

3. The rate at which water flows out of a pipe, in gallons per hour, is given by a differentiable function R of time t . The table above shows the rate as measured every 3 hours for a 24-hour period.

t (hours)	$R(t)$ (gallons per hour)
0	9.6
3	10.4
6	10.8
9	11.2
12	11.4
15	11.3
18	10.7
21	10.2
24	9.6

- (a) Use a midpoint Riemann sum with 4 subdivisions of equal length to approximate $\int_0^{24} R(t) dt$. Using correct units, explain the meaning of your answer in terms of water flow.
- (b) Is there some time t , $0 < t < 24$, such that $R'(t) = 0$? Justify your answer.
- (c) The rate of water flow $R(t)$ can be approximated by $Q(t) = \frac{1}{79}(768 + 23t - t^2)$. Use $Q(t)$ to approximate the average rate of water flow during the 24-hour time period. Indicate units of measure.

(a)
$$\int_0^{24} R(t) dt \approx 6[R(3) + R(9) + R(15) + R(21)]$$

$$= 6[10.4 + 11.2 + 11.3 + 10.2]$$

$$= 258.6 \text{ gallons}$$

This is an approximation to the total flow in gallons of water from the pipe in the 24-hour period.

$$\mathbf{3} \begin{cases} 1: R(3) + R(9) + R(15) + R(21) \\ 1: \text{answer} \\ 1: \text{explanation} \end{cases}$$

- (b) Yes;
 Since $R(0) = R(24) = 9.6$, the Mean Value Theorem guarantees that there is a t , $0 < t < 24$, such that $R'(t) = 0$.

$$\mathbf{2} \begin{cases} 1: \text{answer} \\ 1: \text{MVT or equivalent} \end{cases}$$

(c) Average rate of flow
 \approx average value of $Q(t)$

$$= \frac{1}{24} \int_0^{24} \frac{1}{79}(768 + 23t - t^2) dt$$

$$= 10.785 \text{ gal/hr or } 10.784 \text{ gal/hr}$$

$$\mathbf{3} \begin{cases} 1: \text{limits and average value constant} \\ 1: Q(t) \text{ as integrand} \\ 1: \text{answer} \end{cases}$$

(units) Gallons in part (a) and gallons/hr in part (c), or equivalent.

1: units

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Question 2

The temperature, in degrees Celsius ($^{\circ}\text{C}$), of the water in a pond is a differentiable function W of time t . The table above shows the water temperature as recorded every 3 days over a 15-day period.

t (days)	$W(t)$ ($^{\circ}\text{C}$)
0	20
3	31
6	28
9	24
12	22
15	21

- (a) Use data from the table to find an approximation for $W'(12)$. Show the computations that lead to your answer. Indicate units of measure.
- (b) Approximate the average temperature, in degrees Celsius, of the water over the time interval $0 \leq t \leq 15$ days by using a trapezoidal approximation with subintervals of length $\Delta t = 3$ days.
- (c) A student proposes the function P , given by $P(t) = 20 + 10te^{(-t/3)}$, as a model for the temperature of the water in the pond at time t , where t is measured in days and $P(t)$ is measured in degrees Celsius. Find $P'(12)$. Using appropriate units, explain the meaning of your answer in terms of water temperature.
- (d) Use the function P defined in part (c) to find the average value, in degrees Celsius, of $P(t)$ over the time interval $0 \leq t \leq 15$ days.

- (a) Difference quotient; e.g.

$$W'(12) \approx \frac{W(15) - W(12)}{15 - 12} = -\frac{1}{3} \text{ }^{\circ}\text{C/day or}$$

$$W'(12) \approx \frac{W(12) - W(9)}{12 - 9} = -\frac{2}{3} \text{ }^{\circ}\text{C/day or}$$

$$W'(12) \approx \frac{W(15) - W(9)}{15 - 9} = -\frac{1}{2} \text{ }^{\circ}\text{C/day}$$

- (b) $\frac{3}{2}(20 + 2(31) + 2(28) + 2(24) + 2(22) + 21) = 376.5$

$$\text{Average temperature} \approx \frac{1}{15}(376.5) = 25.1 \text{ }^{\circ}\text{C}$$

- (c) $P'(12) = 10e^{-t/3} - \frac{10}{3}te^{-t/3} \Big|_{t=12}$
 $= -30e^{-4} = -0.549 \text{ }^{\circ}\text{C/day}$

This means that the temperature is decreasing at the rate of $0.549 \text{ }^{\circ}\text{C/day}$ when $t = 12$ days.

