Riemann

AP® CALCULUS BC
2012 SCORING GUIDELINES

Question 1

t (minutes)	0	4	9	15	20
$W(t)$ (degrees Fahrenheit)	55.0	57.1	61.8	67.9	71.0

The temperature of water in a tub at time t is modeled by a strictly increasing, twice-differentiable function W, where $W(t)$ is measured in degrees Fahrenheit and t is measured in minutes. At time $t = 0$, the temperature of the water is 55°F. The water is heated for 30 minutes, beginning at time $t = 0$. Values of $W(t)$ at selected times t for the first 20 minutes are given in the table above.

(a) Use the data in the table to estimate $W'(12)$. Show the computations that lead to your answer. Using correct units, interpret the meaning of your answer in the context of this problem.

(b) Use the data in the table to evaluate $\int_0^{20} W'(t)\,dt$. Using correct units, interpret the meaning of $\int_0^{20} W'(t)\,dt$ in the context of this problem.

(c) For $0 \le t \le 20$, the average temperature of the water in the tub is $\frac{1}{20}\int_0^{20} W(t)\,dt$. Use a left Riemann sum with the four subintervals indicated by the data in the table to approximate $\frac{1}{20}\int_0^{20} W(t)\,dt$. Does this approximation overestimate or underestimate the average temperature of the water over these 20 minutes? Explain your reasoning.

(d) For $20 \le t \le 25$, the function W that models the water temperature has first derivative given by $W'(t) = 0.4\sqrt{t}\cos(0.06t)$. Based on the model, what is the temperature of the water at time $t = 25$?

(a) $W'(12) \approx \dfrac{W(15) - W(9)}{15 - 9} = \dfrac{67.9 - 61.8}{6}$

$= 1.017$ (or 1.016)

The water temperature is increasing at a rate of approximately 1.017 °F per minute at time $t = 12$ minutes.

$2: \begin{cases} 1: \text{estimate} \\ 1: \text{interpretation with units} \end{cases}$

(b) $\int_0^{20} W'(t)\,dt = W(20) - W(0) = 71.0 - 55.0 = 16$

The water has warmed by 16 °F over the interval from $t = 0$ to $t = 20$ minutes.

$2: \begin{cases} 1: \text{value} \\ 1: \text{interpretation with units} \end{cases}$

(c) $\frac{1}{20}\int_0^{20} W(t)\,dt \approx \frac{1}{20}(4 \cdot W(0) + 5 \cdot W(4) + 6 \cdot W(9) + 5 \cdot W(15))$

$= \frac{1}{20}(4 \cdot 55.0 + 5 \cdot 57.1 + 6 \cdot 61.8 + 5 \cdot 67.9)$

$= \frac{1}{20} \cdot 1215.8 = 60.79$

This approximation is an underestimate, because a left Riemann sum is used and the function W is strictly increasing.

$3: \begin{cases} 1: \text{left Riemann sum} \\ 1: \text{approximation} \\ 1: \text{underestimate with reason} \end{cases}$

(d) $W(25) = 71.0 + \int_{20}^{25} W'(t)\,dt$

$= 71.0 + 2.043155 = 73.043$

$2: \begin{cases} 1: \text{integral} \\ 1: \text{answer} \end{cases}$

Question 4

x	1	1.1	1.2	1.3	1.4
$f'(x)$	8	10	12	13	14.5

The function f is twice differentiable for $x > 0$ with $f(1) = 15$ and $f''(1) = 20$. Values of f', the derivative of f, are given for selected values of x in the table above.

(a) Write an equation for the line tangent to the graph of f at $x = 1$. Use this line to approximate $f(1.4)$.

(b) Use a midpoint Riemann sum with two subintervals of equal length and values from the table to approximate $\int_1^{1.4} f'(x)\,dx$. Use the approximation for $\int_1^{1.4} f'(x)\,dx$ to estimate the value of $f(1.4)$. Show the computations that lead to your answer.

