51. A modernistic painting consists of triangles, rectangles, and pentagons, all drawn so as to not overlap or share sides. Within each rectangle are drawn 2 red roses and each pentagon contains 5 carnations. How many triangles, rectangles, and pentagons appear in the painting if the painting contains a total of 40 geometric figures, 153 sides of geometric figures, and 72 flowers?

Group Exercise

52. Group members should develop appropriate functions that model each of the projections shown in Exercise 43.

Preview Exercises

Exercises 53–55 will help you prepare for the material covered in the next section.

53. Subtract: $\frac{3}{x-4} - \frac{2}{x+2}$.

54. Add: $\frac{5x-3}{x^2+1} + \frac{2x}{(x^2+1)^2}$.

55. Solve:

$$\begin{cases} A + B = 3 \\ 2A - 2B + C = 17 \\ 4A - 2C = 14. \end{cases}$$

Section 7.3 Partial Fractions

Objectives

1. Decompose $\frac{P}{Q}$, where Q has only distinct linear factors.
2. Decompose $\frac{P}{Q}$, where Q has repeated linear factors.
3. Decompose $\frac{P}{Q}$, where Q has a nonrepeated prime quadratic factor.
4. Decompose $\frac{P}{Q}$, where Q has a prime, repeated quadratic factor.

The rising and setting of the sun suggest the obvious: Things change over time. Calculus is the study of rates of change, allowing the motion of the rising sun to be measured by "freezing the frame" at one instant in time. If you are given a function, calculus reveals its rate of change at any "frozen" instant. In this section, you will learn an algebraic technique used in calculus to find a function if its rate of change is known. The technique involves expressing a given function in terms of simpler functions.

The Idea behind Partial Fraction Decomposition

We know how to use common denominators to write a sum or difference of rational expressions as a single rational expression. For example,

$$\frac{3}{x-4} - \frac{2}{x+2} = \frac{3(x+2) - 2(x-4)}{(x-4)(x+2)}$$

$$= \frac{3x+6-2x+8}{(x-4)(x+2)} = \frac{x+14}{(x-4)(x+2)}.$$

For solving the kind of calculus problem described in the section opener, we must reverse this process:

$$\frac{x+14}{(x-4)(x+2)} = \underbrace{\overset{\text{Partial fraction}}{\frac{3}{x-4}} + \overset{\text{Partial fraction}}{\frac{-2}{x+2}}}.$$

$\frac{x+14}{(x-4)(x+2)}$ is expressed as the sum of two simpler fractions.

This is the partial fraction decomposition of $\frac{x+14}{(x-4)(x+2)}$.

Each of the two fractions on the right is called a **partial fraction**. The sum of these fractions is called the **partial fraction decomposition** of the rational expression on the left-hand side.

Partial fraction decompositions can be written for rational expressions of the form $\frac{P(x)}{Q(x)}$, where P and Q have no common factors and the highest power in the numerator is less than the highest power in the denominator. In this section, we will show you how to write the partial fraction decompositions for each of the following rational expressions:

$$\frac{9x^2 - 9x + 6}{(2x-1)(x+2)(x-2)}$$

$P(x) = 9x^2 - 9x + 6$; highest power $= 2$

$Q(x) = (2x-1)(x+2)(x-2)$; multiplying factors, highest power $= 3$.

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2}.$$

$P(x) = 5x^3 - 3x^2 + 7x - 3$; highest power $= 3$

$Q(x) = (x^2+1)^2$; squaring the expression, highest power $= 4$.

The partial fraction decomposition of a rational expression depends on the factors of the denominator. We consider four cases involving different kinds of factors in the denominator:

1. The denominator is a product of distinct linear factors.
2. The denominator is a product of linear factors, some of which are repeated.
3. The denominator has prime quadratic factors, none of which is repeated.
4. The denominator has a repeated prime quadratic factor.

