Free-Response Questions Scoring Guidelines

Scoring Guidelines for Question 1: Function Concepts
Part A: Graphing calculator required — 6 points

x	1	2	3	4	5
$f(x)$	1	3	5	3	1

The domain of f consists of the five real numbers 1, 2, 3, 4, and 5. The table defines the function f for these values. The function g is given by $g(x)=2\ln x$.

Model Solution	Scoring	
(A) (i) The function h is defined by $h(x)=(g\circ f)(x)=g(f(x))$. Find the value of $h(4)$ as a decimal approximation, or indicate that it is not defined. (ii) Find all values of x for which $f(x)=3$, or indicate there are no such values.		
(i) $h(4)=g\left(f(4)\right)=g(3)=2\ln 3=2.197$	Value	**1 point** 2.A
(ii) From the table, $f(x)=3$ when $x=2$ and $x=4$.	Values	**1 point** 2.A
	Total for part (A)	**2 points**
(B) (i) Find all values of x, as decimal approximations, for which $g(x)=3$, or indicate there are no such values. (ii) Determine the end behavior of g as x increases without bound. Express your answer using the mathematical notation of a limit.		
(i) $g(x)=3 \Rightarrow 2\ln x=3$ $x=4.482$ (OR 4.481)	Answer	**1 point** 1.A
(ii) The function g is increasing. As x increases without bound, $g(x)$ increases without bound. Therefore, $\lim_{x\to\infty} g(x)=\infty$.	End behavior with limit notation	**1 point** 3.A
	Total for part (B)	**2 points**
(C) (i) Determine if f has an inverse function. (ii) Give a reason for your answer based on the definition of a function and the table of values of $f(x)$.		
(i) f does not have an inverse function on its domain of the five real numbers 1, 2, 3, 4, and 5.	Answer	**1 point** 1.C
(ii) There are output values of f that are not mapped from unique input values. (The function f is not one-to-one.) Because $f(1)=1$ and $f(5)=1$, the inverse function on this domain does not exist. A reason that only states "fails the horizontal line test" is not sufficient.	Reason	**1 point** 3.C
	Total for part (C)	**2 points**
	Total for Question 1	**6 points**

Scoring Guidelines for Question 2: Modeling a Non-Periodic Context
Part A: Graphing calculator required — 6 points

An ecologist began studying a certain type of plant species in a wetlands area in 2013. In 2015 ($t = 2$), there were 59 plants. In 2021 ($t = 8$), there were 118 plants.

The number of plants in this species can be modeled by the function P given by $P(t) = ab^t$, where $P(t)$ is the number of plants during year t, and t is the number of years since 2013.

Model Solution	Scoring	
(A) (i) Use the given data to write two equations that can be used to find the values for constants a and b in the expression for $P(t)$. (ii) Find the values for a and b as decimal approximations.		
(i) Because $P(2) = 59$ and $P(8) = 118$, two equations to find a and b are $ab^2 = 59$ $ab^8 = 118.$	Two equations	**1 point** 1.C
(ii) $a = \frac{59}{b^2} \Rightarrow \left(\frac{59}{b^2}\right)b^8 = 118$ $b = \left(\frac{118}{59}\right)^{1/6} = 1.122462$ $a = 46.828331$ $P(t) = 46.828(1.122)^t$	Values of a and b	**1 point** 1.C
	Total for part (A)	**2 points**
(B) (i) Use the given data to find the average rate of change of the number of plants, in plants per year, from $t = 2$ to $t = 8$ years. Express your answer as a decimal approximation. Show the computations that lead to your answer. (ii) Use the average rate of change found in (i) to estimate the number of plants for $t = 10$ years. Show the work that leads to your answer. (iii) The average rate of change found in (i) can be used to estimate the number of plants during year t for $t > 10$ years. Will these estimates, found using the average rate of change, be less than or greater than the number of plants predicted by the model P during year t for $t > 10$ years? Explain your reasoning.		
(i) $\frac{P(8) - P(2)}{8 - 2} = \frac{(118 - 59)}{6} = 9.833327$ The average rate of change is 9.833 plants per year.	Average rate of change	**1 point** 1.B
(ii) The average rate of change is $r = \frac{P(8) - P(2)}{8 - 2} = 9.833327.$ The secant line between point (2, P(2)) and point	Estimate using average rate of change	**1 point** 3.B