(c) Use Euler's method, starting at $x = 1$ with two steps of equal size, to approximate $f(1.4)$. Show the computations that lead to your answer.

(d) Write the second-degree Taylor polynomial for f about $x = 1$. Use the Taylor polynomial to approximate $f(1.4)$.

(a) $f(1) = 15,\ f'(1) = 8$

An equation for the tangent line is
$y = 15 + 8(x - 1)$.

$f(1.4) \approx 15 + 8(1.4 - 1) = 18.2$

$2 : \begin{cases} 1 : \text{tangent line} \\ 1 : \text{approximation} \end{cases}$

(b) $\int_1^{1.4} f'(x)\,dx \approx (0.2)(10) + (0.2)(13) = 4.6$

$f(1.4) = f(1) + \int_1^{1.4} f'(x)\,dx$

$f(1.4) \approx 15 + 4.6 = 19.6$

$3 : \begin{cases} 1 : \text{midpoint Riemann sum} \\ 1 : \text{Fundamental Theorem of Calculus} \\ 1 : \text{answer} \end{cases}$

(c) $f(1.2) \approx f(1) + (0.2)(8) = 16.6$

$f(1.4) \approx 16.6 + (0.2)(12) = 19.0$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{answer} \end{cases}$

(d) $T_2(x) = 15 + 8(x - 1) + \frac{20}{2!}(x - 1)^2$
$\quad = 15 + 8(x - 1) + 10(x - 1)^2$

$f(1.4) \approx 15 + 8(1.4 - 1) + 10(1.4 - 1)^2 = 19.8$

$2 : \begin{cases} 1 : \text{Taylor polynomial} \\ 1 : \text{approximation} \end{cases}$

Question 2

t (minutes)	0	2	5	9	10
$H(t)$ (degrees Celsius)	66	60	52	44	43

As a pot of tea cools, the temperature of the tea is modeled by a differentiable function H for $0 \le t \le 10$, where time t is measured in minutes and temperature $H(t)$ is measured in degrees Celsius. Values of $H(t)$ at selected values of time t are shown in the table above.

(a) Use the data in the table to approximate the rate at which the temperature of the tea is changing at time $t = 3.5$. Show the computations that lead to your answer.

(b) Using correct units, explain the meaning of $\frac{1}{10}\int_0^{10} H(t)\,dt$ in the context of this problem. Use a trapezoidal sum with the four subintervals indicated by the table to estimate $\frac{1}{10}\int_0^{10} H(t)\,dt$.

(c) Evaluate $\int_0^{10} H'(t)\,dt$. Using correct units, explain the meaning of the expression in the context of this problem.

(d) At time $t = 0$, biscuits with temperature 100°C were removed from an oven. The temperature of the biscuits at time t is modeled by a differentiable function B for which it is known that $B'(t) = -13.84e^{-0.173t}$. Using the given models, at time $t = 10$, how much cooler are the biscuits than the tea?

(a) $H'(3.5) \approx \frac{H(5) - H(2)}{5 - 2}$

$= \frac{52 - 60}{3} = -2.666$ or -2.667 degrees Celsius per minute

1 : answer

(b) $\frac{1}{10}\int_0^{10} H(t)\,dt$ is the average temperature of the tea, in degrees Celsius, over the 10 minutes.

$$\frac{1}{10}\int_0^{10} H(t)\,dt \approx \frac{1}{10}\left(2 \cdot \frac{66+60}{2} + 3 \cdot \frac{60+52}{2} + 4 \cdot \frac{52+44}{2} + 1 \cdot \frac{44+43}{2}\right)$$
$$= 52.95$$

3 : { 1 : meaning of expression; 1 : trapezoidal sum; 1 : estimate }

(c) $\int_0^{10} H'(t)\,dt = H(10) - H(0) = 43 - 66 = -23$

The temperature of the tea drops 23 degrees Celsius from time $t = 0$ to time $t = 10$ minutes.

