13.2 Exponential Decay Functions

Essential Question: How is the graph of $g(x) = ab^{x-h} + k$ where $0 < b < 1$ related to the graph of $f(x) = b^x$?

⊘ Explore 1 Graphing and Analyzing $f(x) = \left(\frac{1}{2}\right)^x$ and $f(x) = \left(\frac{1}{10}\right)^x$

Exponential functions with bases between 0 and 1 can be transformed in a manner similar to exponential functions with bases greater than 1. Begin by plotting the parent functions of two of the more commonly used bases: $\frac{1}{2}$ and $\frac{1}{10}$.

(A) To begin, fill in the table in order to find points along the function $f(x) = \left(\frac{1}{2}\right)^x$. You may need to review the rules of the properties of exponents, including negative exponents.

x	$f(x) = \left(\frac{1}{2}\right)^x$
-3	8
-2	
-1	
0	
1	
2	
3	

(B) What does the end behavior of this function appear to be as x increases?

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(C) Plot the points on the graph and draw a smooth curve through them.
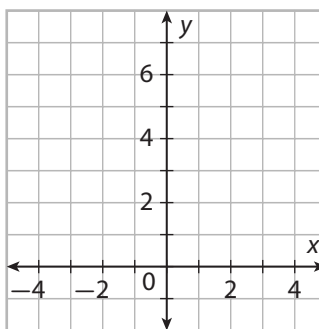
(D) Complete the table for $f(x) = \left(\frac{1}{10}\right)x$.

x	$f(x) = \left(\frac{1}{10}\right)^x$
-3	1000
-2	
-1	
0	
1	
2	
3	

(E) Plot the points on the graph and draw a smooth curve through them.
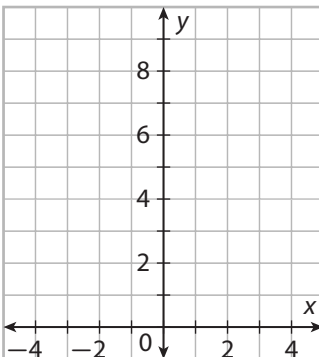
(F) Fill in the following table of properties:

	$f(x) = \left(\frac{1}{2}\right)^x$	$f(x) = \left(\frac{1}{10}\right)^x$
Domain	$\{x \mid -\infty < x < \infty\}$	$\{x \mid \}$
Range	$\{y \mid \}$	$\{y \mid \}$
End behavior as $x \to \infty$	$f(x) \to \square$	$f(x) \to \square$
End behavior as $x \to -\infty$	$f(x) \to \square$	$f(x) \to \square$
y-intercept	$\square \ \square$	$\square \ \square$

(G) Both of these functions [decrease/increase] throughout the domain.

(H) Of the two functions, $f(x) = \left(\dfrac{1}{\square}\right)^x$ decreases faster.

Reflect

1. **Make a Conjecture** Look at the table of properties for the functions. What do you notice? Make a conjecture about these properties for exponential functions of the form $f(x) = \left(\frac{1}{n}\right)^x$, where n is a constant.

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⚙ **Explain 1** **Graphing Combined Transformations of $f(x) = b^x$ Where $0 < b < 1$**

When graphing transformations of $f(x) = b^x$ where $0 < b < 1$, it is helpful to consider the effect of the transformation on two reference points, $(0, 1)$ and $\left(-1, \frac{1}{b}\right)$, as well as the effect on the asymptote, $y = 0$. The table shows these reference points and the asymptote $y = 0$ for $f(x) = b^x$ and the corresponding points and asymptote for the transformed function, $g(x) = ab^{x-h} + k$.

	$f(x) = b^x$	$g(x) = ab^{x-h} + k$
First reference point	$(0, 1)$	$(h, a + k)$
Second reference point	$\left(-1, \dfrac{1}{b}\right)$	$\left(h - 1, \dfrac{a}{b} + k\right)$
Asymptote	$y = 0$	$y = k$

Example 1 The graph of a parent exponential function is shown. Use the reference points and the asymptote shown for the parent graph to graph the given transformed function. Then describe the domain and range of the transformed function using set notation.

