Graph each function.

16. $y = \left(\frac{1}{2}\right)^x$

17. $y = 2^x + 1$

18. $y = \log_3 x$

19. $y = \log_{1/3} x$

Solve.

20. $2^x = 8$

21. $9 = 3^{x-5}$

22. $4^{x-1} = 8^{x+2}$

23. $25^x = 125^{x-1}$

24. $\log_4 16 = x$

25. $\log_{49} 7 = x$

26. $\log_2 x = 5$

27. $\log_x 64 = 3$

28. $\log_x \frac{1}{125} = -3$

29. $\log_3 x = -2$

Write each as a single logarithm.

30. $5 \log_2 x$

31. $x \log_2 5$

32. $3 \log_5 x - 5 \log_5 y$

33. $9 \log_5 x + 3 \log_5 y$

34. $\log_2 x + \log_2(x - 3) - \log_2(x^2 + 4)$

35. $\log_3 y - \log_3(y + 2) + \log_3(y^3 + 11)$

Write each expression as sums or differences of multiples of logarithms.

36. $\log_7 \frac{9x^2}{y}$

37. $\log_6 \frac{5y}{z^2}$

38. An unusually wet spring has caused the size of the mosquito population in a community to increase by 6% each day. If an estimated 100,000 mosquitoes are in the community on April 1, find how many mosquitoes will inhabit the community on April 17. Round to the nearest thousand.

9.7 Common Logarithms, Natural Logarithms, and Change of Base

OBJECTIVES

1. Identify Common Logarithms and Approximate Them by Calculator.
2. Evaluate Common Logarithms of Powers of 10.
3. Identify Natural Logarithms and Approximate Them by Calculator.
4. Evaluate Natural Logarithms of Powers of *e*.
5. Use the Change of Base Formula.

In this section, we look closely at two particular logarithmic bases. These two logarithmic bases are used so frequently that logarithms to their bases are given special names. **Common logarithms** are logarithms to base 10. **Natural logarithms** are logarithms to base *e*, which we introduce in this section. The work in this section is based on the use of a calculator that has both the common "log" LOG and the natural "log" LN keys.

OBJECTIVE 1

Approximating Common Logarithms

Logarithms to base 10, common logarithms, are used frequently because our number system is a base 10 decimal system. The notation $\log x$ means the same as $\log_{10} x$.

Common Logarithms

$$\log x \text{ means } \log_{10} x$$

EXAMPLE 1 Use a calculator to approximate log 7 to four decimal places.

Solution Press the following sequence of keys.

7 LOG or LOG 7 ENTER

To four decimal places,

$$\log 7 \approx 0.8451$$

PRACTICE 1 Use a calculator to approximate log 15 to four decimal places.

OBJECTIVE

2 Evaluating Common Logarithms of Powers of 10

To evaluate the common log of a power of 10, a calculator is not needed. According to the property of logarithms,

$$\log_b b^x = x$$

It follows that if b is replaced with 10, we have

$$\log 10^x = x$$

Helpful Hint

Remember that the understood base here is 10.

EXAMPLE 2 Find the exact value of each logarithm.

a. $\log 10$ **b.** $\log 1000$ **c.** $\log \frac{1}{10}$ **d.** $\log \sqrt{10}$

Solution

a. $\log 10 = \log 10^1 = 1$ **b.** $\log 1000 = \log 10^3 = 3$

c. $\log \frac{1}{10} = \log 10^{-1} = -1$ **d.** $\log \sqrt{10} = \log 10^{1/2} = \frac{1}{2}$ □

PRACTICE

2 Find the exact value of each logarithm.

a. $\log \frac{1}{100}$ **b.** $\log 100{,}000$ **c.** $\log \sqrt[5]{10}$ **d.** $\log 0.001$

As we will soon see, equations containing common logarithms are useful models of many natural phenomena.

EXAMPLE 3 Solve $\log x = 1.2$ for x. Give an exact solution and then approximate the solution to four decimal places.

Solution Remember that the base of a common logarithm is understood to be 10.

$$\log x = 1.2$$

$$10^{1.2} = x \quad \text{Write with exponential notation.}$$

Helpful Hint

The understood base is 10.

