Verifying Trigonometric Identities

What you should learn

- Verify trigonometric identities.

Why you should learn it

You can use trigonometric identities to rewrite trigonometric equations that model real-life situations. For instance, in Exercise 56, you can use trigonometric identities to simplify the equation that models the length of a shadow cast by a gnomon (a device used to tell time).

Introduction

In this section, you will study techniques for verifying trigonometric identities. In the next section, you will study techniques for solving trigonometric equations. The key to verifying identities *and* solving equations is the ability to use the fundamental identities and the rules of algebra to rewrite trigonometric expressions.

Remember that a *conditional equation* is an equation that is true for only some of the values in its domain. For example, the conditional equation

$$\sin x = 0 \qquad \text{Conditional equation}$$

is true only for $x = n\pi$, where n is an integer. When you find these values, you are *solving* the equation.

On the other hand, an equation that is true for all real values in the domain of the variable is an *identity*. For example, the familiar equation

$$\sin^2 x = 1 - \cos^2 x \qquad \text{Identity}$$

is true for all real numbers x. So, it is an identity.

Video

Verifying Trigonometric Identities

Although there are similarities, verifying that a trigonometric equation is an identity is quite different from solving an equation. There is no well-defined set of rules to follow in verifying trigonometric identities, and the process is best learned by practice.

Guidelines for Verifying Trigonometric Identities

1. Work with one side of the equation at a time. It is often better to work with the more complicated side first.
2. Look for opportunities to factor an expression, add fractions, square a binomial, or create a monomial denominator.
3. Look for opportunities to use the fundamental identities. Note which functions are in the final expression you want. Sines and cosines pair up well, as do secants and tangents, and cosecants and cotangents.
4. If the preceding guidelines do not help, try converting all terms to sines and cosines.
5. Always try *something*. Even paths that lead to dead ends provide insights.

Verifying trigonometric identities is a useful process if you need to convert a trigonometric expression into a form that is more useful algebraically. When you verify an identity, you cannot *assume* that the two sides of the equation are equal because you are trying to verify that they *are* equal. As a result, when verifying identities, you cannot use operations such as adding the same quantity to each side of the equation or cross multiplication.

Video

Example 1 Verifying a Trigonometric Identity

Verify the identity $\dfrac{\sec^2\theta - 1}{\sec^2\theta} = \sin^2\theta$.

Solution

Because the left side is more complicated, start with it.

$$\begin{aligned}
\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{(\tan^2\theta + 1) - 1}{\sec^2\theta} && \text{Pythagorean identity} \\
&= \frac{\tan^2\theta}{\sec^2\theta} && \text{Simplify.} \\
&= \tan^2\theta(\cos^2\theta) && \text{Reciprocal identity} \\
&= \frac{\sin^2\theta}{\cancel{(\cos^2\theta)}}\cancel{(\cos^2\theta)} && \text{Quotient identity} \\
&= \sin^2\theta && \text{Simplify.}
\end{aligned}$$

Notice how the identity is verified. You start with the left side of the equation (the more complicated side) and use the fundamental trigonometric identities to simplify it until you obtain the right side.

CHECKPOINT Now try Exercise 5.

There is more than one way to verify an identity. Here is another way to verify the identity in Example 1.

$$\begin{aligned}
\frac{\sec^2\theta - 1}{\sec^2\theta} &= \frac{\sec^2\theta}{\sec^2\theta} - \frac{1}{\sec^2\theta} && \text{Rewrite as the difference of fractions.} \\
&= 1 - \cos^2\theta && \text{Reciprocal identity} \\
&= \sin^2\theta && \text{Pythagorean identity}
\end{aligned}$$

STUDY TIP

Remember that an identity is only true for all real values in the domain of the variable. For instance, in Example 1 the identity is not true when $\theta = \pi/2$ because $\sec^2\theta$ is not defined when $\theta = \pi/2$.

Example 2 Combining Fractions Before Using Identities

Verify the identity $\dfrac{1}{1 - \sin\alpha} + \dfrac{1}{1 + \sin\alpha} = 2\sec^2\alpha$.

Solution

$$\begin{aligned}
\frac{1}{1 - \sin\alpha} + \frac{1}{1 + \sin\alpha} &= \frac{1 + \sin\alpha + 1 - \sin\alpha}{(1 - \sin\alpha)(1 + \sin\alpha)} && \text{Add fractions.} \\
&= \frac{2}{1 - \sin^2\alpha} && \text{Simplify.} \\
&= \frac{2}{\cos^2\alpha} && \text{Pythagorean identity} \\
&= 2\sec^2\alpha && \text{Reciprocal identity}
\end{aligned}$$

CHECKPOINT Now try Exercise 19.

