# Other

## AP® CALCULUS BC
## 2008 SCORING GUIDELINES

### Question 5

The derivative of a function $f$ is given by $f'(x) = (x-3)e^x$ for $x > 0$, and $f(1) = 7$.

(a) The function $f$ has a critical point at $x = 3$. At this point, does $f$ have a relative minimum, a relative maximum, or neither? Justify your answer.

(b) On what intervals, if any, is the graph of $f$ both decreasing and concave up? Explain your reasoning.

(c) Find the value of $f(3)$.

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(a) $f'(x) < 0$ for $0 < x < 3$ and $f'(x) > 0$ for $x > 3$

Therefore, $f$ has a relative minimum at $x = 3$.

$2 : \begin{cases} 1 : \text{minimum at } x = 3 \\ 1 : \text{justification} \end{cases}$

(b) $f''(x) = e^x + (x-3)e^x = (x-2)e^x$

$f''(x) > 0$ for $x > 2$

$f'(x) < 0$ for $0 < x < 3$

Therefore, the graph of $f$ is both decreasing and concave up on the interval $2 < x < 3$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) $f(3) = f(1) + \int_1^3 f'(x)\,dx = 7 + \int_1^3 (x-3)e^x\,dx$

$u = x - 3 \quad dv = e^x\,dx$

$du = dx \quad v = e^x$

$f(3) = 7 + (x-3)e^x\Big|_1^3 - \int_1^3 e^x\,dx$

$= 7 + \left((x-3)e^x - e^x\right)\Big|_1^3$

$= 7 + 3e - e^3$

$4 : \begin{cases} 1 : \text{uses initial condition} \\ 2 : \text{integration by parts} \\ 1 : \text{answer} \end{cases}$

# AP® CALCULUS BC
# 2007 SCORING GUIDELINES

## Question 4

Let $f$ be the function defined for $x > 0$, with $f(e) = 2$ and $f'$, the first derivative of $f$, given by $f'(x) = x^2 \ln x$.

(a) Write an equation for the line tangent to the graph of $f$ at the point $(e, 2)$.

(b) Is the graph of $f$ concave up or concave down on the interval $1 < x < 3$? Give a reason for your answer.

(c) Use antidifferentiation to find $f(x)$.

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(a) $f'(e) = e^2$

An equation for the line tangent to the graph of $f$ at the point $(e, 2)$ is $y - 2 = e^2(x - e)$.

$2 : \begin{cases} 1 : f'(e) \\ 1 : \text{equation of tangent line} \end{cases}$

(b) $f''(x) = x + 2x \ln x$.

For $1 < x < 3$, $x > 0$ and $\ln x > 0$, so $f''(x) > 0$. Thus, the graph of $f$ is concave up on $(1, 3)$.

$3 : \begin{cases} 2 : f''(x) \\ 1 : \text{answer with reason} \end{cases}$

(c) Since $f(x) = \int \left(x^2 \ln x\right) dx$, we consider integration by parts.

$$u = \ln x \qquad dv = x^2\, dx$$
$$du = \frac{1}{x}\, dx \qquad v = \int \left(x^2\right) dx = \frac{1}{3}x^3$$

Therefore,

$$\begin{aligned} f(x) &= \int \left(x^2 \ln x\right) dx \\ &= \frac{1}{3}x^3 \ln x - \int \left(\frac{1}{3}x^3 \cdot \frac{1}{x}\right) dx \\ &= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C. \end{aligned}$$

Since $f(e) = 2$, $2 = \frac{e^3}{3} - \frac{e^3}{9} + C$ and $C = 2 - \frac{2}{9}e^3$.

Thus, $f(x) = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + 2 - \frac{2}{9}e^3$.

$4 : \begin{cases} 2 : \text{antiderivative} \\ 1 : \text{uses } f(e) = 2 \\ 1 : \text{answer} \end{cases}$

## Question 6

The function $f$ is defined by the power series

$$f(x) = -\frac{x}{2} + \frac{2x^2}{3} - \frac{3x^3}{4} + \cdots + \frac{(-1)^n nx^n}{n+1} + \cdots$$

for all real numbers $x$ for which the series converges. The function $g$ is defined by the power series

$$g(x) = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \cdots + \frac{(-1)^n x^n}{(2n)!} + \cdots$$

for all real numbers $x$ for which the series converges.