1 Decompose $\frac{P}{Q}$, where Q has only distinct linear factors.

The Partial Fraction Decomposition of a Rational Expression with Distinct Linear Factors in the Denominator

If the denominator of a rational expression has a linear factor of the form $ax + b$, then the partial fraction decomposition will contain a term of the form

$$\frac{A}{ax+b}.$$

Constant

Linear factor

Each distinct linear factor in the denominator produces a partial fraction of the form *constant over linear factor*. For example,

$$\frac{9x^2 - 9x + 6}{(2x-1)(x+2)(x-2)} = \frac{A}{2x-1} + \frac{B}{x+2} + \frac{C}{x-2}.$$

We write a constant over each linear factor in the denominator.

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ Has Distinct Linear Factors

The form of the partial fraction decomposition for a rational expression with distinct linear factors in the denominator is

$$\frac{P(x)}{(a_1x+b_1)(a_2x+b_2)(a_3x+b_3)\cdots(a_nx+b_n)}$$

$$= \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \frac{A_3}{a_3x+b_3} + \cdots + \frac{A_n}{a_nx+b_n}.$$

EXAMPLE 1 Partial Fraction Decomposition with Distinct Linear Factors

Find the partial fraction decomposition of

$$\frac{x+14}{(x-4)(x+2)}.$$

Solution We begin by setting up the partial fraction decomposition with the unknown constants. Write a constant over each of the two distinct linear factors in the denominator.

$$\frac{x+14}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2}$$

Our goal is to find A and B. We do this by multiplying both sides of the equation by the least common denominator, $(x-4)(x+2)$.

$$(x-4)(x+2)\frac{x+14}{(x-4)(x+2)} = (x-4)(x+2)\left(\frac{A}{x-4} + \frac{B}{x+2}\right)$$

We use the distributive property on the right side.

$$\cancel{(x-4)}\,\cancel{(x+2)}\frac{x+14}{\cancel{(x-4)}\,\cancel{(x+2)}}$$

$$= \cancel{(x-4)}(x+2)\frac{A}{\cancel{(x-4)}} + (x-4)\cancel{(x+2)}\frac{B}{\cancel{(x+2)}}$$

Dividing out common factors in numerators and denominators, we obtain

$$x + 14 = A(x+2) + B(x-4).$$

To find values for A and B that make both sides equal, we'll express the sides in exactly the same form by writing the variable x-terms and then writing the constant terms. Apply the distributive property on the right side.

$$x + 14 = Ax + 2A + Bx - 4B \quad \text{Distribute } A \text{ and } B \text{ over the parentheses.}$$

$$x + 14 = Ax + Bx + 2A - 4B \quad \text{Rearrange terms.}$$

$$1x + 14 = (A + B)x + (2A - 4B) \quad \text{Rewrite to identify the coefficient of } x \text{ and the constant term.}$$

As shown by the arrows, if two polynomials are equal, coefficients of like powers of x must be equal $(A + B = 1)$ and their constant terms must be equal $(2A - 4B = 14)$. Consequently, A and B satisfy the following two equations:

$$\begin{cases} A + B = 1 \\ 2A - 4B = 14. \end{cases}$$

We can use the addition method to solve this linear system in two variables. By multiplying the first equation by -2 and adding equations, we obtain $A = 3$ and $B = -2$. Thus,

$$\frac{x + 14}{(x - 4)(x + 2)} = \frac{A}{x - 4} + \frac{B}{x + 2} = \frac{3}{x - 4} + \frac{-2}{x + 2} \left(\text{or } \frac{3}{x - 4} - \frac{2}{x + 2}\right).$$

Steps in Partial Fraction Decomposition

1. Set up the partial fraction decomposition with the unknown constants A, B, C, etc., in the numerators of the decomposition.
2. Multiply both sides of the resulting equation by the least common denominator.
3. Simplify the right side of the equation.
4. Write both sides in descending powers, equate coefficients of like powers of x, and equate constant terms.
5. Solve the resulting linear system for A, B, C, etc.
6. Substitute the values for A, B, C, etc., into the equation in step 1 and write the partial fraction decomposition.

Study Tip

You will encounter some examples in which the denominator of the given rational expression is not already factored. If necessary, begin by factoring the denominator. Then apply the six steps needed to obtain the partial fraction decomposition.

Check Point 1 Find the partial fraction decomposition of $\frac{5x - 1}{(x - 3)(x + 4)}$.