$(8, P(8))$ is given by $y = y_1 + \left(\frac{P(8)-P(2)}{8-2}\right)(x - x_1)$, where (x_1, y_1) can be either one of the points. Estimates using the average rate of change are given by $y = P(2) + r(x-2)$ OR $y = P(8) + r(x-8)$. Both of these produce the same estimate. For $x = 10$, $y = 59 + r(10-2) = 137.667$. The number of plants for $t = 10$ years was approximately 137 or 138.		
(iii) The estimate using the average rate of change is the y-coordinate of a point on the secant line that passes through $(2, P(2))$ and $(8, P(8))$. Because the graph of P is concave up on the interval $(-\infty, \infty)$, the secant line is below the graph of P outside of the interval $(2, 8)$. Therefore, the estimate using the average rate of change is less than the value of $P(t)$ for $t > 10$.	Answer with explanation	**1 point** 3.C

Total for part (B) **3 points**

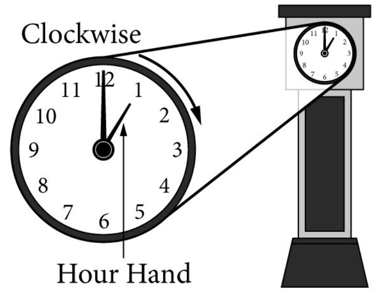
(C) For which t-value, $t = 6$ years or $t = 20$ years, should the ecologist have more confidence in when using the model P? Give a reason for your answer in the context of the problem.

The ecologist should have more confidence in using the model for $t = 6$ years. There is insufficient information to know how many years the exponential model can be extended above the maximum time provided in the data ($t = 8$) to make reasonable predictions. On the other hand, it is appropriate to use the regression model to estimate values at times that fall between the minimum time ($t = 2$) and the maximum time ($t = 8$) provided in the data.	Answer with reason	**1 point** 3.C

Total for part (C) **1 point**

Total for Question 2 **6 points**

Scoring Guidelines for Question 3: Modeling a Periodic Context
Part B: Graphing calculator not allowed — 6 points



Note: Figure not drawn to scale.

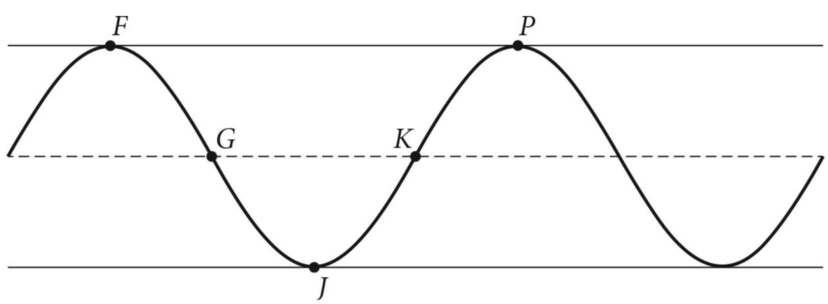
The figure shows a clock standing on a level floor with a close-up view of the clock face. The clock face has a 10-centimeter-long moving hour hand. The center of the clock face is 200 centimeters from the floor. At time $t = 0$ hours, the hour hand is pointing directly up to the 12. The next time the hour hand points directly up to the 12 is at time $t = 12$ hours. As the hour hand moves, the distance of the endpoint of the hour hand from the floor periodically decreases and increases.

The sinusoidal function h models the distance, in centimeters, of the endpoint of the hour hand from the floor as a function of time t in hours.

Model Solution	Scoring

(A) The graph of h and its dashed midline for two full cycles is shown. Five points, F, G, J, K, and P, are labeled on the graph. No scale is indicated, and no axes are presented.

Determine possible coordinates $(t, h(t))$ for the five points: F, G, J, K, and P.



Model Solution	Scoring	Points
F has coordinates $(0, 210)$. G has coordinates $(3, 200)$. J has coordinates $(6, 190)$. K has coordinates $(9, 200)$. P has coordinates $(12, 210)$. Note: t-coordinates will vary. A correct set of coordinates for one full cycle of h as pictured is acceptable.	$h(t)$-coordinates	**1 point** 2.B
	t-coordinates	**1 point** 2.B

Total for part (A) **2 points**

(B) The function h can be written in the form $h(t) = a\sin(b(t+c)) + d$. Find values of constants a, b, c, and d.

$h(t) = a\sin(b(t+c)) + d$ $a = 10$ $\frac{2\pi}{b} = 12$, so $b = \frac{2\pi}{12} = \frac{\pi}{6}$ $c = -9$ $d = 200$ $h(t) = 10\sin\left(\frac{\pi}{6}(t-9)\right) + 200$ OR $a = -10$ $\frac{2\pi}{b} = 12$, so $b = \frac{2\pi}{12} = \frac{\pi}{6}$ $c = 9$ $d = 200$ $h(t) = -10\sin\left(\frac{\pi}{6}(t+9)\right) + 200$ Note: Based on horizontal shifts and reflections, there are other correct forms for $h(t)$.	Vertical transformations: Values of a and d Horizontal transformations: Values of b and c	**1 point** 1.C **1 point** 1.C

Total for part (B) **2 points**

(C) Refer to the graph of h in part (A). The t-coordinate of J is t_1, and the t-coordinate of K is t_2.

