3.6 Dividing Polynomials

3.6 OBJECTIVES

1. Find the quotient when a polynomial is divided by a monomial
2. Find the quotient of two polynomials

In Section 1.7, we introduced the second property of exponents, which was used to divide one monomial by another monomial. Let's review that process.

Step by Step: To Divide a Monomial by a Monomial

Step 1 Divide the coefficients.

Step 2 Use the second property of exponents to combine the variables.

NOTE The second property says: If x is not zero,

$$\frac{x^m}{x^n} = x^{m-n}$$

Example 1

Dividing Monomials

Divide: $\frac{8}{2} = 4$

(a) $\frac{8x^4}{2x^2} = 4x^{4-2}$

Subtract the exponents.

$= 4x^2$

(b) $\frac{45a^5b^3}{9a^2b} = 5a^3b^2$

CHECK YOURSELF 1

Divide.

(a) $\frac{16a^5}{8a^3}$

(b) $\frac{28m^4n^3}{7m^3n}$

Now let's look at how this can be extended to divide any polynomial by a monomial. For example, to divide $12a^3 + 8a^2$ by $4a$, proceed as follows:

NOTE Technically, this step depends on the distributive property and the definition of division.

$$\frac{12a^3 + 8a^2}{4a} = \frac{12a^3}{4a} + \frac{8a^2}{4a}$$

Divide each term in the numerator by the denominator, $4a$.

Now do each division.

$= 3a^2 + 2a$

The work above leads us to the following rule.

Step by Step: To Divide a Polynomial by a Monomial

1. Divide each term of the polynomial by the monomial.
2. Simplify the results.

Example 2

Dividing by Monomials

Divide each term by 2.

(a) $$\frac{4a^2 + 8}{2} = \frac{4a^2}{2} + \frac{8}{2}$$

$$= 2a^2 + 4$$

Divide each term by 6*y*.

(b) $$\frac{24y^3 + (-18y^2)}{6y} = \frac{24y^3}{6y} + \frac{-18y^2}{6y}$$

$$= 4y^2 - 3y$$

Remember the rules for signs in division.

(c) $$\frac{15x^2 + 10x}{-5x} = \frac{15x^2}{-5x} + \frac{10x}{-5x}$$

$$= -3x - 2$$

NOTE With practice you can write just the quotient.

(d) $$\frac{14x^4 + 28x^3 - 21x^2}{7x^2} = \frac{14x^4}{7x^2} + \frac{28x^3}{7x^2} - \frac{21x^2}{7x^2}$$

$$= 2x^2 + 4x - 3$$

(e) $$\frac{9a^3b^4 - 6a^2b^3 + 12ab^4}{3ab} = \frac{9a^3b^4}{3ab} - \frac{6a^2b^3}{3ab} + \frac{12ab^4}{3ab}$$

$$= 3a^2b^3 - 2ab^2 + 4b^3$$

CHECK YOURSELF 2

Divide.

(a) $\dfrac{20y^3 - 15y^2}{5y}$

(b) $\dfrac{8a^3 - 12a^2 + 4a}{-4a}$

(c) $\dfrac{16m^4n^3 - 12m^3n^2 + 8mn}{4mn}$

We are now ready to look at dividing one polynomial by another polynomial (with more than one term). The process is very much like long division in arithmetic, as Example 3 illustrates.

Example 3

Dividing by Binomials

Divide $x^2 + 7x + 10$ by $x + 2$.

NOTE The first term in the dividend, x^2, is divided by the first term in the divisor, x.

Step 1

$$\begin{array}{r}
x\phantom{{}+7x+10}\\
x+2\,\overline{)\,x^2+7x+10}
\end{array}$$

Divide x^2 by x to get x.

Step 2

$$\begin{array}{r}
x\phantom{{}+7x+10}\\
x+2\,\overline{)\,x^2+7x+10}\\
\underline{x^2+2x}\phantom{{}+10}
\end{array}$$

Multiply the divisor, $x + 2$, by x.

REMEMBER To subtract $x^2 + 2x$, mentally change each sign to $-x^2 - 2x$, and add. Take your time and be careful here. It's where most errors are made.

Step 3

$$\begin{array}{r}
x\phantom{{}+7x+10}\\
x+2\,\overline{)\,x^2+7x+10}\\
\underline{x^2+2x}\phantom{{}+10}\\
5x+10
\end{array}$$

Subtract and bring down 10.

Step 4

$$\begin{array}{r}
x+5\\
x+2\,\overline{)\,x^2+7x+10}\\
\underline{x^2+2x}\phantom{{}+10}\\
5x+10
\end{array}$$

Divide $5x$ by x to get 5.