2 Decompose $\frac{P}{Q}$, where Q has repeated linear factors.

The Partial Fraction Decomposition of a Rational Expression with Linear Factors in the Denominator, Some of Which Are Repeated

Suppose that $(ax + b)^n$ is a factor of the denominator. This means that the linear factor $ax + b$ is repeated n times. When this occurs, the partial fraction decomposition will contain a sum of n fractions for this factor of the denominator.

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ Has Repeated Linear Factors

The form of the partial fraction decomposition for a rational expression containing the linear factor $ax + b$ occurring n times as its denominator is

$$\frac{P(x)}{(ax + b)^n} = \frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \cdots + \frac{A_n}{(ax + b)^n}.$$

Include one fraction with a constant numerator for each power of $ax + b$.

EXAMPLE 2 Partial Fraction Decomposition with Repeated Linear Factors

Find the partial fraction decomposition of $\dfrac{x-18}{x(x-3)^2}$.

Study Tip

Avoid this common error:

$$\cancel{\frac{x-18}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x-3}}$$

Listing $x-3$ twice does not take into account $(x-3)^2$.

Solution

Step 1 Set up the partial fraction decomposition with the unknown constants. Because the linear factor $x-3$ occurs twice, we must include one fraction with a constant numerator for each power of $x-3$.

$$\frac{x-18}{x(x-3)^2} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}$$

Step 2 Multiply both sides of the resulting equation by the least common denominator. We clear fractions, multiplying both sides by $x(x-3)^2$, the least common denominator.

$$x(x-3)^2\left[\frac{x-18}{x(x-3)^2}\right] = x(x-3)^2\left[\frac{A}{x} + \frac{B}{x-3} + \frac{C}{(x-3)^2}\right]$$

We use the distributive property on the right side.

$$\cancel{x}\,\cancel{(x-3)^2}\cdot\frac{x-18}{\cancel{x}\,\cancel{(x-3)^2}} = \cancel{x}(x-3)^2\cdot\frac{A}{\cancel{x}} + x(x-3)^2\cdot\frac{B}{(x-3)} + x\cancel{(x-3)^2}\cdot\frac{C}{\cancel{(x-3)^2}}$$

Dividing out common factors in numerators and denominators, we obtain

$$x-18 = A(x-3)^2 + Bx(x-3) + Cx.$$

Step 3 Simplify the right side of the equation. Square $x-3$. Then apply the distributive property.

$$x-18 = A(x^2-6x+9) + Bx(x-3) + Cx$$ Square $x-3$ using $(A-B)^2 = A^2 - 2AB + B^2$.

$$x-18 = Ax^2 - 6Ax + 9A + Bx^2 - 3Bx + Cx$$ Apply the distributive property.

Step 4 Write both sides in descending powers, equate coefficients of like powers of x, and equate constant terms. The left side, $x-18$, is in descending powers of x: $x - 18x^0$. We will write the right side in descending powers of x.

$$x-18 = Ax^2 + Bx^2 - 6Ax - 3Bx + Cx + 9A$$ Rearrange terms on the right side.

Express both sides in the same form.

$$0x^2 + 1x - 18 = (A+B)x^2 + (-6A-3B+C)x + 9A$$ Rewrite to identify coefficients and the constant term.

Equating coefficients of like powers of x and equating constant terms results in the following system of linear equations:

$$\begin{cases} A + B = 0 \\ -6A - 3B + C = 1 \\ 9A = -18. \end{cases}$$

Step 5 Solve the resulting system for A, B, and C. Dividing both sides of the last equation by 9, we obtain $A = -2$. Substituting -2 for A in the first equation, $A + B = 0$, gives $-2 + B = 0$, so $B = 2$. We find C by substituting -2 for A and 2 for B in the middle equation, $-6A - 3B + C = 1$. We obtain $C = -5$.

Step 6 Substitute the values of A, B, and C, and write the partial fraction decomposition. With $A = -2$, $B = 2$, and $C = -5$, the required partial fraction decomposition is

$$\frac{x - 18}{x(x - 3)^2} = \frac{A}{x} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} = -\frac{2}{x} + \frac{2}{x - 3} - \frac{5}{(x - 3)^2}.$$

Check Point 2 Find the partial fraction decomposition of $\dfrac{x + 2}{x(x - 1)^2}$.