The exact solution is $10^{1.2}$. To four decimal places, $x \approx 15.8489$. □

PRACTICE

3 Solve $\log x = 3.4$ for x. Give an exact solution, and then approximate the solution to four decimal places.

The Richter scale measures the intensity, or magnitude, of an earthquake. The formula for the magnitude R of an earthquake is $R = \log\left(\frac{a}{T}\right) + B$, where a is the amplitude in micrometers of the vertical motion of the ground at the recording station, T is the number of seconds between successive seismic waves, and B is an adjustment factor that takes into account the weakening of the seismic wave as the distance increases from the epicenter of the earthquake.

EXAMPLE 4 **Finding the Magnitude of an Earthquake**

Find an earthquake's magnitude on the Richter scale if a recording station measures an amplitude of 300 micrometers and 2.5 seconds between waves. Assume that B is 4.2. Approximate the solution to the nearest tenth.

Solution Substitute the known values into the formula for earthquake intensity.

$$\begin{aligned} R &= \log\left(\frac{a}{T}\right) + B && \text{Richter scale formula} \\ &= \log\left(\frac{300}{2.5}\right) + 4.2 && \text{Let } a = 300, T = 2.5, \text{ and } B = 4.2. \\ &= \log(120) + 4.2 \\ &\approx 2.1 + 4.2 && \text{Approximate log 120 by 2.1.} \\ &= 6.3 \end{aligned}$$

This earthquake had a magnitude of 6.3 on the Richter scale. □

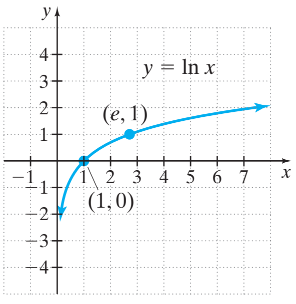
PRACTICE

4 Find an earthquake's magnitude on the Richter scale if a recording station measures an amplitude of 450 micrometers and 4.2 seconds between waves with $B = 3.6$. Approximate the solution to the nearest tenth.

OBJECTIVE

3 Approximating Natural Logarithms

Natural logarithms are also frequently used, especially to describe natural events hence the label "natural logarithm." Natural logarithms are logarithms to the base e, which is a constant approximately equal to 2.7183. The number e is an irrational number, as is π. The notation $\log_e x$ is usually abbreviated to $\ln x$. (The abbreviation ln is read "el en.")



Natural Logarithms

$$\ln x \text{ means } \log_e x$$

The graph of $y = \ln x$ is shown to the left.

EXAMPLE 5 Use a calculator to approximate ln 8 to four decimal places.

Solution Press the following sequence of keys.

$\boxed{8}\ \boxed{\text{LN}}$ or $\boxed{\text{LN}}\ \boxed{8}\ \boxed{\text{ENTER}}$

To four decimal places,

$$\ln 8 \approx 2.0794$$ □

PRACTICE

5 Use a calculator to approximate ln 13 to four decimal places.

OBJECTIVE

4 Evaluating Natural Logarithms of Powers of e.

As a result of the property $\log_b b^x = x$, we know that $\log_e e^x = x$, or $\mathbf{\ln e^x = x}$.

Since $\ln e^x = x$, $\ln e^5 = 5$, $\ln e^{22} = 22$, and so on. Also,

$$\ln e^1 = 1 \text{ or simply } \ln e = 1.$$

That is why the graph of $y = \ln x$ shown above in the margin passes through $(e, 1)$.

If $x = e$, then $y = \ln e = 1$, thus the ordered pair is $(e, 1)$.

EXAMPLE 6 Find the exact value of each natural logarithm.

a. $\ln e^3$ **b.** $\ln \sqrt[5]{e}$

Solution

a. $\ln e^3 = 3$ **b.** $\ln \sqrt[5]{e} = \ln e^{1/5} = \dfrac{1}{5}$ □

PRACTICE 6 Find the exact value of each natural logarithm.

a. $\ln e^4$ **b.** $\ln \sqrt[3]{e}$

EXAMPLE 7 Solve $\ln 3x = 5$. Give an exact solution and then approximate the solution to four decimal places.