(a) Find the interval of convergence of the power series for $f$. Justify your answer.

(b) The graph of $y = f(x) - g(x)$ passes through the point $(0, -1)$. Find $y'(0)$ and $y''(0)$. Determine whether $y$ has a relative minimum, a relative maximum, or neither at $x = 0$. Give a reason for your answer.

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(a) $$\left|\frac{(-1)^{n+1}(n+1)x^{n+1}}{n+2} \cdot \frac{n+1}{(-1)^n nx^n}\right| = \frac{(n+1)^2}{(n+2)(n)} \cdot |x|$$

$$\lim_{n\to\infty} \frac{(n+1)^2}{(n+2)(n)} \cdot |x| = |x|$$

The series converges when $-1 < x < 1$.

When $x = 1$, the series is $-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

When $x = -1$, the series is $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \cdots$

This series does not converge, because the limit of the individual terms is not zero.

Thus, the interval of convergence is $-1 < x < 1$.

5 :
- 1 : sets up ratio
- 1 : computes limit of ratio
- 1 : identifies radius of convergence
- 1 : considers both endpoints
- 1 : analysis/conclusion for both endpoints

(b) $f'(x) = -\frac{1}{2} + \frac{4}{3}x - \frac{9}{4}x^2 + \cdots$ and $f'(0) = -\frac{1}{2}$.

$g'(x) = -\frac{1}{2!} + \frac{2}{4!}x - \frac{3}{6!}x^2 + \cdots$ and $g'(0) = -\frac{1}{2}$.

$y'(0) = f'(0) - g'(0) = 0$

$f''(0) = \frac{4}{3}$ and $g''(0) = \frac{2}{4!} = \frac{1}{12}$.

Thus, $y''(0) = \frac{4}{3} - \frac{1}{12} > 0$.

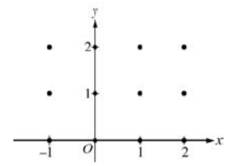
Since $y'(0) = 0$ and $y''(0) > 0$, $y$ has a relative minimum at $x = 0$.

4 :
- 1 : $y'(0)$
- 1 : $y''(0)$
- 1 : conclusion
- 1 : reasoning

# AP® CALCULUS BC
# 2005 SCORING GUIDELINES

## Question 4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.




(a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and sketch the solution curve that passes through the point $(0, 1)$. (**Note: Use the axes provided in the pink test booklet.**)

(b) The solution curve that passes through the point $(0, 1)$ has a local minimum at $x = \ln\left(\frac{3}{2}\right)$. What is the $y$-coordinate of this local minimum?

(c) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(0) = 1$. Use Euler's method, starting at $x = 0$ with two steps of equal size, to approximate $f(-0.4)$. Show the work that leads to your answer.

(d) Find $\frac{d^2y}{dx^2}$ in terms of $x$ and $y$. Determine whether the approximation found in part (c) is less than or greater than $f(-0.4)$. Explain your reasoning.

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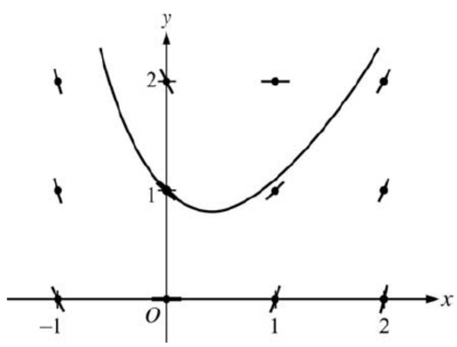
(a)




$3 : \begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \\ 1 : \text{curve through } (0, 1) \end{cases}$

(b) $\frac{dy}{dx} = 0$ when $2x = y$

The $y$-coordinate is $2\ln\left(\frac{3}{2}\right)$.