3 Decompose $\frac{P}{Q}$, where Q has a nonrepeated prime quadratic factor.

The Partial Fraction Decomposition of a Rational Expression with Prime, Nonrepeated Quadratic Factors in the Denominator

Our final two cases of partial fraction decomposition involve prime quadratic factors of the form $ax^2 + bx + c$. Based on our work with the discriminant, we know that $ax^2 + bx + c$ is prime and cannot be factored over the integers if $b^2 - 4ac < 0$ or if $b^2 - 4ac$ is not a perfect square.

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ Has a Nonrepeated, Prime Quadratic Factor

If $ax^2 + bx + c$ is a prime quadratic factor of $Q(x)$, the partial fraction decomposition will contain a term of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Linear numerator

Quadratic factor

The voice balloons in the box show that each distinct prime quadratic factor in the denominator produces a partial fraction of the form *linear numerator over quadratic factor*. For example,

$$\frac{3x^2 + 17x + 14}{(x - 2)(x^2 + 2x + 4)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 2x + 4}.$$

We write a constant over the linear factor in the denominator.

We write a linear numerator over the prime quadratic factor in the denominator.

Our next example illustrates how a linear system in three variables is used to determine values for A, B, and C.

EXAMPLE 3 Partial Fraction Decomposition

Find the partial fraction decomposition of

$$\frac{3x^2 + 17x + 14}{(x-2)(x^2+2x+4)}.$$

Solution

Step 1 Set up the partial fraction decomposition with the unknown constants. We put a constant (A) over the linear factor and a linear expression $(Bx + C)$ over the prime quadratic factor.

$$\frac{3x^2 + 17x + 14}{(x-2)(x^2+2x+4)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}$$

Step 2 Multiply both sides of the resulting equation by the least common denominator. We clear fractions, multiplying both sides by $(x-2)(x^2+2x+4)$, the least common denominator.

$$(x-2)(x^2+2x+4)\left[\frac{3x^2 + 17x + 14}{(x-2)(x^2+2x+4)}\right] = (x-2)(x^2+2x+4)\left[\frac{A}{x-2} + \frac{Bx+C}{x^2+2x+4}\right]$$

We use the distributive property on the right side.

$$\cancel{(x-2)(x^2+2x+4)}\cdot\frac{3x^2 + 17x + 14}{\cancel{(x-2)(x^2+2x+4)}}$$

$$= \cancel{(x-2)}(x^2+2x+4)\cdot\frac{A}{\cancel{x-2}} + (x-2)\cancel{(x^2+2x+4)}\cdot\frac{Bx+C}{\cancel{x^2+2x+4}}$$

Dividing out common factors in numerators and denominators, we obtain

$$3x^2 + 17x + 14 = A(x^2+2x+4) + (Bx+C)(x-2).$$

Step 3 Simplify the right side of the equation. We simplify on the right side by distributing A over each term in parentheses and multiplying $(Bx + C)(x - 2)$ using the FOIL method.

$$3x^2 + 17x + 14 = Ax^2 + 2Ax + 4A + Bx^2 - 2Bx + Cx - 2C$$

Step 4 Write both sides in descending powers, equate coefficients of like powers of x, and equate constant terms. The left side, $3x^2 + 17x + 14$, is in descending powers of x. We write the right side in descending powers of x

$$3x^2 + 17x + 14 = Ax^2 + Bx^2 + 2Ax - 2Bx + Cx + 4A - 2C$$

and express both sides in the same form.

$$3x^2 + 17x + 14 = (A + B)x^2 + (2A - 2B + C)x + (4A - 2C)$$

Equating coefficients of like powers of x and equating constant terms results in the following system of linear equations:

$$\begin{cases} A + B = 3 \\ 2A - 2B + C = 17 \\ 4A - 2C = 14. \end{cases}$$

Step 5 Solve the resulting system for $A, B,$ and C. Because the first equation involves A and B, we can obtain another equation in A and B by eliminating C from the second and third equations. Multiply the second equation by 2 and add equations. Solving in this manner, we obtain $A = 5$, $B = -2$, and $C = 3$.