Solution Remember that the base of a natural logarithm is understood to be e.

> **Helpful Hint**
> The understood base is e.

$$\ln 3x = 5$$

$$e^5 = 3x \quad \text{Write with exponential notation.}$$

$$\frac{e^5}{3} = x \quad \text{Solve for } x.$$

The exact solution is $\dfrac{e^5}{3}$. To four decimal places,

$$x \approx 49.4711.$$ □

PRACTICE 7 Solve $\ln 5x = 8$. Give an exact solution and then approximate the solution to four decimal places.

Recall from Section 9.3 the formula $A = P\left(1 + \dfrac{r}{n}\right)^{nt}$ for compound interest, where n represents the number of compoundings per year. When interest is compounded continuously, the formula $A = Pe^{rt}$ is used, where r is the annual interest rate, and interest is compounded continuously for t years.

EXAMPLE 8 **Finding Final Loan Payment**

Find the amount owed at the end of 5 years if \$1600 is loaned at a rate of 9% compounded continuously.

Solution Use the formula $A = Pe^{rt}$, where

$$P = \$1600 \text{ (the amount of the loan)}$$

$$r = 9\% = 0.09 \text{ (the rate of interest)}$$

$$t = 5 \text{ (the 5-year duration of the loan)}$$

$$A = Pe^{rt}$$

$$= 1600e^{0.09(5)} \quad \text{Substitute in known values.}$$

$$= 1600e^{0.45}$$

Now we can use a calculator to approximate the solution.

$$A \approx 2509.30$$

The total amount of money owed is \$2509.30. □

PRACTICE 8 Find the amount owed at the end of 4 years if \$2400 is borrowed at a rate of 6% compounded continuously.

OBJECTIVE 5

Using the Change of Base Formula

Calculators are handy tools for approximating natural and common logarithms. Unfortunately, some calculators cannot be used to approximate logarithms to bases other than e or 10—at least not directly. In such cases, we use the change of base formula.

Change of Base

If a, b, and c are positive real numbers and neither b nor c is 1, then

$$\log_b a = \frac{\log_c a}{\log_c b}$$

EXAMPLE 9 Approximate $\log_5 3$ to four decimal places.

Solution Use the change of base property to write $\log_5 3$ as a quotient of logarithms to base 10.

$$\log_5 3 = \frac{\log 3}{\log 5}$$ Use the change of base property. In the change of base property, we let $a = 3$, $b = 5$, and $c = 10$.

$$\approx \frac{0.4771213}{0.69897}$$ Approximate logarithms by calculator.

$$\approx 0.6826063$$ Simplify by calculator.

To four decimal places, $\log_5 3 \approx 0.6826$. □

PRACTICE 9 Approximate $\log_8 5$ to four decimal places.

✓CONCEPT CHECK

If a graphing calculator cannot directly evaluate logarithms to base 5, describe how you could use the graphing calculator to graph the function $f(x) = \log_5 x$.

Vocabulary, Readiness & Video Check

Use the choices to fill in each blank.

1. The base of log 7 is ________.
   a. e **b.** 7 **c.** 10 **d.** no answer

2. The base of ln 7 is __.
   a. e **b.** 7 **c.** 10 **d.** no answer

3. $\log_{10} 10^7 =$ ________.
   a. e **b.** 7 **c.** 10 **d.** no answer

4. $\log_7 1 =$ ________.
   a. e **b.** 7 **c.** 10 **d.** 0

5. $\log_e e^5 =$ ________.
   a. e **b.** 5 **c.** 0 **d.** 1

6. Study exercise 5 to the left. Then answer: $\ln e^5 =$ ________.
   a. e **b.** 5 **c.** 0 **d.** 1

7. $\log_2 7 =$ ________ (There may be more than one answer.)
   a. $\frac{\log 7}{\log 2}$ **b.** $\frac{\ln 7}{\ln 2}$ **c.** $\frac{\log 2}{\log 7}$ **d.** $\log \frac{7}{2}$

Answer to Concept Check:
$f(x) = \frac{\log x}{\log 5}$

Martin-Gay Interactive Videos

See Video 9.7

Watch the section lecture video and answer the following questions.