(A) $g(x) = 3\left(\dfrac{1}{2}\right)^{x-2} - 2$

Identify parameters: $\qquad a = 3 \qquad b = \dfrac{1}{2} \qquad h = 2 \qquad k = -2$

Find reference points:

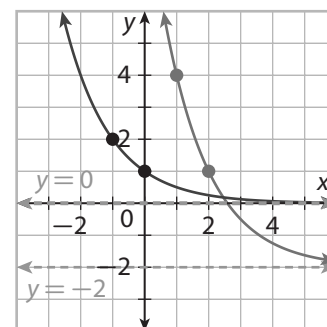$$\left(h, a + k\right) = (2, 3 - 2) = (2, 1)$$

$$\left(h - 1, \dfrac{a}{b} + k\right) = \left(2 - 1, \dfrac{3}{\frac{1}{2}} - 2\right) = (1, 4)$$

Find the asymptote: $y = -2$

Plot the points and draw the asymptote. Then connect the points with a smooth curve that approaches the asymptote without crossing it.

Domain: $\left\{x \mid -\infty < x < \infty\right\}$

Range: $\left\{y \mid y > -2\right\}$

(B) $g(x) = -\left(\dfrac{1}{10}\right)^{x+2} + 8$

Identify parameters:

$a = \boxed{} \qquad b = \boxed{} \qquad h = \boxed{} \qquad k = \boxed{}$

Find reference points:

$$\left(h, \boxed{}\right) = (-2, -1 + 8) = (-2, 7)$$

$$\left(h - 1, \dfrac{a}{b} + k\right) = \left(-2 - 1, \dfrac{\boxed{}}{\boxed{}} + 8\right) = \left(\boxed{}, \boxed{}\right)$$
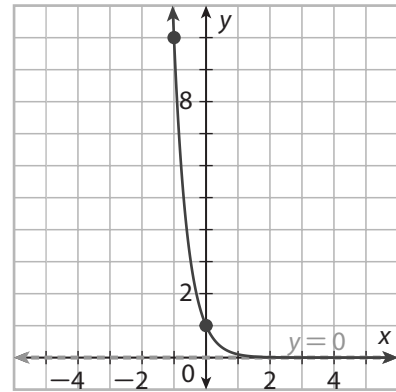
Find the asymptote:

$$y = \boxed{}$$

Plot the points and draw the asymptote. Then connect the points with a smooth curve that approaches the asymptote without crossing it.

Domain: $\left\{ x \mid \boxed{} \right\}$

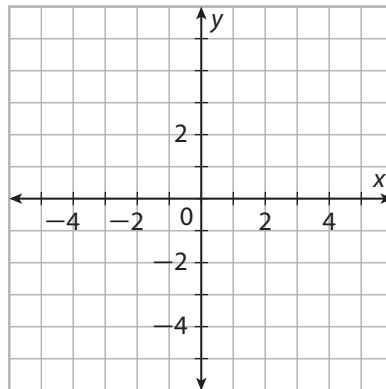Range: $\left\{ y \mid \boxed{} \right\}$

Reflect

2. Which parameters make the domain and range of $g(x)$ differ from those of the parent function? Write the transformed domain and range for $g(x)$ in set notation.

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Your Turn

Graph the given transformed function. Then describe the domain and range of the transformed function using set notation.

3. $g(x) = 3\left(\dfrac{1}{3}\right)^{x+2} - 4$

Writing Equations for Combined Transformations of $f(x) = b^x$ where $0 < b < 1$

Given a graph of an exponential function, $g(x) = ab^{x-h} + k$, the reference points and the asymptote can be used to identify the transformation parameters in order to write the function rule.

Example 2 Write the function represented by this graph and state the domain and range using set notation.

(A)
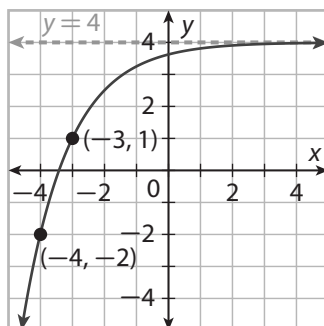
Find k from the asymptote: $k = 4$.

The first reference point is at $(-3,1)$.

