt$$

$$\ln|W - 300| = \frac{1}{25}t + C$$

$$\ln(1400 - 300) = \frac{1}{25}(0) + C \Rightarrow \ln(1100) = C$$

$$W - 300 = 1100e^{\frac{1}{25}t}$$

$$W(t) = 300 + 1100e^{\frac{1}{25}t}, \quad 0 \leq t \leq 20$$

$$2 : \begin{cases} 1 : \frac{dW}{dt} \text{ at } t = 0 \\ 1 : \text{answer} \end{cases}$$

$$2 : \begin{cases} 1 : \frac{d^2W}{dt^2} \\ 1 : \text{answer with reason} \end{cases}$$

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } W \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

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2010 SCORING GUIDELINES

Question 5

Consider the differential equation $\frac{dy}{dx} = 1 - y$. Let $y = f(x)$ be the particular solution to this differential equation with the initial condition $f(1) = 0$. For this particular solution, $f(x) < 1$ for all values of x .

- (a) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(0)$. Show the work that leads to your answer.
- (b) Find $\lim_{x \rightarrow 1} \frac{f(x)}{x^3 - 1}$. Show the work that leads to your answer.
- (c) Find the particular solution $y = f(x)$ to the differential equation $\frac{dy}{dx} = 1 - y$ with the initial condition $f(1) = 0$.

(a)
$$f\left(\frac{1}{2}\right) \approx f(1) + \left(\frac{dy}{dx}\bigg|_{(1,0)}\right) \cdot \Delta x$$

$$= 0 + 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2}$$

$$f(0) \approx f\left(\frac{1}{2}\right) + \left(\frac{dy}{dx}\bigg|_{\left(\frac{1}{2}, -\frac{1}{2}\right)}\right) \cdot \Delta x$$

$$\approx -\frac{1}{2} + \frac{3}{2} \cdot \left(-\frac{1}{2}\right) = -\frac{5}{4}$$

2 : $\begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

- (b) Since f is differentiable at $x = 1$, f is continuous at $x = 1$. So,

$\lim_{x \rightarrow 1} f(x) = 0 = \lim_{x \rightarrow 1} (x^3 - 1)$ and we may apply L'Hospital's Rule.

$$\lim_{x \rightarrow 1} \frac{f(x)}{x^3 - 1} = \lim_{x \rightarrow 1} \frac{f'(x)}{3x^2} = \frac{\lim_{x \rightarrow 1} f'(x)}{\lim_{x \rightarrow 1} 3x^2} = \frac{1}{3}$$

2 : $\begin{cases} 1 : \text{use of L'Hospital's Rule} \\ 1 : \text{answer} \end{cases}$

(c) $\frac{dy}{dx} = 1 - y$

$$\int \frac{1}{1-y} dy = \int 1 dx$$

$$-\ln|1-y| = x + C$$

$$-\ln 1 = 1 + C \Rightarrow C = -1$$

$$\ln|1-y| = 1-x$$

$$|1-y| = e^{1-x}$$

$$f(x) = 1 - e^{1-x}$$

5 : $\begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

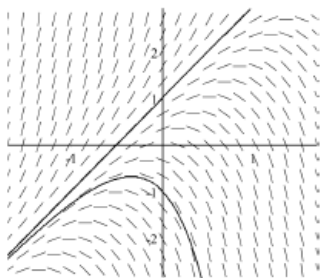
AP[®] CALCULUS BC 2002 SCORING GUIDELINES

Question 5

Consider the differential equation $\frac{dy}{dx} = 2y - 4x$.

- (a) The slope field for the given differential equation is provided. Sketch the solution curve that passes through the point $(0,1)$ and sketch the solution curve that passes through the point $(0,-1)$.
- (b) Let f be the function that satisfies the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with a step size of 0.1 , to approximate $f(0.2)$. Show the work that leads to your answer.
- (c) Find the value of b for which $y = 2x + b$ is a solution to the given differential equation. Justify your answer.
- (d) Let g be the function that satisfies the given differential equation with the initial condition $g(0) = 0$. Does the graph of g have a local extremum at the point $(0,0)$? If so, is the point a local maximum or a local minimum? Justify your answer.