$2 : \begin{cases} 1 : \text{sets } \frac{dy}{dx} = 0 \\ 1 : \text{answer} \end{cases}$

(c)
$$\begin{aligned} f(-0.2) &\approx f(0) + f'(0)(-0.2) \\ &= 1 + (-1)(-0.2) = 1.2 \\ f(-0.4) &\approx f(-0.2) + f'(-0.2)(-0.2) \\ &\approx 1.2 + (-1.6)(-0.2) = 1.52 \end{aligned}$$

$2 : \begin{cases} 1 : \text{Euler's method with two steps} \\ 1 : \text{Euler approximation to } f(-0.4) \end{cases}$

(d) $\frac{d^2y}{dx^2} = 2 - \frac{dy}{dx} = 2 - 2x + y$

$\frac{d^2y}{dx^2}$ is positive in quadrant II because $x < 0$ and $y > 0$.

$1.52 < f(-0.4)$ since all solution curves in quadrant II are concave up.

$2 : \begin{cases} 1 : \frac{d^2y}{dx^2} \\ 1 : \text{answer with reason} \end{cases}$

## Question 5

A population is modeled by a function $P$ that satisfies the logistic differential equation

$$\frac{dP}{dt} = \frac{P}{5}\left(1 - \frac{P}{12}\right).$$

(a) If $P(0) = 3$, what is $\lim_{t\to\infty} P(t)$?

If $P(0) = 20$, what is $\lim_{t\to\infty} P(t)$?

(b) If $P(0) = 3$, for what value of $P$ is the population growing the fastest?

(c) A different population is modeled by a function $Y$ that satisfies the separable differential equation

$$\frac{dY}{dt} = \frac{Y}{5}\left(1 - \frac{t}{12}\right).$$

Find $Y(t)$ if $Y(0) = 3$.

(d) For the function $Y$ found in part (c), what is $\lim_{t\to\infty} Y(t)$?

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| | |
|---|---|
| (a) For this logistic differential equation, the carrying capacity is 12.<br>If $P(0) = 3$, $\lim_{t\to\infty} P(t) = 12$.<br>If $P(0) = 20$, $\lim_{t\to\infty} P(t) = 12$. | $2 : \begin{cases} 1 : \text{answer} \\ 1 : \text{answer} \end{cases}$ |
| (b) The population is growing the fastest when $P$ is half the carrying capacity. Therefore, $P$ is growing the fastest when $P = 6$. | 1 : answer |
| (c) $\frac{1}{Y}dY = \frac{1}{5}\left(1 - \frac{t}{12}\right)dt = \left(\frac{1}{5} - \frac{t}{60}\right)dt$<br>$\ln\lvert Y\rvert = \frac{t}{5} - \frac{t^2}{120} + C$<br>$Y(t) = Ke^{\frac{t}{5} - \frac{t^2}{120}}$<br>$K = 3$<br>$Y(t) = 3e^{\frac{t}{5} - \frac{t^2}{120}}$ | $5 : \begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } Y \\ \quad 0/1 \text{ if } Y \text{ is not exponential} \end{cases}$<br>Note: max 2/5 [1-1-0-0-0] if no constant of integration<br>Note: 0/5 if no separation of variables |
| (d) $\lim_{t\to\infty} Y(t) = 0$ | 1 : answer<br>$0/1$ if $Y$ is not exponential |

# AP® CALCULUS BC
# 2003 SCORING GUIDELINES

### Question 6

The function $f$ is defined by the power series

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots + \frac{(-1)^n x^{2n}}{(2n+1)!} + \cdots$$

for all real numbers $x$.

(a) Find $f'(0)$ and $f''(0)$. Determine whether $f$ has a local maximum, a local minimum, or neither at $x = 0$. Give a reason for your answer.

(b) Show that $1 - \frac{1}{3!}$ approximates $f(1)$ with error less than $\frac{1}{100}$.

(c) Show that $y = f(x)$ is a solution to the differential equation $xy' + y = \cos x$.

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(a) $f'(0) =$ coefficient of $x$ term $= 0$

$f''(0) = 2$ (coefficient of $x^2$ term) $= 2\left(-\frac{1}{3!}\right) = -\frac{1}{3}$

$f$ has a local maximum at $x = 0$ because $f'(0) = 0$ and $f''(0) < 0$.