Step 6 Substitute the values of $A, B,$ and C, and write the partial fraction decomposition. With $A = 5$, $B = -2$, and $C = 3$, the required partial fraction decomposition is

$$\frac{3x^2 + 17x + 14}{(x - 2)(x^2 + 2x + 4)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 2x + 4} = \frac{5}{x - 2} + \frac{-2x + 3}{x^2 + 2x + 4}.$$

Technology

Numeric Connections

You can use the $\boxed{\text{TABLE}}$ feature of a graphing utility to check a partial fraction decomposition. To check the result of Example 3, enter the given rational function and its partial fraction decomposition:

$$y_1 = \frac{3x^2 + 17x + 14}{(x - 2)(x^2 + 2x + 4)}$$

$$y_2 = \frac{5}{x - 2} + \frac{-2x + 3}{x^2 + 2x + 4}.$$

X	Y1	Y2
-3	.28571	.28571
-2	.5	.5
-1	0	0
0	-1.75	-1.75
1	-4.857	-4.857
2	ERROR	ERROR
3	4.8421	4.8421

X= -3

No matter how far up or down we scroll, $y_1 = y_2$, so the decomposition appears to be correct.

Check Point 3 Find the partial fraction decomposition of

$$\frac{8x^2 + 12x - 20}{(x + 3)(x^2 + x + 2)}.$$

4 Decompose $\frac{P}{Q}$, where Q has a prime, repeated quadratic factor.

The Partial Fraction Decomposition of a Rational Expression with a Prime, Repeated Quadratic Factor in the Denominator

Suppose that $(ax^2 + bx + c)^n$ is a factor of the denominator and that $ax^2 + bx + c$ cannot be factored further. This means that the quadratic factor $ax^2 + bx + c$ occurs n times. When this occurs, the partial fraction decomposition will contain a linear numerator for each power of $ax^2 + bx + c$.

The Partial Fraction Decomposition of $\frac{P(x)}{Q(x)}$: $Q(x)$ Has a Prime, Repeated Quadratic Factor

The form of the partial fraction decomposition for a rational expression containing the prime factor $ax^2 + bx + c$ occurring n times as its denominator is

$$\frac{P(x)}{(ax^2 + bx + c)^n} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \frac{A_3x + B_3}{(ax^2 + bx + c)^3} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n}.$$

Include one fraction with a linear numerator for each power of $ax^2 + bx + c$.

EXAMPLE 4 Partial Fraction Decomposition with a Repeated Quadratic Factor

Find the partial fraction decomposition of

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2}.$$

Solution

Step 1 Set up the partial fraction decomposition with the unknown constants. Because the quadratic factor $x^2 + 1$ occurs twice, we must include one fraction with a linear numerator for each power of $x^2 + 1$.

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

Step 2 Multiply both sides of the resulting equation by the least common denominator. We clear fractions, multiplying both sides by $(x^2 + 1)^2$, the least common denominator.

$$(x^2 + 1)^2\left[\frac{5x^3 - 3x^2 + 7x - 3}{(x^2 + 1)^2}\right] = (x^2 + 1)^2\left[\frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}\right]$$

Now we multiply and simplify.

$$5x^3 - 3x^2 + 7x - 3 = (x^2 + 1)(Ax + B) + Cx + D$$

Step 3 Simplify the right side of the equation. We multiply $(x^2 + 1)(Ax + B)$ using the FOIL method.

$$5x^3 - 3x^2 + 7x - 3 = Ax^3 + Bx^2 + Ax + B + Cx + D$$

Step 4 Write both sides in descending powers, equate coefficients of like powers of x, and equate constant terms.

$$5x^3 - 3x^2 + 7x - 3 = Ax^3 + Bx^2 + Ax + Cx + B + D$$

$$5x^3 - 3x^2 + 7x - 3 = Ax^3 + Bx^2 + (A + C)x + (B + D)$$

Equating coefficients of like powers of x and equating constant terms results in the following system of linear equations:

$$\begin{cases} A = 5 \\ B = -3 \\ A + C = 7 & \text{With } A = 5\text{, we immediately obtain } C = 2. \\ B + D = -3. & \text{With } B = -3\text{, we immediately obtain } D = 0. \end{cases}$$

Study Tip

When the denominator of a rational expression contains a **power of a linear factor**, set up the partial fraction decomposition with **constant numerators** (A, B, C, etc.). When the denominator of a rational expression contains a **power of a prime quadratic factor**, set up the partial fraction decomposition with **linear numerators** ($Ax + B$, $Cx + D$, etc.).