OBJECTIVE 1
8. From Example 1 and the lecture before, what is the understood base of a common logarithm?

OBJECTIVE 2
9. From Example 2, why can you find exact values of common logarithms of powers of 10?

OBJECTIVE 3
10. From Example 4 and the lecture before, what is the understood base of a natural logarithm?

OBJECTIVE 4
11. In Examples 5 and 6, consider how the expression is rewritten and the resulting answer. What logarithm property is actually used here?

OBJECTIVE 5
12. From Example 8, what two equivalent fractions will give you the exact value of $\log_6 4$?

9.7 Exercise Set

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MIXED PRACTICE

Use a calculator to approximate each logarithm to four decimal places. See Examples 1 and 5.

1. $\log 8$
2. $\log 6$
3. $\log 2.31$
4. $\log 4.86$
5. $\ln 2$
6. $\ln 3$
7. $\ln 0.0716$
8. $\ln 0.0032$
9. $\log 12.6$
10. $\log 25.9$
11. $\ln 5$
12. $\ln 7$
13. $\log 41.5$
14. $\ln 41.5$

MIXED PRACTICE

Find the exact value. See Examples 2 and 6.

15. $\log 100$
16. $\log 10{,}000$
17. $\log \frac{1}{1000}$
18. $\log \frac{1}{100}$
19. $\ln e^2$
20. $\ln e^4$
21. $\ln \sqrt[4]{e}$
22. $\ln \sqrt[5]{e}$
23. $\log 10^3$
24. $\log 10^7$
25. $\ln e^{-7}$
26. $\ln e^{-5}$
27. $\log 0.0001$
28. $\log 0.001$
29. $\ln \sqrt{e}$
30. $\log \sqrt{10}$

Solve each equation for x. Give an exact solution and a four-decimal-place approximation. See Examples 3 and 7.

31. $\ln 2x = 7$
32. $\ln 5x = 9$
33. $\log x = 1.3$
34. $\log x = 2.1$
35. $\log 2x = 1.1$
36. $\log 3x = 1.3$
37. $\ln x = 1.4$
38. $\ln x = 2.1$
39. $\ln(3x - 4) = 2.3$
40. $\ln(2x + 5) = 3.4$
41. $\log x = 2.3$
42. $\log x = 3.1$
43. $\ln x = -2.3$
44. $\ln x = -3.7$
45. $\log(2x + 1) = -0.5$
46. $\log(3x - 2) = -0.8$
47. $\ln 4x = 0.18$
48. $\ln 3x = 0.76$

Approximate each logarithm to four decimal places. See Example 9.

49. $\log_2 3$
50. $\log_3 2$
51. $\log_{1/2} 5$
52. $\log_{1/3} 2$
53. $\log_4 9$
54. $\log_9 4$
55. $\log_3 \frac{1}{6}$
56. $\log_6 \frac{2}{3}$
57. $\log_8 6$
58. $\log_6 8$

Use the formula $R = \log\left(\frac{a}{T}\right) + B$ to find the intensity R on the Richter scale of the earthquakes that fit the descriptions given. Round answers to one decimal place. See Example 4.

59. Amplitude a is 200 micrometers, time T between waves is 1.6 seconds, and B is 2.1.
60. Amplitude a is 150 micrometers, time T between waves is 3.6 seconds, and B is 1.9.
61. Amplitude a is 400 micrometers, time T between waves is 2.6 seconds, and B is 3.1.
62. Amplitude a is 450 micrometers, time T between waves is 4.2 seconds, and B is 2.7.

Use the formula $A = Pe^{rt}$ to solve. See Example 8.

63. Find how much money Dana Jones has after 12 years if \$1400 is invested at 8% interest compounded continuously.
64. Determine the amount in an account in which \$3500 earns 6% interest compounded continuously for 1 year.

65. Find the amount of money Barbara Mack owes at the end of 4 years if 6% interest is compounded continuously on her \$2000 debt.