2 : { 1 : value of integral; 1 : meaning of expression }

(d) $B(10) = 100 + \int_0^{10} B'(t)\,dt = 34.18275;\quad H(10) - B(10) = 8.817$

The biscuits are 8.817 degrees Celsius cooler than the tea.

3 : { 1 : integrand; 1 : uses $B(0) = 100$; 1 : answer }

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Question 2

t (hours)	0	2	5	7	8
$E(t)$ (hundreds of entries)	0	4	13	21	23

A zoo sponsored a one-day contest to name a new baby elephant. Zoo visitors deposited entries in a special box between noon $(t = 0)$ and 8 P.M. $(t = 8)$. The number of entries in the box t hours after noon is modeled by a differentiable function E for $0 \le t \le 8$. Values of $E(t)$, in hundreds of entries, at various times t are shown in the table above.

(a) Use the data in the table to approximate the rate, in hundreds of entries per hour, at which entries were being deposited at time $t = 6$. Show the computations that lead to your answer.

(b) Use a trapezoidal sum with the four subintervals given by the table to approximate the value of $\frac{1}{8}\int_0^8 E(t)\,dt$. Using correct units, explain the meaning of $\frac{1}{8}\int_0^8 E(t)\,dt$ in terms of the number of entries.

(c) At 8 P.M., volunteers began to process the entries. They processed the entries at a rate modeled by the function P, where $P(t) = t^3 - 30t^2 + 298t - 976$ hundreds of entries per hour for $8 \le t \le 12$. According to the model, how many entries had not yet been processed by midnight $(t = 12)$?

(d) According to the model from part (c), at what time were the entries being processed most quickly? Justify your answer.

(a) $E'(6) \approx \dfrac{E(7) - E(5)}{7 - 5} = 4$ hundred entries per hour

1 : answer

(b) $\frac{1}{8}\int_0^8 E(t)\,dt \approx$

$\frac{1}{8}\left(2\cdot\frac{E(0)+E(2)}{2} + 3\cdot\frac{E(2)+E(5)}{2} + 2\cdot\frac{E(5)+E(7)}{2} + 1\cdot\frac{E(7)+E(8)}{2}\right)$

$= 10.687$ or 10.688

$\frac{1}{8}\int_0^8 E(t)\,dt$ is the average number of hundreds of entries in the box between noon and 8 P.M.

3 : { 1 : trapezoidal sum; 1 : approximation; 1 : meaning }

(c) $23 - \int_8^{12} P(t)\,dt = 23 - 16 = 7$ hundred entries

2 : { 1 : integral; 1 : answer }

(d) $P'(t) = 0$ when $t = 9.183503$ and $t = 10.816497$.

t	$P(t)$
8	0
9.183503	5.088662
10.816497	2.911338
12	8

Entries are being processed most quickly at time $t = 12$.

3 : { 1 : considers $P'(t) = 0$; 1 : identifies candidates; 1 : answer with justification }

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Question 5

x	2	3	5	8	13
$f(x)$	1	4	-2	3	6

Let f be a function that is twice differentiable for all real numbers. The table above gives values of f for selected points in the closed interval $2 \le x \le 13$.

(a) Estimate $f'(4)$. Show the work that leads to your answer.

(b) Evaluate $\int_2^{13} (3 - 5f'(x))\,dx$. Show the work that leads to your answer.

(c) Use a left Riemann sum with subintervals indicated by the data in the table to approximate $\int_2^{13} f(x)\,dx$. Show the work that leads to your answer.

(d) Suppose $f'(5) = 3$ and $f''(x) < 0$ for all x in the closed interval $5 \le x \le 8$. Use the line tangent to the graph of f at $x = 5$ to show that $f(7) \le 4$. Use the secant line for the graph of f on $5 \le x \le 8$ to show that $f(7) \ge \frac{4}{3}$.