Step 5 Solve the resulting system for A, B, C, and D. Based on our observations in step 4, $A = 5$, $B = -3$, $C = 2$, and $D = 0$.

Step 6 Substitute the values of A, B, C, and D, and write the partial fraction decomposition.

$$\frac{5x^3 - 3x^2 + 7x - 3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} = \frac{5x-3}{x^2+1} + \frac{2x}{(x^2+1)^2}$$

Check Point 4 Find the partial fraction decomposition of $\dfrac{2x^3 + x + 3}{(x^2+1)^2}$.

Study Tip

When a rational expression contains a power of a factor in the denominator, be sure to set up the partial fraction decomposition to allow for every natural-number power of that factor less than or equal to the power. Example:

$$\frac{2x+1}{(x-5)^2x^3}$$

$$= \frac{A}{x-5} + \frac{B}{(x-5)^2} + \frac{C}{x} + \frac{D}{x^2} + \frac{E}{x^3}$$

Although $(x-5)^2$ and x^2 are quadratic, they are still expressed as powers of linear factors, $x - 5$ and x. Thus, the numerators are constant.

Exercise Set 7.3

Practice Exercises

In Exercises 1–8, write the form of the partial fraction decomposition of the rational expression. It is not necessary to solve for the constants.

1. $\dfrac{11x - 10}{(x-2)(x+1)}$

2. $\dfrac{5x+7}{(x-1)(x+3)}$

3. $\dfrac{6x^2 - 14x - 27}{(x+2)(x-3)^2}$

4. $\dfrac{3x+16}{(x+1)(x-2)^2}$

5. $\dfrac{5x^2 - 6x + 7}{(x-1)(x^2+1)}$

6. $\dfrac{5x^2 - 9x + 19}{(x-4)(x^2+5)}$

7. $\dfrac{x^3 + x^2}{(x^2+4)^2}$

8. $\dfrac{7x^2 - 9x + 3}{(x^2+7)^2}$

In Exercises 9–42, write the partial fraction decomposition of each rational expression.