4 :
- 1 : $f'(0)$
- 1 : $f''(0)$
- 1 : critical point answer
- 1 : reason

(b) $f(1) = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots + \frac{(-1)^n}{(2n+1)!} + \cdots$

This is an alternating series whose terms decrease in absolute value with limit 0. Thus, the error is less than the first omitted term, so $\left| f(1) - \left(1 - \frac{1}{3!}\right) \right| \le \frac{1}{5!} = \frac{1}{120} < \frac{1}{100}$.

1 : error bound $< \frac{1}{100}$

(c)

$$y' = -\frac{2x}{3!} + \frac{4x^3}{5!} - \frac{6x^5}{7!} + \cdots + \frac{(-1)^n 2nx^{2n-1}}{(2n+1)!} + \cdots$$

$$xy' = -\frac{2x^2}{3!} + \frac{4x^4}{5!} - \frac{6x^6}{7!} + \cdots + \frac{(-1)^n 2nx^{2n}}{(2n+1)!} + \cdots$$

$$xy' + y = 1 - \left(\frac{2}{3!} + \frac{1}{3!}\right)x^2 + \left(\frac{4}{5!} + \frac{1}{5!}\right)x^4 - \left(\frac{6}{7!} + \frac{1}{7!}\right)x^6 + \cdots + (-1)^n\left(\frac{2n}{(2n+1)!} + \frac{1}{(2n+1)!}\right)x^{2n} + \cdots$$

$$= 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots$$

$$= \cos x$$

4 :
- 1 : series for $y'$
- 1 : series for $xy'$
- 1 : series for $xy' + y$
- 1 : identifies series as $\cos x$

OR

$$xy = xf(x) = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} + \cdots$$

$$= \sin x$$

$$xy' + y = (xy)' = (\sin x)' = \cos x$$

OR

4 :
- 1 : series for $xf(x)$
- 1 : identifies series as $\sin x$
- 1 : handles $xy' + y$
- 1 : makes connection

## Question 4

Let $h$ be a function defined for all $x \neq 0$ such that $h(4) = -3$ and the derivative of $h$ is given by $h'(x) = \dfrac{x^2 - 2}{x}$ for all $x \neq 0$.

(a) Find all values of $x$ for which the graph of $h$ has a horizontal tangent, and determine whether $h$ has a local maximum, a local minimum, or neither at each of these values. Justify your answers.

(b) On what intervals, if any, is the graph of $h$ concave up? Justify your answer.

(c) Write an equation for the line tangent to the graph of $h$ at $x = 4$.

(d) Does the line tangent to the graph of $h$ at $x = 4$ lie above or below the graph of $h$ for $x > 4$? Why?

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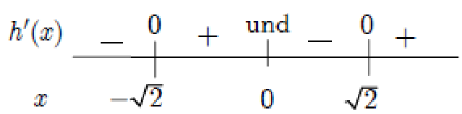
(a) $h'(x) = 0$ at $x = \pm\sqrt{2}$




Local minima at $x = -\sqrt{2}$ and at $x = \sqrt{2}$

4 : { 1 : $x = \pm\sqrt{2}$; 1 : analysis; 2 : conclusions; $< -1 >$ not dealing with discontinuity at 0 }

(b) $h''(x) = 1 + \dfrac{2}{x^2} > 0$ for all $x \neq 0$. Therefore, the graph of $h$ is concave up for all $x \neq 0$.

3 : { 1 : $h''(x)$; 1 : $h''(x) > 0$; 1 : answer }

(c) $h'(4) = \dfrac{16 - 2}{4} = \dfrac{7}{2}$

$y + 3 = \dfrac{7}{2}(x - 4)$

1 : tangent line equation

(d) The tangent line is below the graph because the graph of $h$ is concave up for $x > 4$.

1 : answer with reason

## Question 6

A function $f$ is defined by

$$f(x) = \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots$$

for all $x$ in the interval of convergence of the given power series.

(a) Find the interval of convergence for this power series. Show the work that leads to your answer.

(b) Find $\displaystyle\lim_{x \to 0} \frac{f(x) - \frac{1}{3}}{x}$.