- (d) $\frac{1}{15} \int_0^{15} (20 + 10te^{-t/3}) dt = 25.757 \text{ }^{\circ}\text{C}$

$$2 : \begin{cases} 1 : \text{difference quotient} \\ 1 : \text{answer (with units)} \end{cases}$$

$$2 : \begin{cases} 1 : \text{trapezoidal method} \\ 1 : \text{answer} \end{cases}$$

$$2 : \begin{cases} 1 : P'(12) \text{ (with or without units)} \\ 1 : \text{interpretation} \end{cases}$$

$$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and} \\ \quad \text{average value constant} \\ 1 : \text{answer} \end{cases}$$

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2003 SCORING GUIDELINES (Form B)

Question 3

A blood vessel is 360 millimeters (mm) long with circular cross sections of varying diameter. The table above gives the measurements of the diameter of the blood vessel at selected points

Distance x (mm)	0	60	120	180	240	300	360
Diameter $B(x)$ (mm)	24	30	28	30	26	24	26

along the length of the blood vessel, where x represents the distance from one end of the blood vessel and $B(x)$ is a twice-differentiable function that represents the diameter at that point.

- (a) Write an integral expression in terms of $B(x)$ that represents the average radius, in mm, of the blood vessel between $x = 0$ and $x = 360$.
- (b) Approximate the value of your answer from part (a) using the data from the table and a midpoint Riemann sum with three subintervals of equal length. Show the computations that lead to your answer.
- (c) Using correct units, explain the meaning of $\pi \int_{125}^{275} \left(\frac{B(x)}{2}\right)^2 dx$ in terms of the blood vessel.
- (d) Explain why there must be at least one value x , for $0 < x < 360$, such that $B''(x) = 0$.

(a) $\frac{1}{360} \int_0^{360} \frac{B(x)}{2} dx$

2 : $\left\{ \begin{array}{l} 1 : \text{limits and constant} \\ 1 : \text{integrand} \end{array} \right.$

(b) $\frac{1}{360} \left[120 \left(\frac{B(60)}{2} + \frac{B(180)}{2} + \frac{B(300)}{2} \right) \right] =$
 $\frac{1}{360} [60(30 + 30 + 24)] = 14$

2 : $\left\{ \begin{array}{l} 1 : B(60) + B(180) + B(300) \\ 1 : \text{answer} \end{array} \right.$

(c) $\frac{B(x)}{2}$ is the radius, so $\pi \left(\frac{B(x)}{2}\right)^2$ is the area of the cross section at x . The expression is the volume in mm^3 of the blood vessel between 125 mm and 275 mm from the end of the vessel.

2 : $\left\{ \begin{array}{l} 1 : \text{volume in } \text{mm}^3 \\ 1 : \text{between } x = 125 \text{ and} \\ \quad x = 275 \end{array} \right.$

(d) By the MVT, $B'(c_1) = 0$ for some c_1 in (60, 180) and $B'(c_2) = 0$ for some c_2 in (240, 360). The MVT applied to $B'(x)$ shows that $B''(x) = 0$ for some x in the interval (c_1, c_2) .

3 : $\left\{ \begin{array}{l} 2 : \text{explains why there are two} \\ \quad \text{values of } x \text{ where } B'(x) \text{ has} \\ \quad \text{the same value} \\ 1 : \text{explains why that means} \\ \quad B''(x) = 0 \text{ for } 0 < x < 360 \end{array} \right.$

Note: max 1/3 if only explains why $B'(x) = 0$ at some x in $(0, 360)$.