Example 3 Verifying Trigonometric Identity

Verify the identity $(\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x$.

Algebraic Solution

By applying identities before multiplying, you obtain the following.

$$\begin{aligned}
(\tan^2 x + 1)(\cos^2 x - 1) &= (\sec^2 x)(-\sin^2 x) && \text{Pythagorean identities}\\
&= -\frac{\sin^2 x}{\cos^2 x} && \text{Reciprocal identity}\\
&= -\left(\frac{\sin x}{\cos x}\right)^2 && \text{Rule of exponents}\\
&= -\tan^2 x && \text{Quotient identity}
\end{aligned}$$

CHECKPOINT Now try Exercise 39.

Numerical Solution

Use the *table* feature of a graphing utility set in *radian* mode to create a table that shows the values of $y_1 = (\tan^2 x + 1)(\cos^2 x - 1)$ and $y_2 = -\tan^2 x$ for different values of x, as shown in Figure 2. From the table you can see that the values of y_1 and y_2 appear to be identical, so $(\tan^2 x + 1)(\cos^2 x - 1) = -\tan^2 x$ appears to be an identity.

X	Y1	Y2
-.5	-.2984	-.2984
-.25	-.0652	-.0652
0	0	0
.25	-.0652	-.0652
.5	-.2984	-.2984
.75	-.8679	-.8679
1	-2.426	-2.426

X=-.5

FIGURE 2

Example 4 Converting to Sines and Cosines

Verify the identity $\tan x + \cot x = \sec x \csc x$.

Solution

Try converting the left side into sines and cosines.

$$\begin{aligned}
\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} && \text{Quotient identities}\\
&= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x} && \text{Add fractions.}\\
&= \frac{1}{\cos x \sin x} && \text{Pythagorean identity}\\
&= \frac{1}{\cos x} \cdot \frac{1}{\sin x} = \sec x \csc x && \text{Reciprocal identities}
\end{aligned}$$

CHECKPOINT Now try Exercise 29.

> **STUDY TIP**
>
> Although a graphing utility can be useful in helping to verify an identity, you must use algebraic techniques to produce a *valid* proof.

Recall from algebra that *rationalizing the denominator* using conjugates is, on occasion, a powerful simplification technique. A related form of this technique, shown below, works for simplifying trigonometric expressions as well.

$$\begin{aligned}
\frac{1}{1 - \cos x} &= \frac{1}{1 - \cos x}\left(\frac{1 + \cos x}{1 + \cos x}\right) = \frac{1 + \cos x}{1 - \cos^2 x} = \frac{1 + \cos x}{\sin^2 x}\\
&= \csc^2 x(1 + \cos x)
\end{aligned}$$

This technique is demonstrated in the next example.

> **STUDY TIP**
>
> As shown at the right, $\csc^2 x(1 + \cos x)$ is considered a simplified form of $1/(1 - \cos x)$ because the expression does not contain any fractions.

Example 5 **Verifying Trigonometric Identities**

Verify the identity $\sec y + \tan y = \dfrac{\cos y}{1 - \sin y}$.

Solution

Begin with the *right* side, because you can create a monomial denominator by multiplying the numerator and denominator by $1 + \sin y$.

$$\frac{\cos y}{1 - \sin y} = \frac{\cos y}{1 - \sin y}\left(\frac{1 + \sin y}{1 + \sin y}\right) \quad \text{Multiply numerator and denominator by } 1 + \sin y.$$

$$= \frac{\cos y + \cos y \sin y}{1 - \sin^2 y} \quad \text{Multiply.}$$

$$= \frac{\cos y + \cos y \sin y}{\cos^2 y} \quad \text{Pythagorean identity}$$

$$= \frac{\cos y}{\cos^2 y} + \frac{\cos y \sin y}{\cos^2 y} \quad \text{Write as separate fractions.}$$

$$= \frac{1}{\cos y} + \frac{\sin y}{\cos y} \quad \text{Simplify.}$$

$$= \sec y + \tan y \quad \text{Identities}$$

CHECKPOINT Now try Exercise 33.