(c) Write the first three nonzero terms and the general term for an infinite series that represents $\int_0^1 f(x)\,dx$.

(d) Find the sum of the series determined in part (c).

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(a) $\displaystyle\lim_{n\to\infty} \left| \frac{\frac{(n+2)x^{n+1}}{3^{n+2}}}{\frac{(n+1)x^n}{3^{n+1}}} \right| = \lim_{n\to\infty} \left| \frac{(n+2)}{(n+1)} \frac{x}{3} \right| = \left| \frac{x}{3} \right| < 1$

At $x = -3$, the series is $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{3}$, which diverges.

At $x = 3$, the series is $\displaystyle\sum_{n=0}^{\infty} \frac{n+1}{3}$, which diverges.

Therefore, the interval of convergence is $-3 < x < 3$.

4 :
- 1 : sets up ratio test
- 1 : computes limit
- 1 : conclusion of ratio test
- 1 : endpoint conclusion

(b) $\displaystyle\lim_{x\to 0} \frac{f(x) - \frac{1}{3}}{x} = \lim_{x\to 0} \left( \frac{2}{3^2} + \frac{3}{3^3}x + \frac{4}{3^4}x^2 + \cdots \right) = \frac{2}{9}$

1 : answer

(c) $$\int_0^1 f(x)\,dx = \int_0^1 \left( \frac{1}{3} + \frac{2}{3^2}x + \frac{3}{3^3}x^2 + \cdots + \frac{n+1}{3^{n+1}}x^n + \cdots \right) dx$$

$$= \left( \frac{1}{3}x + \frac{1}{3^2}x^2 + \frac{1}{3^3}x^3 + \cdots + \frac{1}{3^{n+1}}x^{n+1} + \cdots \right) \Bigg|_{x=0}^{x=1}$$

$$= \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{n+1}} + \cdots$$

3 :
- 1 : antidifferentiation of series
- 1 : first three terms for definite integral series
- 1 : general term

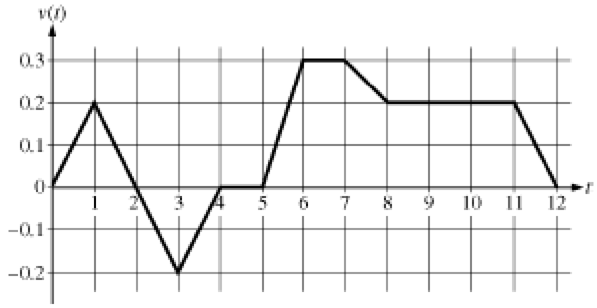
(d) The series representing $\int_0^1 f(x)\,dx$ is a geometric series.

Therefore, $\displaystyle\int_0^1 f(x)\,dx = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}$.

1 : answer

# AP® CALCULUS BC
# 2009 SCORING GUIDELINES

## Question 1




Caren rides her bicycle along a straight road from home to school, starting at home at time $t = 0$ minutes and arriving at school at time $t = 12$ minutes. During the time interval $0 \le t \le 12$ minutes, her velocity $v(t)$, in miles per minute, is modeled by the piecewise-linear function whose graph is shown above.

(a) Find the acceleration of Caren's bicycle at time $t = 7.5$ minutes. Indicate units of measure.

(b) Using correct units, explain the meaning of $\int_0^{12} |v(t)|\,dt$ in terms of Caren's trip. Find the value of $\int_0^{12} |v(t)|\,dt$.

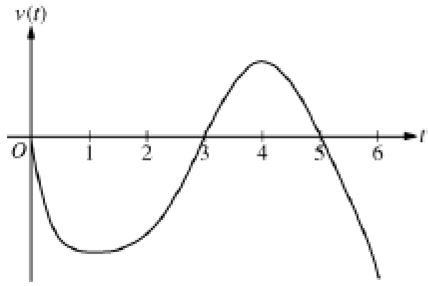
(c) Shortly after leaving home, Caren realizes she left her calculus homework at home, and she returns to get it. At what time does she turn around to go back home? Give a reason for your answer.