NOTE Notice that we repeat the process until the degree of the remainder is less than that of the divisor or until there is no remainder.

Step 5

$$\begin{array}{r}
x+5\\
x+2\,\overline{)\,x^2+7x+10}\\
\underline{x^2+2x}\phantom{{}+10}\\
5x+10\\
\underline{5x+10}\\
0
\end{array}$$

Multiply $x + 2$ by 5 and then subtract.

The quotient is $x + 5$.

CHECK YOURSELF 3

Divide $x^2 + 9x + 20$ *by* $x + 4$.

In Example 3, we showed all the steps separately to help you see the process. In practice, the work can be shortened.

Example 4

Dividing by Binomials

Divide $x^2 + x - 12$ by $x - 3$.

NOTE You might want to write out a problem like 408 ÷ 17, to compare the steps.

$$\begin{array}{r} x + 4 \\ x - 3\overline{)x^2 + x - 12} \\ \underline{x^2 - 3x\phantom{{}-12}} \\ 4x - 12 \\ \underline{4x - 12} \\ 0 \end{array}$$

Step 1 Divide x^2 by x to get x, the first term of the quotient.
Step 2 Multiply $x - 3$ by x.
Step 3 Subtract and bring down -12. Remember to mentally change the signs to $-x^2 + 3x$ and add.
Step 4 Divide $4x$ by x to get 4, the second term of the quotient.
Step 5 Multiply $x - 3$ by 4 and subtract.

The quotient is $x + 4$.

CHECK YOURSELF 4

Divide.

$(x^2 + 2x - 24) \div (x - 4)$

You may have a remainder in algebraic long division just as in arithmetic. Consider Example 5.

Dividing by Binomials

Divide $4x^2 - 8x + 11$ by $2x - 3$.

$$\begin{array}{r} \overbrace{2x - 1} \quad \text{Quotient} \\ \underbrace{2x - 3}_{\text{Divisor}}\overline{)4x^2 - 8x + 11} \\ \underline{4x^2 - 6x\phantom{{}+11}} \\ -2x + 11 \\ \underline{-2x + 3} \\ \underset{\substack{\uparrow \\ \text{Remainder}}}{8} \end{array}$$

This result can be written as

$$\frac{4x^2 - 8x + 11}{2x - 3}$$

$$= \underbrace{2x - 1}_{\text{Quotient}} + \frac{8 \leftarrow \text{Remainder}}{2x - 3 \leftarrow \text{Divisor}}$$

CHECK YOURSELF 5

Divide.

$(6x^2 - 7x + 15) \div (3x - 5)$

The division process shown in our previous examples can be extended to dividends of a higher degree. The steps involved in the division process are exactly the same, as Example 6 illustrates.

Example 6

Dividing by Binomials

Divide $6x^3 + x^2 - 4x - 5$ by $3x - 1$.

$$
\begin{array}{rrrrr}
 & & 2x^2 & +\; x & -\; 1 \\
3x - 1 \,\big) & \overline{6x^3} & \overline{+\; x^2} & \overline{-\; 4x} & \overline{-\; 5} \\
 & \underline{6x^3} & \underline{-\; 2x^2} & & \\
 & & 3x^2 & -\; 4x & \\
 & & \underline{3x^2} & \underline{-\; x} & \\
 & & & -3x & -\; 5 \\
 & & & \underline{-3x} & \underline{+\; 1} \\
 & & & & -\; 6
\end{array}
$$

The result can be written as

$$\frac{6x^3 + x^2 - 4x - 5}{3x - 1} = 2x^2 + x - 1 + \frac{-6}{3x - 1}$$

CHECK YOURSELF 6

Divide $4x^3 - 2x^2 + 2x + 15$ *by* $2x + 3$.

Suppose that the dividend is "missing" a term in some power of the variable. You can use 0 as the coefficient for the missing term. Consider Example 7.

Dividing by Binomials

Divide $x^3 - 2x^2 + 5$ by $x + 3$.

$$
\begin{array}{rrrrr}
 & & x^2 & -\; 5x & +\; 15 \\
x + 3 \,\big) & \overline{x^3} & \overline{-\; 2x^2} & \overline{+\; 0x} & \overline{+\; 5} \\
 & \underline{x^3} & \underline{+\; 3x^2} & & \\
 & & -\; 5x^2 & +\; 0x & \\
 & & \underline{-\; 5x^2} & \underline{-\; 15x} & \\
 & & & 15x & +\; 5 \\
 & & & \underline{15x} & \underline{+\; 45} \\
 & & & & -\; 40
\end{array}
$$

Write $0x$ for the "missing" term in x.