9. $\dfrac{x}{(x-3)(x-2)}$

10. $\dfrac{1}{x(x-1)}$

11. $\dfrac{3x+50}{(x-9)(x+2)}$

12. $\dfrac{5x-1}{(x-2)(x+1)}$

13. $\dfrac{7x-4}{x^2 - x - 12}$

14. $\dfrac{9x+21}{x^2+2x-15}$

15. $\dfrac{4}{2x^2 - 5x - 3}$

16. $\dfrac{x}{x^2+2x-3}$

17. $\dfrac{4x^2 + 13x - 9}{x(x-1)(x+3)}$

18. $\dfrac{4x^2 - 5x - 15}{x(x+1)(x-5)}$

19. $\dfrac{4x^2 - 7x - 3}{x^3 - x}$

20. $\dfrac{2x^2 - 18x - 12}{x^3 - 4x}$

21. $\dfrac{6x - 11}{(x-1)^2}$

22. $\dfrac{x}{(x+1)^2}$

23. $\dfrac{x^2 - 6x + 3}{(x-2)^3}$

24. $\dfrac{2x^2 + 8x + 3}{(x+1)^3}$

25. $\dfrac{x^2 + 2x + 7}{x(x-1)^2}$

26. $\dfrac{3x^2 + 49}{x(x+7)^2}$

27. $\dfrac{x^2}{(x-1)^2(x+1)}$

28. $\dfrac{x^2}{(x-1)^2(x+1)^2}$

29. $\dfrac{5x^2 - 6x + 7}{(x - 1)(x^2 + 1)}$

30. $\dfrac{5x^2 - 9x + 19}{(x - 4)(x^2 + 5)}$

31. $\dfrac{5x^2 + 6x + 3}{(x + 1)(x^2 + 2x + 2)}$

32. $\dfrac{9x + 2}{(x - 2)(x^2 + 2x + 2)}$

33. $\dfrac{x + 4}{x^2(x^2 + 4)}$

34. $\dfrac{10x^2 + 2x}{(x - 1)^2(x^2 + 2)}$

35. $\dfrac{6x^2 - x + 1}{x^3 + x^2 + x + 1}$

36. $\dfrac{3x^2 - 2x + 8}{x^3 + 2x^2 + 4x + 8}$

37. $\dfrac{x^3 + x^2 + 2}{(x^2 + 2)^2}$

38. $\dfrac{x^2 + 2x + 3}{(x^2 + 4)^2}$

39. $\dfrac{x^3 - 4x^2 + 9x - 5}{(x^2 - 2x + 3)^2}$

40. $\dfrac{3x^3 - 6x^2 + 7x - 2}{(x^2 - 2x + 2)^2}$

41. $\dfrac{4x^2 + 3x + 14}{x^3 - 8}$

42. $\dfrac{3x - 5}{x^3 - 1}$

Practice Plus

In Exercises 43–46, perform each long division and write the partial fraction decomposition of the remainder term.

43. $\dfrac{x^5 + 2}{x^2 - 1}$

44. $\dfrac{x^5}{x^2 - 4x + 4}$

45. $\dfrac{x^4 - x^2 + 2}{x^3 - x^2}$

46. $\dfrac{x^4 + 2x^3 - 4x^2 + x - 3}{x^2 - x - 2}$

In Exercises 47–50, write the partial fraction decomposition of each rational expression.

47. $\dfrac{1}{x^2 - c^2} \quad (c \neq 0)$

48. $\dfrac{ax + b}{x^2 - c^2} \quad (c \neq 0)$

49. $\dfrac{ax + b}{(x - c)^2} \quad (c \neq 0)$

50. $\dfrac{1}{x^2 - ax - bx + ab} \quad (a \neq b)$

Application Exercises

51. Find the partial fraction decomposition for $\dfrac{1}{x(x + 1)}$ and use the result to find the following sum:

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{99 \cdot 100}.$$

52. Find the partial fraction decomposition for $\dfrac{2}{x(x + 2)}$ and use the result to find the following sum:

$$\frac{2}{1 \cdot 3} + \frac{2}{3 \cdot 5} + \frac{2}{5 \cdot 7} + \cdots + \frac{2}{99 \cdot 101}.$$

Writing in Mathematics

53. Explain what is meant by the partial fraction decomposition of a rational expression.

54. Explain how to find the partial fraction decomposition of a rational expression with distinct linear factors in the denominator.

55. Explain how to find the partial fraction decomposition of a rational expression with a repeated linear factor in the denominator.

56. Explain how to find the partial fraction decomposition of a rational expression with a prime quadratic factor in the denominator.

57. Explain how to find the partial fraction decomposition of a rational expression with a repeated, prime quadratic factor in the denominator.

58. How can you verify your result for the partial fraction decomposition for a given rational expression without using a graphing utility?

Technology Exercise

59. Use the $\boxed{\text{TABLE}}$ feature of a graphing utility to verify any three of the decompositions that you obtained in Exercises 9–42.

Critical Thinking Exercises

Make Sense? *In Exercises 60–63, determine whether each statement makes sense or does not make sense, and explain your reasoning.*

60. Partial fraction decomposition involves finding a single rational expression for a given sum or difference of rational expressions.

61. I apply partial fraction decompositions for rational expressions of the form $\dfrac{P(x)}{Q(x)}$, where P and Q have no common factors and the degree of P is greater than the degree of Q.