(d) Larry also rides his bicycle along a straight road from home to school in 12 minutes. His velocity is modeled by the function $w$ given by $w(t) = \frac{\pi}{15}\sin\left(\frac{\pi}{12}t\right)$, where $w(t)$ is in miles per minute for $0 \le t \le 12$ minutes. Who lives closer to school: Caren or Larry? Show the work that leads to your answer.

| | |
|---|---|
| (a) $a(7.5) = v'(7.5) = \dfrac{v(8) - v(7)}{8 - 7} = -0.1$ miles/minute$^2$ | 2 : { 1 : answer<br>1 : units |
| (b) $\int_0^{12} |v(t)|\,dt$ is the total distance, in miles, that Caren rode during the 12 minutes from $t = 0$ to $t = 12$.<br>$\int_0^{12} |v(t)|\,dt = \int_0^{2} v(t)\,dt - \int_2^{4} v(t)\,dt + \int_4^{12} v(t)\,dt$<br>$= 0.2 + 0.2 + 1.4 = 1.8$ miles | 2 : { 1 : meaning of integral<br>1 : value of integral |
| (c) Caren turns around to go back home at time $t = 2$ minutes. This is the time at which her velocity changes from positive to negative. | 2 : { 1 : answer<br>1 : reason |
| (d) $\int_0^{12} w(t)\,dt = 1.6$; Larry lives 1.6 miles from school.<br>$\int_0^{12} v(t)\,dt = 1.4$; Caren lives 1.4 miles from school.<br>Therefore, Caren lives closer to school. | 3 : { 2 : Larry's distance from school<br>1 : integral<br>1 : value<br>1 : Caren's distance from school and conclusion |

# AP® CALCULUS BC
# 2008 SCORING GUIDELINES

## Question 4




Graph of $v$

A particle moves along the $x$-axis so that its velocity at time $t$, for $0 \le t \le 6$, is given by a differentiable function $v$ whose graph is shown above. The velocity is 0 at $t = 0$, $t = 3$, and $t = 5$, and the graph has horizontal tangents at $t = 1$ and $t = 4$. The areas of the regions bounded by the $t$-axis and the graph of $v$ on the intervals $[0, 3]$, $[3, 5]$, and $[5, 6]$ are 8, 3, and 2, respectively. At time $t = 0$, the particle is at $x = -2$.

(a) For $0 \le t \le 6$, find both the time and the position of the particle when the particle is farthest to the left. Justify your answer.

(b) For how many values of $t$, where $0 \le t \le 6$, is the particle at $x = -8$ ? Explain your reasoning.

(c) On the interval $2 < t < 3$, is the speed of the particle increasing or decreasing? Give a reason for your answer.

(d) During what time intervals, if any, is the acceleration of the particle negative? Justify your answer.

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| | |
|---|---|
| (a) Since $v(t) < 0$ for $0 < t < 3$ and $5 < t < 6$, and $v(t) > 0$ for $3 < t < 5$, we consider $t = 3$ and $t = 6$.<br>$x(3) = -2 + \int_0^3 v(t)\,dt = -2 - 8 = -10$<br>$x(6) = -2 + \int_0^6 v(t)\,dt = -2 - 8 + 3 - 2 = -9$<br>Therefore, the particle is farthest left at time $t = 3$ when its position is $x(3) = -10$. | 3 : { 1 : identifies $t = 3$ as a candidate<br>1 : considers $\int_0^6 v(t)\,dt$<br>1 : conclusion |
| (b) The particle moves continuously and monotonically from $x(0) = -2$ to $x(3) = -10$. Similarly, the particle moves continuously and monotonically from $x(3) = -10$ to $x(5) = -7$ and also from $x(5) = -7$ to $x(6) = -9$.<br>By the Intermediate Value Theorem, there are three values of $t$ for which the particle is at $x(t) = -8$. | 3 : { 1 : positions at $t = 3$, $t = 5$, and $t = 6$<br>1 : description of motion<br>1 : conclusion |
| (c) The speed is decreasing on the interval $2 < t < 3$ since on this interval $v < 0$ and $v$ is increasing. | 1 : answer with reason |
| (d) The acceleration is negative on the intervals $0 < t < 1$ and $4 < t < 6$ since velocity is decreasing on these intervals. | 2 : { 1 : answer<br>1 : justification |