In Examples 1 through 5, you have been verifying trigonometric identities by working with one side of the equation and converting to the form given on the other side. On occasion, it is practical to work with each side *separately*, to obtain one common form equivalent to both sides. This is illustrated in Example 6.

Example 6 **Working with Each Side Separately**

Verify the identity $\dfrac{\cot^2 \theta}{1 + \csc \theta} = \dfrac{1 - \sin \theta}{\sin \theta}$.

Solution

Working with the left side, you have

$$\frac{\cot^2 \theta}{1 + \csc \theta} = \frac{\csc^2 \theta - 1}{1 + \csc \theta} \quad \text{Pythagorean identity}$$

$$= \frac{(\csc \theta - 1)\cancel{(\csc \theta + 1)}}{\cancel{1 + \csc \theta}} \quad \text{Factor.}$$

$$= \csc \theta - 1. \quad \text{Simplify.}$$

Now, simplifying the right side, you have

$$\frac{1 - \sin \theta}{\sin \theta} = \frac{1}{\sin \theta} - \frac{\sin \theta}{\sin \theta} \quad \text{Write as separate fractions.}$$

$$= \csc \theta - 1. \quad \text{Reciprocal identity}$$

The identity is verified because both sides are equal to $\csc \theta - 1$.

CHECKPOINT Now try Exercise 47.

In Example 7, powers of trigonometric functions are rewritten as more complicated sums of products of trigonometric functions. This is a common procedure used in calculus.

Example 7 **Three Examples from Calculus**

Verify each identity.

a. $\tan^4 x = \tan^2 x \sec^2 x - \tan^2 x$

b. $\sin^3 x \cos^4 x = (\cos^4 x - \cos^6 x) \sin x$

c. $\csc^4 x \cot x = \csc^2 x(\cot x + \cot^3 x)$

Solution

a.
$$\begin{aligned} \tan^4 x &= (\tan^2 x)(\tan^2 x) && \text{Write as separate factors.} \\ &= \tan^2 x(\sec^2 x - 1) && \text{Pythagorean identity} \\ &= \tan^2 x \sec^2 x - \tan^2 x && \text{Multiply.} \end{aligned}$$

b.
$$\begin{aligned} \sin^3 x \cos^4 x &= \sin^2 x \cos^4 x \sin x && \text{Write as separate factors.} \\ &= (1 - \cos^2 x)\cos^4 x \sin x && \text{Pythagorean identity} \\ &= (\cos^4 x - \cos^6 x) \sin x && \text{Multiply.} \end{aligned}$$

c.
$$\begin{aligned} \csc^4 x \cot x &= \csc^2 x \csc^2 x \cot x && \text{Write as separate factors.} \\ &= \csc^2 x(1 + \cot^2 x) \cot x && \text{Pythagorean identity} \\ &= \csc^2 x(\cot x + \cot^3 x) && \text{Multiply.} \end{aligned}$$

CHECKPOINT Now try Exercise 49.

WRITING ABOUT MATHEMATICS

Error Analysis You are tutoring a student in trigonometry. One of the homework problems your student encounters asks whether the following statement is an identity.

$$\tan^2 x \sin^2 x \stackrel{?}{=} \frac{5}{6} \tan^2 x$$

Your student does not attempt to verify the equivalence algebraically, but mistakenly uses only a graphical approach. Using range settings of

Xmin $= -3\pi$	Ymin $= -20$
Xmax $= 3\pi$	Ymax $= 20$
Xscl $= \pi/2$	Yscl $= 1$

your student graphs both sides of the expression on a graphing utility and concludes that the statement is an identity.

What is wrong with your student's reasoning? Explain. Discuss the limitations of verifying identities graphically.

Exercises

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system.

Click on S to view the complete solution of the exercise.

Click on M to print an enlarged copy of the graph.

Click on D to view the Make a Decision exercise.

VOCABULARY CHECK: Glossary

In Exercises 1 and 2, fill in the blanks.

1. An equation that is true for all real values in its domain is called an ________.
2. An equation that is true for only some values in its domain is called a ________ ________.

In Exercises 3–8, fill in the blank to complete the trigonometric identity.

3. $\dfrac{1}{\cot u} =$ ________
4. $\dfrac{\cos u}{\sin u} =$ ________
5. $\sin^2 u +$ ________ $= 1$
6. $\cos\left(\dfrac{\pi}{2} - u\right) =$ ________
7. $\csc(-u) =$ ________
8. $\sec(-u) =$ ________

In Exercises 1–38, verify the identity.