This result can be written as

$$\frac{x^3 - 2x^2 + 5}{x + 3} = x^2 - 5x + 15 + \frac{-40}{x + 3}$$

CHECK YOURSELF 7

Divide.

$(4x^3 + x + 10) \div (2x - 1)$

You should always arrange the terms of the divisor and dividend in descending-exponent form before starting the long division process, as illustrated in Example 8.

Example 8

Dividing by Binomials

Divide $5x^2 - x + x^3 - 5$ by $-1 + x^2$.

Write the divisor as $x^2 - 1$ and the dividend as $x^3 + 5x^2 - x - 5$.

$$\begin{array}{r} x + 5 \\ x^2 - 1\overline{)x^3 + 5x^2 - x - 5} \\ \underline{x^3 \qquad\quad - x} \qquad \\ 5x^2 \qquad - 5 \\ \underline{5x^2 \qquad - 5} \\ 0 \end{array}$$

Write $x^3 - x$, the product of x and $x^2 - 1$, so that like terms fall in the same columns.

CHECK YOURSELF 8

Divide:

$(5x^2 + 10 + 2x^3 + 4x) \div (2 + x^2)$

CHECK YOURSELF ANSWERS

1. **(a)** $2a^2$; **(b)** $4mn^2$ **2.** **(a)** $4y^2 - 3y$; **(b)** $-2a^2 + 3a - 1$; **(c)** $4m^3n^2 - 3m^2n + 2$

3. $x + 5$ **4.** $x + 6$ **5.** $2x + 1 + \dfrac{20}{3x - 5}$ **6.** $2x^2 - 4x + 7 + \dfrac{-6}{2x + 3}$

7. $2x^2 + x + 1 + \dfrac{11}{2x - 1}$ **8.** $2x + 5$

Name ______________________

Section __________ Date __________

3.6 Exercises

Divide.

1. $\frac{18x^6}{9x^2}$

2. $\frac{20a^7}{5a^5}$

3. $\frac{35m^3n^2}{7mn^2}$

4. $\frac{42x^5y^2}{6x^3y}$

5. $\frac{3a + 6}{3}$

6. $\frac{4x - 8}{4}$

7. $\frac{9b^2 - 12}{3}$

8. $\frac{10m^2 + 5m}{5}$

9. $\frac{16a^3 - 24a^2}{4a}$

10. $\frac{9x^3 + 12x^2}{3x}$

11. $\frac{12m^2 + 6m}{-3m}$

12. $\frac{20b^3 - 25b^2}{-5b}$

13. $\frac{18a^4 + 12a^3 - 6a^2}{6a}$

14. $\frac{21x^5 - 28x^4 + 14x^3}{7x}$

15. $\frac{20x^4y^2 - 15x^2y^3 + 10x^3y}{5x^2y}$

16. $\frac{16m^3n^3 + 24m^2n^2 - 40mn^3}{8mn^2}$

Perform the indicated divisions.

17. $\frac{x^2 + 5x + 6}{x + 2}$

18. $\frac{x^2 + 8x + 15}{x + 3}$

19. $\frac{x^2 - x - 20}{x + 4}$

20. $\frac{x^2 - 2x - 35}{x + 5}$

21. $\frac{2x^2 + 5x - 3}{2x - 1}$

22. $\frac{3x^2 + 20x - 32}{3x - 4}$

23. $\frac{2x^2 - 3x - 5}{x - 3}$

24. $\frac{3x^2 + 17x - 12}{x + 6}$

ANSWERS

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25. $\frac{4x^2 - 18x - 15}{x - 5}$