62. Because $x + 5$ is linear and $x^2 - 3x + 2$ is quadratic, I set up the following partial fraction decomposition:

$$\frac{7x^2 + 9x + 3}{(x + 5)(x^2 - 3x + 2)} = \frac{A}{x + 5} + \frac{Bx + C}{x^2 - 3x + 2}.$$

63. Because $(x + 3)^2$ consists of two factors of $x + 3$, I set up the following partial fraction decomposition:

$$\frac{5x + 2}{(x + 3)^2} = \frac{A}{x + 3} + \frac{B}{x + 3}.$$

64. Use an extension of the Study Tip on page 764 to describe how to set up the partial fraction decomposition of a rational expression that contains powers of a prime cubic factor in the denominator. Give an example of such a decomposition.

65. Find the partial fraction decomposition of

$$\frac{4x^2 + 5x - 9}{x^3 - 6x - 9}.$$

Preview Exercises

Exercises 66–68 will help you prepare for the material covered in the next section.

66. Solve by the substitution method:

$$\begin{cases} 4x + 3y = 4 \\ y = 2x - 7. \end{cases}$$

67. Solve by the addition method:

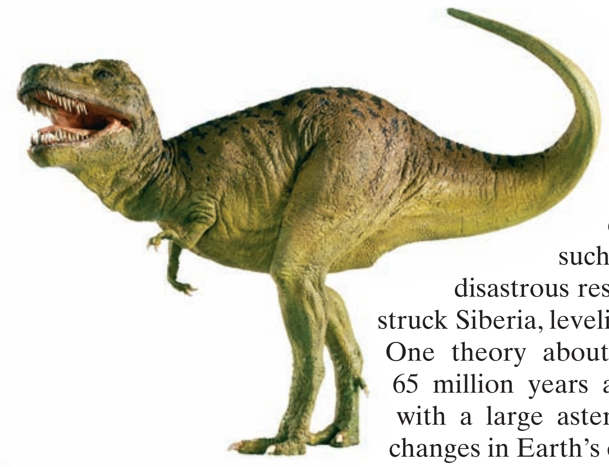
$$\begin{cases} 2x + 4y = -4 \\ 3x + 5y = -3. \end{cases}$$

68. Graph $x - y = 3$ and $(x - 2)^2 + (y + 3)^2 = 4$ in the same rectangular coordinate system. What are the two intersection points? Show that each of these ordered pairs satisfies both equations.

Section 7.4 Systems of Nonlinear Equations in Two Variables

Objectives

1. Recognize systems of nonlinear equations in two variables.
2. Solve nonlinear systems by substitution.
3. Solve nonlinear systems by addition.
4. Solve problems using systems of nonlinear equations.



Scientists debate the probability that a "doomsday rock" will collide with Earth. It has been estimated that an asteroid, a tiny planet that revolves around the sun, crashes into Earth about once every 250,000 years, and that such a collision would have disastrous results. In 1908, a small fragment struck Siberia, leveling thousands of acres of trees. One theory about the extinction of dinosaurs 65 million years ago involves Earth's collision with a large asteroid and the resulting drastic changes in Earth's climate.

Understanding the path of Earth and the path of a comet is essential to detecting threatening space debris. Orbits about the sun are not described by linear equations in the form $Ax + By = C$. The ability to solve systems that contain nonlinear equations provides NASA scientists watching for troublesome asteroids with a way to locate possible collision points with Earth's orbit.

1 Recognize systems of nonlinear equations in two variables.

Systems of Nonlinear Equations and Their Solutions

A **system of** two **nonlinear equations** in two variables, also called a **nonlinear system**, contains at least one equation that cannot be expressed in the form $Ax + By = C$. Here are two examples:

$$\begin{cases} x^2 = 2y + 10 \\ 3x - y = 9 \end{cases}$$

Not in the form $Ax + By = C$. The term x^2 is not linear.

$$\begin{cases} y = x^2 + 3 \\ x^2 + y^2 = 9. \end{cases}$$

Neither equation is in the form $Ax + By = C$. The terms x^2 and y^2 are not linear.

A **solution** of a nonlinear system in two variables is an ordered pair of real numbers that satisfies both equations in the system. The **solution set** of the system is the set of all such ordered pairs. As with linear systems in two variables, the solution of a nonlinear system (if there is one) corresponds to the intersection point(s) of the graphs of the equations in the system. Unlike linear systems, the graphs can be circles, parabolas, or anything other than two lines. We will solve nonlinear systems using the substitution method and the addition method.