1. $\sin t \csc t = 1$
2. $\sec y \cos y = 1$
3. $(1 + \sin \alpha)(1 - \sin \alpha) = \cos^2 \alpha$
4. $\cot^2 y(\sec^2 y - 1) = 1$
5. $\cos^2 \beta - \sin^2 \beta = 1 - 2\sin^2 \beta$
6. $\cos^2 \beta - \sin^2 \beta = 2\cos^2 \beta - 1$
7. $\sin^2 \alpha - \sin^4 \alpha = \cos^2 \alpha - \cos^4 \alpha$
8. $\cos x + \sin x \tan x = \sec x$
9. $\dfrac{\csc^2 \theta}{\cot \theta} = \csc \theta \sec \theta$
10. $\dfrac{\cot^3 t}{\csc t} = \cos t(\csc^2 t - 1)$
11. $\dfrac{\cot^2 t}{\csc t} = \csc t - \sin t$
12. $\dfrac{1}{\tan \beta} + \tan \beta = \dfrac{\sec^2 \beta}{\tan \beta}$
13. $\sin^{1/2} x \cos x - \sin^{5/2} x \cos x = \cos^3 x\sqrt{\sin x}$
14. $\sec^6 x(\sec x \tan x) - \sec^4 x(\sec x \tan x) = \sec^5 x \tan^3 x$
15. $\dfrac{1}{\sec x \tan x} = \csc x - \sin x$
16. $\dfrac{\sec \theta - 1}{1 - \cos \theta} = \sec \theta$
17. $\csc x - \sin x = \cos x \cot x$
18. $\sec x - \cos x = \sin x \tan x$
19. $\dfrac{1}{\tan x} + \dfrac{1}{\cot x} = \tan x + \cot x$
20. $\dfrac{1}{\sin x} - \dfrac{1}{\csc x} = \csc x - \sin x$
21. $\dfrac{\cos \theta \cot \theta}{1 - \sin \theta} - 1 = \csc \theta$
22. $\dfrac{1 + \sin \theta}{\cos \theta} + \dfrac{\cos \theta}{1 + \sin \theta} = 2\sec \theta$
23. $\dfrac{1}{\sin x + 1} + \dfrac{1}{\csc x + 1} = 1$
24. $\cos x - \dfrac{\cos x}{1 - \tan x} = \dfrac{\sin x \cos x}{\sin x - \cos x}$
25. $\tan\left(\dfrac{\pi}{2} - \theta\right)\tan \theta = 1$
26. $\dfrac{\cos[(\pi/2) - x]}{\sin[(\pi/2) - x]} = \tan x$
27. $\dfrac{\csc(-x)}{\sec(-x)} = -\cot x$
28. $(1 + \sin y)[1 + \sin(-y)] = \cos^2 y$
29. $\dfrac{\tan x \cot x}{\cos x} = \sec x$
30. $\dfrac{\tan x + \tan y}{1 - \tan x \tan y} = \dfrac{\cot x + \cot y}{\cot x \cot y - 1}$
31. $\dfrac{\tan x + \cot y}{\tan x \cot y} = \tan y + \cot x$
32. $\dfrac{\cos x - \cos y}{\sin x + \sin y} + \dfrac{\sin x - \sin y}{\cos x + \cos y} = 0$
33. $\sqrt{\dfrac{1 + \sin \theta}{1 - \sin \theta}} = \dfrac{1 + \sin \theta}{|\cos \theta|}$
34. $\sqrt{\dfrac{1 - \cos \theta}{1 + \cos \theta}} = \dfrac{1 - \cos \theta}{|\sin \theta|}$
35. $\cos^2 \beta + \cos^2\left(\dfrac{\pi}{2} - \beta\right) = 1$
36. $\sec^2 y - \cot^2\left(\dfrac{\pi}{2} - y\right) = 1$
37. $\sin t \csc\left(\dfrac{\pi}{2} - t\right) = \tan t$
38. $\sec^2\left(\dfrac{\pi}{2} - x\right) - 1 = \cot^2 x$

In Exercises 39–46, (a) use a graphing utility to graph each side of the equation to determine whether the equation is an identity, (b) use the *table* feature of a graphing utility to determine whether the equation is an identity, and (c) confirm the results of parts (a) and (b) algebraically.