66. Find the amount of money for which a \$2500 certificate of deposit is redeemable if it has been paying 10% interest compounded continuously for 3 years.

REVIEW AND PREVIEW

Solve each equation for x. See Sections 2.1, 2.3, and 5.8.

67. $6x - 3(2 - 5x) = 6$

68. $2x + 3 = 5 - 2(3x - 1)$

69. $2x + 3y = 6x$

70. $4x - 8y = 10x$

71. $x^2 + 7x = -6$

72. $x^2 + 4x = 12$

Solve each system of equations. See Section 4.1.

73. $\begin{cases} x + 2y = -4 \\ 3x - y = 9 \end{cases}$

74. $\begin{cases} 5x + y = 5 \\ -3x - 2y = -10 \end{cases}$

CONCEPT EXTENSIONS

75. Use a calculator to try to approximate log 0. Describe what happens and explain why.

76. Use a calculator to try to approximate ln 0. Describe what happens and explain why.

77. Without using a calculator, explain which of log 50 or ln 50 must be larger and why.

78. Without using a calculator, explain which of $\log 50^{-1}$ or $\ln 50^{-1}$ must be larger and why.

Graph each function by finding ordered pair solutions, plotting the solutions, and then drawing a smooth curve through the plotted points.

79. $f(x) = e^x$

80. $f(x) = e^{2x}$

81. $f(x) = e^{-3x}$

82. $f(x) = e^{-x}$

83. $f(x) = e^x + 2$

84. $f(x) = e^x - 3$

85. $f(x) = e^{x-1}$

86. $f(x) = e^{x+4}$

87. $f(x) = 3e^x$

88. $f(x) = -2e^x$

89. $f(x) = \ln x$

90. $f(x) = \log x$

91. $f(x) = -2 \log x$

92. $f(x) = 3 \ln x$

93. $f(x) = \log(x + 2)$

94. $f(x) = \log(x - 2)$

95. $f(x) = \ln x - 3$

96. $f(x) = \ln x + 3$

97. Graph $f(x) = e^x$ (Exercise 79), $f(x) = e^x + 2$ (Exercise 83), and $f(x) = e^x - 3$ (Exercise 84) on the same screen. Discuss any trends shown on the graphs.

98. Graph $f(x) = \ln x$ (Exercise 89), $f(x) = \ln x - 3$ (Exercise 95),and $f(x) = \ln x + 3$ (Exercise 96) on the same screen. Discuss any trends shown on the graphs.

9.8 Exponential and Logarithmic Equations and Problem Solving

OBJECTIVES

1. Solve Exponential Equations.
2. Solve Logarithmic Equations.
3. Solve Problems That Can Be Modeled by Exponential and Logarithmic Equations.

OBJECTIVE 1

Solving Exponential Equations

In Section 9.3 we solved exponential equations such as $2^x = 16$ by writing 16 as a power of 2 and applying the uniqueness of b^x.

$$2^x = 16$$
$$2^x = 2^4 \quad \text{Write 16 as } 2^4.$$
$$x = 4 \quad \text{Use the uniqueness of } b^x.$$

Solving the equation in this manner is possible since 16 is a power of 2. If solving an equation such as $2^x =$ *a number,* where the number is not a power of 2, we use logarithms. For example, to solve an equation such as $3^x = 7$, we use the fact that $f(x) = \log_b x$ is a one-to-one function. Another way of stating this fact is as a property of equality.

Logarithm Property of Equality

Let a, b, and c be real numbers such that $\log_b a$ and $\log_b c$ are real numbers and b is not 1. Then

$$\log_b a = \log_b c \text{ is equivalent to } a = c$$

EXAMPLE 1 Solve: $3^x = 7$.

Solution To solve, we use the logarithm property of equality and take the logarithm of both sides. For this example, we use the common logarithm.

$$3^x = 7$$
$$\log 3^x = \log 7 \quad \text{Take the common logarithm of both sides.}$$
$$x \log 3 = \log 7 \quad \text{Apply the power property of logarithms.}$$
$$x = \frac{\log 7}{\log 3} \quad \text{Divide both sides by log 3.}$$