Question 3

A test plane flies in a straight line with positive velocity $v(t)$, in miles per minute at time t minutes, where v is a differentiable function of t . Selected values of $v(t)$ for $0 \leq t \leq 40$ are shown in the table above.

t (min)	0	5	10	15	20	25	30	35	40
$v(t)$ (mpm)	7.0	9.2	9.5	7.0	4.5	2.4	2.4	4.3	7.3

- (a) Use a midpoint Riemann sum with four subintervals of equal length and values from the table to approximate $\int_0^{40} v(t) dt$. Show the computations that lead to your answer. Using correct units, explain the meaning of $\int_0^{40} v(t) dt$ in terms of the plane's flight.
- (b) Based on the values in the table, what is the smallest number of instances at which the acceleration of the plane could equal zero on the open interval $0 < t < 40$? Justify your answer.
- (c) The function f , defined by $f(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$, is used to model the velocity of the plane, in miles per minute, for $0 \leq t \leq 40$. According to this model, what is the acceleration of the plane at $t = 23$? Indicate units of measure.
- (d) According to the model f , given in part (c), what is the average velocity of the plane, in miles per minute, over the time interval $0 \leq t \leq 40$?

- (a) Midpoint Riemann sum is
 $10 \cdot [v(5) + v(15) + v(25) + v(35)]$
 $= 10 \cdot [9.2 + 7.0 + 2.4 + 4.3] = 229$

The integral gives the total distance in miles that the plane flies during the 40 minutes.

$$3 : \begin{cases} 1 : v(5) + v(15) + v(25) + v(35) \\ 1 : \text{answer} \\ 1 : \text{meaning with units} \end{cases}$$

- (b) By the Mean Value Theorem, $v'(t) = 0$ somewhere in the interval $(0, 15)$ and somewhere in the interval $(25, 30)$. Therefore the acceleration will equal 0 for at least two values of t .

$$2 : \begin{cases} 1 : \text{two instances} \\ 1 : \text{justification} \end{cases}$$

- (c) $f'(23) = -0.407$ or -0.408 miles per minute²

$$1 : \text{answer with units}$$

- (d) Average velocity $= \frac{1}{40} \int_0^{40} f(t) dt$
 $= 5.916$ miles per minute

$$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$$

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Question 4

t (seconds)	0	10	20	30	40	50	60	70	80
$v(t)$ (feet per second)	5	14	22	29	35	40	44	47	49

Rocket A has positive velocity $v(t)$ after being launched upward from an initial height of 0 feet at time $t = 0$ seconds. The velocity of the rocket is recorded for selected values of t over the interval $0 \leq t \leq 80$ seconds, as shown in the table above.

- (a) Find the average acceleration of rocket A over the time interval $0 \leq t \leq 80$ seconds. Indicate units of measure.
- (b) Using correct units, explain the meaning of $\int_{10}^{70} v(t) dt$ in terms of the rocket's flight. Use a midpoint Riemann sum with 3 subintervals of equal length to approximate $\int_{10}^{70} v(t) dt$.
- (c) Rocket B is launched upward with an acceleration of $a(t) = \frac{3}{\sqrt{t+1}}$ feet per second per second. At time $t = 0$ seconds, the initial height of the rocket is 0 feet, and the initial velocity is 2 feet per second. Which of the two rockets is traveling faster at time $t = 80$ seconds? Explain your answer.

- (a) Average acceleration of rocket A is

$$\frac{v(80) - v(0)}{80 - 0} = \frac{49 - 5}{80} = \frac{11}{20} \text{ ft/sec}^2$$

- (b) Since the velocity is positive, $\int_{10}^{70} v(t) dt$ represents the distance, in feet, traveled by rocket A from $t = 10$ seconds to $t = 70$ seconds.

A midpoint Riemann sum is

$$20[v(20) + v(40) + v(60)]$$

$$= 20[22 + 35 + 44] = 2020 \text{ ft}$$

- (c) Let $v_B(t)$ be the velocity of rocket B at time t .

$$v_B(t) = \int \frac{3}{\sqrt{t+1}} dt = 6\sqrt{t+1} + C$$

$$2 = v_B(0) = 6 + C$$

$$v_B(t) = 6\sqrt{t+1} - 4$$

$$v_B(80) = 50 > 49 = v(80)$$

Rocket B is traveling faster at time $t = 80$ seconds.

Units of ft/sec^2 in (a) and ft in (b)

1 : answer

3 : $\left\{ \begin{array}{l} 1 : \text{explanation} \\ 1 : \text{uses } v(20), v(40), v(60) \\ 1 : \text{value} \end{array} \right.$

4 : $\left\{ \begin{array}{l} 1 : 6\sqrt{t+1} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{finds } v_B(80), \text{ compares to } v(80), \\ \text{and draws a conclusion} \end{array} \right.$

1 : units in (a) and (b)

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Question 2

t (hours)	0	1	3	4	7	8	9
$L(t)$ (people)	120	156	176	126	150	80	0

Concert tickets went on sale at noon ($t = 0$) and were sold out within 9 hours. The number of people waiting in line to purchase tickets at time t is modeled by a twice-differentiable function L for $0 \leq t \leq 9$. Values of $L(t)$ at various times t are shown in the table above.

- (a) Use the data in the table to estimate the rate at which the number of people waiting in line was changing at 5:30 P.M. ($t = 5.5$). Show the computations that lead to your answer. Indicate units of measure.
- (b) Use a trapezoidal sum with three subintervals to estimate the average number of people waiting in line during the first 4 hours that tickets were on sale.
- (c) For $0 \leq t \leq 9$, what is the fewest number of times at which $L'(t)$ must equal 0? Give a reason for your answer.
- (d) The rate at which tickets were sold for $0 \leq t \leq 9$ is modeled by $r(t) = 550te^{-t/2}$ tickets per hour. Based on the model, how many tickets were sold by 3 P.M. ($t = 3$), to the nearest whole number?

(a) $L'(5.5) \approx \frac{L(7) - L(4)}{7 - 4} = \frac{150 - 126}{3} = 8$ people per hour

2 : $\begin{cases} 1 : \text{estimate} \\ 1 : \text{units} \end{cases}$

(b) The average number of people waiting in line during the first 4 hours is approximately

$$\frac{1}{4} \left(\frac{L(0) + L(1)}{2} (1 - 0) + \frac{L(1) + L(3)}{2} (3 - 1) + \frac{L(3) + L(4)}{2} (4 - 3) \right) = 155.25 \text{ people}$$

2 : $\begin{cases} 1 : \text{trapezoidal sum} \\ 1 : \text{answer} \end{cases}$

(c) L is differentiable on $[0, 9]$ so the Mean Value Theorem implies $L'(t) > 0$ for some t in $(1, 3)$ and some t in $(4, 7)$. Similarly, $L'(t) < 0$ for some t in $(3, 4)$ and some t in $(7, 8)$. Then, since L' is continuous on $[0, 9]$, the Intermediate Value Theorem implies that $L'(t) = 0$ for at least three values of t in $[0, 9]$.

3 : $\begin{cases} 1 : \text{considers change in sign of } L' \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

OR

The continuity of L on $[1, 4]$ implies that L attains a maximum value there. Since $L(3) > L(1)$ and $L(3) > L(4)$, this maximum occurs on $(1, 4)$. Similarly, L attains a minimum on $(3, 7)$ and a maximum on $(4, 8)$. L is differentiable, so $L'(t) = 0$ at each relative extreme point on $(0, 9)$. Therefore $L'(t) = 0$ for at least three values of t in $[0, 9]$.

OR

3 : $\begin{cases} 1 : \text{considers relative extrema of } L \text{ on } (0, 9) \\ 1 : \text{analysis} \\ 1 : \text{conclusion} \end{cases}$

[Note: There is a function L that satisfies the given conditions with $L'(t) = 0$ for exactly three values of t .]

(d) $\int_0^3 r(t) dt = 972.784$

There were approximately 973 tickets sold by 3 P.M.

2 : $\begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and answer} \end{cases}$