39. $2\sec^2 x - 2\sec^2 x \sin^2 x - \sin^2 x - \cos^2 x = 1$

40. $\csc x(\csc x - \sin x) + \dfrac{\sin x - \cos x}{\sin x} + \cot x = \csc^2 x$

41. $2 + \cos^2 x - 3\cos^4 x = \sin^2 x(3 + 2\cos^2 x)$

42. $\tan^4 x + \tan^2 x - 3 = \sec^2 x(4\tan^2 x - 3)$

43. $\csc^4 x - 2\csc^2 x + 1 = \cot^4 x$

44. $(\sin^4 \beta - 2\sin^2 \beta + 1)\cos \beta = \cos^5 \beta$

45. $\dfrac{\cos x}{1 - \sin x} = \dfrac{1 - \sin x}{\cos x}$

46. $\dfrac{\cot \alpha}{\csc \alpha + 1} = \dfrac{\csc \alpha + 1}{\cot \alpha}$

In Exercises 47–50, verify the identity.

47. $\tan^5 x = \tan^3 x \sec^2 x - \tan^3 x$

48. $\sec^4 x \tan^2 x = (\tan^2 x + \tan^4 x)\sec^2 x$

49. $\cos^3 x \sin^2 x = (\sin^2 x - \sin^4 x)\cos x$

50. $\sin^4 x + \cos^4 x = 1 - 2\cos^2 x + 2\cos^4 x$

In Exercises 51–54, use the cofunction identities to evaluate the expression without the aid of a calculator.

51. $\sin^2 25° + \sin^2 65°$

52. $\cos^2 55° + \cos^2 35°$

53. $\cos^2 20° + \cos^2 52° + \cos^2 38° + \cos^2 70°$

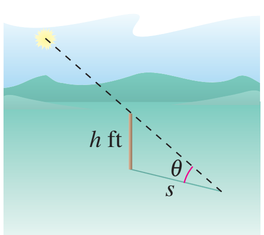
54. $\sin^2 12° + \sin^2 40° + \sin^2 50° + \sin^2 78°$

55. ***Rate of Change*** The rate of change of the function $f(x) = \sin x + \csc x$ with respect to change in the variable x is given by the expression $\cos x - \csc x \cot x$. Show that the expression for the rate of change can also be $-\cos x \cot^2 x$.

Model It

56. ***Shadow Length*** The length s of a shadow cast by a vertical gnomon (a device used to tell time) of height h when the angle of the sun above the horizon is θ (see figure) can be modeled by the equation

$$s = \frac{h\sin(90° - \theta)}{\sin \theta}.$$



Model It (continued)

(a) Verify that the equation for s is equal to $h \cot \theta$.

(b) Use a graphing utility to complete the table. Let $h = 5$ feet.

θ	10°	20°	30°	40°	50°
s					

θ	60°	70°	80°	90°
s				

(c) Use your table from part (b) to determine the angles of the sun for which the length of the shadow is the greatest and the least.

(d) Based on your results from part (c), what time of day do you think it is when the angle of the sun above the horizon is 90°?

Synthesis

True or False? **In Exercises 57 and 58, determine whether the statement is true or false. Justify your answer.**

57. The equation $\sin^2 \theta + \cos^2 \theta = 1 + \tan^2 \theta$ is an identity, because $\sin^2(0) + \cos^2(0) = 1$ and $1 + \tan^2(0) = 1$.

58. The equation $1 + \tan^2 \theta = 1 + \cot^2 \theta$ is *not* an identity, because it is true that $1 + \tan^2(\pi/6) = 1\frac{1}{3}$, and $1 + \cot^2(\pi/6) = 4$.

Think About It **In Exercises 59 and 60, explain why the equation is not an identity and find one value of the variable for which the equation is not true.**

59. $\sin \theta = \sqrt{1 - \cos^2 \theta}$

60. $\tan \theta = \sqrt{\sec^2 \theta - 1}$

Skills Review

In Exercises 61–64, perform the operation and simplify.

61. $(2 + 3i) - \sqrt{-26}$

62. $(2 - 5i)^2$

63. $\sqrt{-16}\left(1 + \sqrt{-4}\right)$

64. $(3 + 2i)^3$

In Exercises 65–68, use the Quadratic Formula to solve the quadratic equation.

65. $x^2 + 6x - 12 = 0$

66. $x^2 + 5x - 7 = 0$

67. $3x^2 - 6x - 12 = 0$

68. $8x^2 - 4x - 3 = 0$