26. $\frac{3x^2 - 18x - 32}{x - 8}$

27. $\frac{6x^2 - x - 10}{3x - 5}$

28. $\frac{4x^2 + 6x - 25}{2x + 7}$

29. $\frac{x^3 + x^2 - 4x - 4}{x + 2}$

30. $\frac{x^3 - 2x^2 + 4x - 21}{x - 3}$

31. $\frac{4x^3 + 7x^2 + 10x + 5}{4x - 1}$

32. $\frac{2x^3 - 3x^2 + 4x + 4}{2x + 1}$

33. $\frac{x^3 - x^2 + 5}{x - 2}$

34. $\frac{x^3 + 4x - 3}{x + 3}$

35. $\frac{25x^3 + x}{5x - 2}$

36. $\frac{8x^3 - 6x^2 + 2x}{4x + 1}$

37. $\frac{2x^2 - 8 - 3x + x^3}{x - 2}$

38. $\frac{x^2 - 18x + 2x^3 + 32}{x + 4}$

39. $\frac{x^4 - 1}{x - 1}$

40. $\frac{x^4 + x^2 - 16}{x + 2}$

41. $\frac{x^3 - 3x^2 - x + 3}{x^2 - 1}$

42. $\frac{x^3 + 2x^2 + 3x + 6}{x^2 + 3}$

43. $\frac{x^4 + 2x^2 - 2}{x^2 + 3}$

44. $\frac{x^4 + x^2 - 5}{x^2 - 2}$

45. $\frac{y^3 + 1}{y + 1}$

46. $\frac{y^3 - 8}{y - 2}$

47. $\frac{x^4 - 1}{x^2 - 1}$

48. $\frac{x^6 - 1}{x^3 - 1}$

49. Find the value of c so that $\dfrac{y^2 - y + c}{y + 1} = y - 2$

50. Find the value of c so that $\dfrac{x^3 + x^2 + x + c}{x^2 + 1} = x + 1$

51. Write a summary of your work with polynomials. Explain how a polynomial is recognized, and explain the rules for the arithmetic of polynomials—how to add, subtract, multiply, and divide. What parts of this chapter do you feel you understand very well, and what part(s) do you still have questions about, or feel unsure of? Exchange papers with another student and compare your questions.

52. A funny (and useful) thing about division of polynomials: To find out about this funny thing, do this division. Compare your answer with another student's.

$(x - 2)\overline{)2x^2 + 3x - 5}$ Is there a remainder?

Now, evaluate the polynomial $2x^2 + 3x - 5$ when $x = 2$. Is this value the same as the remainder?

Try $(x + 3)\overline{)5x^2 - 2x + 1}$ Is there a remainder?

Evaluate the polynomial $5x^2 - 2x + 1$ when $x = -3$. Is this value the same as the remainder?

What happens when there is no remainder?

Try $(x - 6)\overline{)3x^3 + 14x^2 - 23x + 6}$ Is the remainder zero?

Evaluate the polynomial $3x^3 + 14x - 23x + 6$ when $x = 6$. Is this value zero? Write a description of the patterns you see. When does the pattern hold? Make up several more examples, and test your conjecture.

53. (a) Divide $\dfrac{x^2 - 1}{x - 1}$ (b) Divide $\dfrac{x^3 - 1}{x - 1}$ (c) Divide $\dfrac{x^4 - 1}{x - 1}$

(d) Based on your results to (a), (b), and (c), predict $\dfrac{x^{50} - 1}{x - 1}$

54. (a) Divide $\dfrac{x^2 + x + 1}{x - 1}$ (b) Divide $\dfrac{x^3 + x^2 + x + 1}{x - 1}$

(c) Divide $\dfrac{x^4 + x^3 + x^2 + x + 1}{x - 1}$

(d) Based on your results to (a), (b), and (c), predict $\dfrac{x^{10} + x^9 + x^8 + \cdots + x + 1}{x - 1}$

ANSWERS

49. ____

50. ____

51. ____

52. ____

53. **(a)** ____

(b) ____

(c) ____

(d) ____

54. **(a)** ____

(b) ____

(c) ____

(d) ____

Answers

1. $2x^4$ **3.** $5m^2$ **5.** $a + 2$ **7.** $3b^2 - 4$ **9.** $4a^2 - 6a$ **11.** $-4m - 2$

13. $3a^3 + 2a^2 - a$ **15.** $4x^2y - 3y^2 + 2x$ **17.** $x + 3$ **19.** $x - 5$

21. $x + 3$ **23.** $2x + 3 + \frac{4}{x - 3}$ **25.** $4x + 2 + \frac{-5}{x - 5}$

27. $2x + 3 + \frac{5}{3x - 5}$ **29.** $x^2 - x - 2$ **31.** $x^2 + 2x + 3 + \frac{8}{4x - 1}$

33. $x^2 + x + 2 + \frac{9}{x - 2}$ **35.** $5x^2 + 2x + 1 + \frac{2}{5x - 2}$

37. $x^2 + 4x + 5 + \frac{2}{x - 2}$ **39.** $x^3 + x^2 + x + 1$ **41.** $x - 3$

43. $x^2 - 1 + \frac{1}{x^2 + 3}$ **45.** $y^2 - y + 1$ **47.** $x^2 + 1$ **49.** $c = -2$

51. **53.** **(a)** $x + 1$; **(b)** $x^2 + x + 1$; **(c)** $x^3 + x^2 + x + 1$;

(d) $x^{49} + x^{48} + \cdots + x + 1$