(i) On the interval (t_1, t_2), which of the following is true about h?

a. h is positive and increasing.
b. h is positive and decreasing.
c. h is negative and increasing.
d. h is negative and decreasing.

(ii) Describe how the rate of change of h is changing on the interval (t_1, t_2).

(i) Choice a.	Function behavior	**1 point** 2.A
(ii) Because the graph of h is concave up on the interval (t_1, t_2), the rate of change of h is increasing on the interval (t_1, t_2).	Change in rate of change	**1 point** 3.A

Total for part (C) **2 points**

Total for Question 3 **6 points**

Scoring Guidelines for Question 4: Symbolic Manipulations
Part B: Graphing calculator not allowed — 6 points

Directions:

- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number. Angle measures for trigonometric functions are assumed to be in radians.
- Solutions to equations must be real numbers. Determine the exact value of any expression that can be obtained without a calculator. For example, $\log_2 8$, $\cos\left(\frac{\pi}{2}\right)$, and $\sin^{-1}(1)$ can be evaluated without a calculator.
- Unless otherwise specified, combine terms using algebraic methods and rules for exponents and logarithms, where applicable. For example, $2x+3x$, $5^2 \cdot 5^3$, $\frac{x^5}{x^2}$, and $\ln 3 + \ln 5$ should be rewritten in equivalent forms.
- For each part of the question, show the work that leads to your answers.

Model Solution	Scoring	

(A) The functions g and h are given by

$$g(x) = \log_4(2x)$$

$$h(x) = \frac{(e^x)^5}{e^{1/4}}.$$

(i) Solve $g(x) = 3$ for values of x in the domain of g.

(ii) Solve $h(x) = e^{1/2}$ for values of x in the domain of h.

Model Solution	Scoring	Points
(i) $g(x)=3$ $\log_4(2x)=3$ $4^3 = 2x$ $x = \frac{4^3}{2} = 32$	Solution to $g(x)=3$	**1 point** 1.A
(ii) $h(x)=e^{1/2}$ $\frac{(e^x)^5}{e^{1/4}} = e^{1/2}$ $e^{(5x-1/4)} = e^{1/2}$ $5x - \frac{1}{4} = \frac{1}{2}$ $5x = \frac{3}{4}$ $x = \frac{3}{20}$	Solution to $h(x)=e^{1/2}$	**1 point** 1.A
	Total for part (A)	**2 points**

(B) The functions j and k are given by

$$j(x)=\log_{10}(x+1)-5\log_{10}(2-x)+\log_{10}3$$
$$k(x)=\sec x-\cos x.$$

(i) Rewrite $j(x)$ as a single logarithm base 10 without negative exponents in any part of the expression. Your result should be of the form $\log_{10}(\text{expression})$.

(ii) Rewrite $k(x)$ as a product involving $\tan x$ and $\sin x$ and no other trigonometric functions.

(i) $j(x)=\log_{10}(x+1)-5\log_{10}(2-x)+\log_{10}3$ $j(x)=\log_{10}(x+1)-\log_{10}(2-x)^5+\log_{10}3$ $j(x)=\log_{10}\left(\frac{3(x+1)}{(2-x)^5}\right),\ -1<x<2$	Expression for $j(x)$	**1 point** 1.B
(ii) $k(x)=\sec x-\cos x$ $k(x)=\frac{1}{\cos x}-\cos x$ $k(x)=\frac{1-\cos^2 x}{\cos x}$ $k(x)=\frac{\sin^2 x}{\cos x}=\tan x\sin x,\ \cos x\neq 0$	Expression for $k(x)$	**1 point** 1.B

Total for part (B) **2 points**

(C) The function m is given by

$$m(x)=2\tan^{-1}(\sqrt{3}\pi x).$$

Find all input values in the domain of m that yield an output value of $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$.

$m(x)=\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\Rightarrow 2\tan^{-1}(\sqrt{3}\pi x)=\frac{\pi}{3}$ $\tan^{-1}(\sqrt{3}\pi x)=\frac{\pi}{6}$ $\sqrt{3}\pi x=\tan\left(\frac{\pi}{6}\right)$ $\sqrt{3}\pi x=\frac{1}{\sqrt{3}}$ $x=\frac{1}{3\pi}$	$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)=\frac{\pi}{3}$	**1 point** 1.A
	Input value	**1 point** 1.A

Total for part (C) **2 points**

Total for Question 4 **6 points**