(a)



(b) $f(0.1) \approx f(0) + f'(0)(0.1)$
 $= 1 + (2 - 0)(0.1) = 1.2$
 $f(0.2) \approx f(0.1) + f'(0.1)(0.1)$
 $\approx 1.2 + (2.4 - 0.4)(0.1) = 1.4$

(c) Substitute $y = 2x + b$ in the DE:
 $2 = 2(2x + b) - 4x = 2b$, so $b = 1$
 OR
 Guess $b = 1$, $y = 2x + 1$
 Verify: $2y - 4x = (4x + 2) - 4x = 2 = \frac{dy}{dx}$.

(d) g has local maximum at $(0, 0)$.
 $g'(0) = \left. \frac{dy}{dx} \right|_{(0,0)} = 2(0) - 4(0) = 0$, and
 $g''(x) = \frac{d^2y}{dx^2} = 2 \frac{dy}{dx} - 4$, so
 $g''(0) = 2g'(0) - 4 = -4 < 0$.

$$2 \left\{ \begin{array}{l} 1: \text{ solution curve through } (0,1) \\ 1: \text{ solution curve through } (0,-1) \end{array} \right.$$

Curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.

$$2 \left\{ \begin{array}{l} 1: \text{ Euler's method equations or} \\ \text{equivalent table applied to (at least)} \\ \text{two iterations} \\ 1: \text{ Euler approximation to } f(0.2) \\ \text{(not eligible without first point)} \end{array} \right.$$

$$2 \left\{ \begin{array}{l} 1: \text{ uses } \frac{d}{dx}(2x + b) = 2 \text{ in DE} \\ 1: b = 1 \end{array} \right.$$

$$3 \left\{ \begin{array}{l} 1: g'(0) = 0 \\ 1: \text{ shows } g''(0) = -4 \\ 1: \text{ conclusion} \end{array} \right.$$

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2001 SCORING GUIDELINES**

Question 5

Let f be the function satisfying $f'(x) = -3xf(x)$, for all real numbers x , with $f(1) = 4$ and $\lim_{x \rightarrow \infty} f(x) = 0$.

- (a) Evaluate $\int_1^{\infty} -3xf(x) dx$. Show the work that leads to your answer.
- (b) Use Euler's method, starting at $x = 1$ with a step size of 0.5, to approximate $f(2)$.
- (c) Write an expression for $y = f(x)$ by solving the differential equation $\frac{dy}{dx} = -3xy$ with the initial condition $f(1) = 4$.

(a)
$$\int_1^{\infty} -3xf(x) dx$$

$$= \int_1^{\infty} f'(x) dx = \lim_{b \rightarrow \infty} \int_1^b f'(x) dx = \lim_{b \rightarrow \infty} f(x) \Big|_1^b$$

$$= \lim_{b \rightarrow \infty} f(b) - f(1) = 0 - 4 = -4$$

2 : $\left\{ \begin{array}{l} 1 : \text{use of FTC} \\ 1 : \text{answer from limiting process} \end{array} \right.$

(b)
$$f(1.5) \approx f(1) + f'(1)(0.5)$$

$$= 4 - 3(1)(4)(0.5) = -2$$

$$f(2) \approx -2 + f'(1.5)(0.5)$$

$$\approx -2 - 3(1.5)(-2)(0.5) = 2.5$$

2 : $\left\{ \begin{array}{l} 1 : \text{Euler's method equations or} \\ \quad \text{equivalent table} \\ 1 : \text{Euler approximation to } f(2) \\ \quad \text{(not eligible without first point)} \end{array} \right.$

(c)
$$\frac{1}{y} dy = -3x dx$$

$$\ln y = -\frac{3}{2}x^2 + k$$

$$y = Ce^{-\frac{3}{2}x^2}$$

$$4 = Ce^{-\frac{3}{2}} ; C = 4e^{\frac{3}{2}}$$

$$y = 4e^{\frac{3}{2}} e^{-\frac{3}{2}x^2}$$

5 : $\left\{ \begin{array}{l} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } f(1) = 4 \\ 1 : \text{solves for } y \end{array} \right.$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables