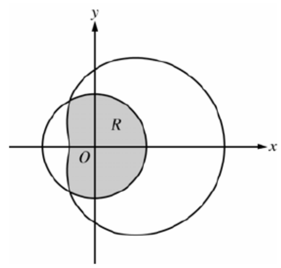
Polar

AP® CALCULUS BC
2007 SCORING GUIDELINES

Question 3



The graphs of the polar curves $r = 2$ and $r = 3 + 2\cos\theta$ are shown in the figure above. The curves intersect when $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

(a) Let R be the region that is inside the graph of $r = 2$ and also inside the graph of $r = 3 + 2\cos\theta$, as shaded in the figure above. Find the area of R.

(b) A particle moving with nonzero velocity along the polar curve given by $r = 3 + 2\cos\theta$ has position $(x(t), y(t))$ at time t, with $\theta = 0$ when $t = 0$. This particle moves along the curve so that $\frac{dr}{dt} = \frac{dr}{d\theta}$. Find the value of $\frac{dr}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(c) For the particle described in part (b), $\frac{dy}{dt} = \frac{dy}{d\theta}$. Find the value of $\frac{dy}{dt}$ at $\theta = \frac{\pi}{3}$ and interpret your answer in terms of the motion of the particle.

(a) Area $= \frac{2}{3}\pi(2)^2 + \frac{1}{2}\int_{2\pi/3}^{4\pi/3}(3 + 2\cos\theta)^2\,d\theta$

$= 10.370$

4 :
- 1 : area of circular sector
- 2 : integral for section of limaçon
  - 1 : integrand
  - 1 : limits and constant
- 1 : answer

(b) $\left.\frac{dr}{dt}\right|_{\theta=\pi/3} = \left.\frac{dr}{d\theta}\right|_{\theta=\pi/3} = -1.732$

The particle is moving closer to the origin, since $\frac{dr}{dt} < 0$ and $r > 0$ when $\theta = \frac{\pi}{3}$.

2 :
- 1 : $\left.\frac{dr}{dt}\right|_{\theta=\pi/3}$
- 1 : interpretation

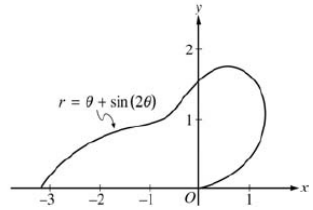
(c) $y = r\sin\theta = (3 + 2\cos\theta)\sin\theta$

$\left.\frac{dy}{dt}\right|_{\theta=\pi/3} = \left.\frac{dy}{d\theta}\right|_{\theta=\pi/3} = 0.5$

The particle is moving away from the x-axis, since $\frac{dy}{dt} > 0$ and $y > 0$ when $\theta = \frac{\pi}{3}$.

3 :
- 1 : expression for y in terms of θ
- 1 : $\left.\frac{dy}{dt}\right|_{\theta=\pi/3}$
- 1: interpretation

Question 2



The curve above is drawn in the xy-plane and is described by the equation in polar coordinates $r = \theta + \sin(2\theta)$ for $0 \le \theta \le \pi$, where r is measured in meters and θ is measured in radians. The derivative of r with respect to θ is given by $\frac{dr}{d\theta} = 1 + 2\cos(2\theta)$.

(a) Find the area bounded by the curve and the x-axis.

(b) Find the angle θ that corresponds to the point on the curve with x-coordinate -2.

(c) For $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, $\frac{dr}{d\theta}$ is negative. What does this fact say about r? What does this fact say about the curve?

(d) Find the value of θ in the interval $0 \le \theta \le \frac{\pi}{2}$ that corresponds to the point on the curve in the first quadrant with greatest distance from the origin. Justify your answer.

(a) Area $= \frac{1}{2}\int_0^{\pi} r^2 \, d\theta$

$= \frac{1}{2}\int_0^{\pi} (\theta + \sin(2\theta))^2 \, d\theta = 4.382$

3 : { 1 : limits and constant; 1 : integrand; 1 : answer }

(b) $-2 = r\cos(\theta) = (\theta + \sin(2\theta))\cos(\theta)$

$\theta = 2.786$

2 : { 1 : equation; 1 : answer }

(c) Since $\frac{dr}{d\theta} < 0$ for $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$, r is decreasing on this interval. This means the curve is getting closer to the origin.

2 : { 1 : information about r; 1 : information about the curve }

(d) The only value in $\left[0, \frac{\pi}{2}\right]$ where $\frac{dr}{d\theta} = 0$ is $\theta = \frac{\pi}{3}$.

θ	r
0	0
$\frac{\pi}{3}$	1.913
$\frac{\pi}{2}$	1.571

The greatest distance occurs when $\theta = \frac{\pi}{3}$.

2 : { 1 : $\theta = \frac{\pi}{3}$ or 1.047; 1 : answer with justification }

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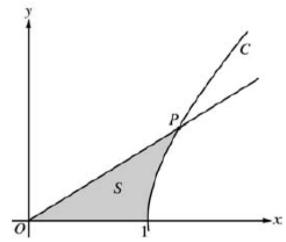
Question 3



The figure above shows the graphs of the line $x = \frac{5}{3}y$ and the curve C given by $x = \sqrt{1+y^2}$. Let S be the shaded region bounded by the two graphs and the x-axis. The line and the curve intersect at point P.

(a) Find the coordinates of point P and the value of $\frac{dx}{dy}$ for curve C at point P.

(b) Set up and evaluate an integral expression with respect to y that gives the area of S.

(c) Curve C is a part of the curve $x^2 - y^2 = 1$. Show that $x^2 - y^2 = 1$ can be written as the polar equation $r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$.

(d) Use the polar equation given in part (c) to set up an integral expression with respect to the polar angle θ that represents the area of S.

(a) At P, $\frac{5}{3}y = \sqrt{1+y^2}$, so $y = \frac{3}{4}$.

Since $x = \frac{5}{3}y$, $x = \frac{5}{4}$.

$$\frac{dx}{dy} = \frac{y}{\sqrt{1+y^2}} = \frac{y}{x}. \text{ At } P, \frac{dx}{dy} = \frac{3/4}{5/4} = \frac{3}{5}.$$

$2: \begin{cases} 1: \text{coordinates of } P \\ 1: \frac{dx}{dy} \text{ at } P \end{cases}$

(b)
$$\text{Area} = \int_0^{3/4}\left(\sqrt{1+y^2} - \frac{5}{3}y\right)dy$$
$$= 0.346 \text{ or } 0.347$$

$3: \begin{cases} 1: \text{limits} \\ 1: \text{integrand} \\ 1: \text{answer} \end{cases}$

(c) $x = r\cos\theta$; $y = r\sin\theta$

$$x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1$$
$$r^2 = \frac{1}{\cos^2\theta - \sin^2\theta}$$

$2: \begin{cases} 1: \text{substitutes } x = r\cos\theta \text{ and } y = r\sin\theta \text{ into } x^2 - y^2 = 1 \\ 1: \text{isolates } r^2 \end{cases}$

(d) Let β be the angle that segment OP makes with the x-axis. Then $\tan\beta = \frac{y}{x} = \frac{3/4}{5/4} = \frac{3}{5}$.

$$\text{Area} = \int_0^{\tan^{-1}(3/5)} \frac{1}{2}r^2\,d\theta$$
$$= \frac{1}{2}\int_0^{\tan^{-1}(3/5)} \frac{1}{\cos^2\theta - \sin^2\theta}\,d\theta$$

$2: \begin{cases} 1: \text{limits} \\ 1: \text{integrand and constant} \end{cases}$