(a) $f'(4) \approx \dfrac{f(5) - f(3)}{5 - 3} = -3$

1 : answer

(b) $\int_2^{13} (3 - 5f'(x))\,dx = \int_2^{13} 3\,dx - 5\int_2^{13} f'(x)\,dx$
$= 3(13 - 2) - 5(f(13) - f(2)) = 8$

2 : { 1 : uses Fundamental Theorem of Calculus; 1 : answer }

(c) $\int_2^{13} f(x)\,dx \approx f(2)(3 - 2) + f(3)(5 - 3) + f(5)(8 - 5) + f(8)(13 - 8) = 18$

2 : { 1 : left Riemann sum; 1 : answer }

(d) An equation for the tangent line is $y = -2 + 3(x - 5)$.
Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the line tangent to the graph of $y = f(x)$ at $x = 5$ lies above the graph for all x in the interval $5 < x \le 8$.
Therefore, $f(7) \le -2 + 3 \cdot 2 = 4$.

An equation for the secant line is $y = -2 + \frac{5}{3}(x - 5)$.
Since $f''(x) < 0$ for all x in the interval $5 \le x \le 8$, the secant line connecting $(5, f(5))$ and $(8, f(8))$ lies below the graph of $y = f(x)$ for all x in the interval $5 < x < 8$.
Therefore, $f(7) \ge -2 + \frac{5}{3} \cdot 2 = \frac{4}{3}$.

4 : { 1 : tangent line; 1 : shows $f(7) \le 4$; 1 : secant line; 1 : shows $f(7) \ge \frac{4}{3}$ }

Question 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon $(t = 0)$ and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \le t \le 9$. Values of $L(t)$ at various times t are shown in the table above.

(a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. $(t = 5.5)$. Show the computations that lead to your answer. Indicate units of measure.

(b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.

(c) For $0 \le t \le 9$, what is the fewest number of times at which $L'(t)$ must equal 0 ? Give a reason for your answer.

(d) The rate at which tickets were sold for $0 \le t \le 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. $(t = 3)$, to the nearest whole number?

(a) $L'(5.5) \approx \dfrac{L(7) - L(4)}{7 - 4} = \dfrac{150 - 126}{3} = 8$ people per hour

$2: \begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The average number of people waiting in line during the first 4 hours is approximately

$$\frac{1}{4}\left(\frac{L(0) + L(1)}{2}(1 - 0) + \frac{L(1) + L(3)}{2}(3 - 1) + \frac{L(3) + L(4)}{2}(4 - 3)\right)$$

$= 155.25$ people

$2: \begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{answer} \end{cases}$

(c) L is differentiable on $[0, 9]$ so the Mean Value Theorem implies $L'(t) > 0$ for some t in $(1, 3)$ and some t in $(4, 7)$. Similarly, $L'(t) < 0$ for some t in $(3, 4)$ and some t in $(7, 8)$. Then, since L' is continuous on $[0, 9]$, the Intermediate Value Theorem implies that $L'(t) = 0$ for at least three values of t in $[0, 9]$.

$3: \begin{cases} 1 : \text{considers change in sign of } L' \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

OR

The continuity of L on $[1, 4]$ implies that L attains a maximum value there. Since $L(3) > L(1)$ and $L(3) > L(4)$, this maximum occurs on $(1, 4)$. Similarly, L attains a minimum on $(3, 7)$ and a maximum on $(4, 8)$. L is differentiable, so $L'(t) = 0$ at each relative extreme point on $(0, 9)$. Therefore $L'(t) = 0$ for at least three values of t in $[0, 9]$.

OR

$3: \begin{cases} 1 : \text{considers relative extrema of } L \text{ on } (0, 9) \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

[*Note: There is a function L that satisfies the given conditions with $L'(t) = 0$ for exactly three values of t.*]

(d) $\int_0^3 r(t)\,dt = 972.784$

There were approximately 973 tickets sold by 3 P.M.

$2: \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and answer} \end{cases}$

Question 5

t (minutes)	0	2	5	7	11	12
$r'(t)$ (feet per minute)	5.7	4.0	2.0	1.2	0.6	0.5

The volume of a spherical hot air balloon expands as the air inside the balloon is heated. The radius of the balloon, in feet, is modeled by a twice-differentiable function r of time t, where t is measured in minutes. For $0 < t < 12$, the graph of r is concave down. The table above gives selected values of the rate of change, $r'(t)$, of the radius of the balloon over the time interval $0 \le t \le 12$. The radius of the balloon is 30 feet when $t = 5$. (Note: The volume of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.)

(a) Estimate the radius of the balloon when $t = 5.4$ using the tangent line approximation at $t = 5$. Is your estimate greater than or less than the true value? Give a reason for your answer.

(b) Find the rate of change of the volume of the balloon with respect to time when $t = 5$. Indicate units of measure.

(c) Use a right Riemann sum with the five subintervals indicated by the data in the table to approximate $\int_0^{12} r'(t)\,dt$. Using correct units, explain the meaning of $\int_0^{12} r'(t)\,dt$ in terms of the radius of the balloon.

(d) Is your approximation in part (c) greater than or less than $\int_0^{12} r'(t)\,dt$? Give a reason for your answer.

(a) $r(5.4) \approx r(5) + r'(5)\Delta t = 30 + 2(0.4) = 30.8$ ft Since the graph of r is concave down on the interval $5 < t < 5.4$, this estimate is greater than $r(5.4)$.	2 : { 1 : estimate 1 : conclusion with reason
(b) $\frac{dV}{dt} = 3\left(\frac{4}{3}\right)\pi r^2 \frac{dr}{dt}$ $\left.\frac{dV}{dt}\right\vert_{t=5} = 4\pi(30)^2 2 = 7200\pi$ ft^3/min	3 : { 2 : $\frac{dV}{dt}$ 1 : answer
(c) $\int_0^{12} r'(t)\,dt \approx 2(4.0) + 3(2.0) + 2(1.2) + 4(0.6) + 1(0.5)$ $= 19.3$ ft $\int_0^{12} r'(t)\,dt$ is the change in the radius, in feet, from $t = 0$ to $t = 12$ minutes.	2 : { 1 : approximation 1 : explanation
(d) Since r is concave down, r' is decreasing on $0 < t < 12$. Therefore, this approximation, 19.3 ft, is less than $\int_0^{12} r'(t)\,dt$.	1 : conclusion with reason
Units of ft^3/min in part (b) and ft in part (c)	1 : units in (b) and (c)

Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \le t \le 80$ seconds, as shown in the table above.

(a) Find the average acceleration of rocket A over the time interval $0 \le t \le 80$ seconds. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_{10}^{70} v(t)\,dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t)\,dt$.

(c) Rocket B is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

(a) Average acceleration of rocket A is

$$\frac{v(80) - v(0)}{80 - 0} = \frac{49 - 5}{80} = \frac{11}{20} \text{ ft/sec}^2$$

1 : answer

(b) Since the velocity is positive, $\int_{10}^{70} v(t)\,dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds.

A midpoint Riemann sum is

$$20[v(20) + v(40) + v(60)]$$
$$= 20[22 + 35 + 44] = 2020 \text{ ft}$$

3 :
- 1 : explanation
- 1 : uses $v(20)$, $v(40)$, $v(60)$
- 1 : value

(c) Let $v_B(t)$ be the velocity of rocket B at time t.

$$v_B(t) = \int \frac{3}{\sqrt{t+1}}\,dt = 6\sqrt{t+1} + C$$
$$2 = v_B(0) = 6 + C$$
$$v_B(t) = 6\sqrt{t+1} - 4$$
$$v_B(80) = 50 > 49 = v(80)$$

Rocket B is traveling faster at time $t = 80$ seconds.

4 :
- 1 : $6\sqrt{t+1}$
- 1 : constant of integration
- 1 : uses initial condition
- 1 : finds $v_B(80)$, compares to $v(80)$, and draws a conclusion

Units of ft/sec^2 in (a) and ft in (b)

1 : units in (a) and (b)