Equate point value with parameters-based expression. $(-3, 1) = (h,\ a + k)$

Use the x-coordinate to solve for h. $h = -3$

Use the y-coordinate to solve for a. $a = 1 - k$

$= -3$

The second reference point is at $(-4, -2)$.

Equate point value with parameters-based expression. $(-4, -2) = \left(h - 1, \dfrac{a}{b} + k\right)$

Equate y-coordinate with parameters. $\dfrac{-3}{b} + 4 = -2$

Solve for b. $\dfrac{-3}{b} = -6$
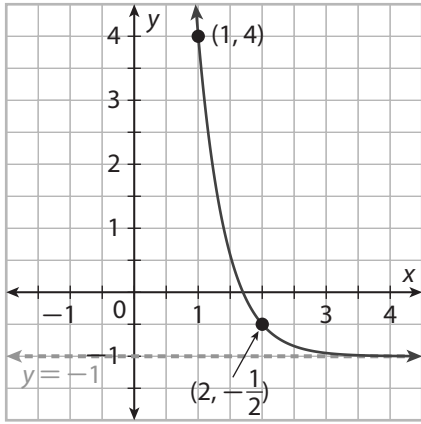
$b = \dfrac{-3}{-6}$

$= \dfrac{1}{2}$

$g(x) = -3\left(\dfrac{1}{2}\right)^{x+3} + 4$

Domain: $\left\{x \mid -\infty < x < \infty\right\}$

Range: $\left\{y \mid y < 4\right\}$

(B)

Find k from the asymptote: $k = \boxed{}$.

The first reference point is at $\left(\boxed{}, \boxed{} \right)$, so $\left(\boxed{}, -\frac{1}{2} \right) = \left(h, \boxed{} \right)$

$h = \boxed{}$ $a = \boxed{} - k$

$ = \boxed{}$

The second reference point is at $\left(\boxed{}, \boxed{} \right)$, so $\left(\boxed{}, 4 \right) = \left(h - 1, \boxed{} \right)$

$\dfrac{\boxed{}}{b} - 1 = \boxed{}$

$\dfrac{\frac{1}{2}}{b} = \boxed{}$

$b = \dfrac{\frac{1}{2}}{5}$

$ = \boxed{}$

$g(x) = \boxed{} \left(\dfrac{1}{10} \right)^{x - \boxed{}} - \boxed{}$

Domain: $\left\{ x \mid -\infty < x < \infty \right\}$

Range: $\left\{ y \mid y \boxed{} -1 \right\}$

4. Compare the *y*-intercept and the asymptote of the function shown in this table to the function plotted in Example 2A.

x	−5	−4	−3	−2	−1	0	1	2
g(x)	−10	−4	−4	$\frac{1}{2}$	$1\frac{1}{4}$	$1\frac{5}{8}$	$1\frac{13}{16}$	$1\frac{29}{32}$

5. Compare the *y*-intercept and the asymptote of the function shown in this table to the function plotted in Example 2B.

x	−3	−2	−1	0	1	2
g(x)	49	4	−0.5	−0.95	−0.995	−0.9995

Write the function represented by this graph and state the domain and range using set notation.

6.
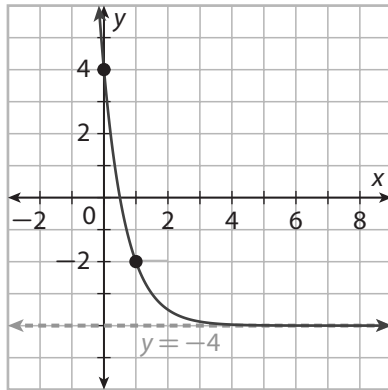
⚙ Explain 3 Modeling with Exponential Decay Functions

An **exponential decay function** has the form $f(t) = a(1 - r)^t$ where a > 0 and r is a constant percent decrease (expressed as a decimal) for each unit increase in time t. That is, since $f(t + 1) = (1 - r) \cdot f(t) = f(t) - r \cdot f(t)$, the value of the function decreases by $r \cdot f(t)$ on the interval $[t, t + 1]$. The base 1 – r of an exponential decay function is called the **decay factor**, and the constant percent decrease r, in decimal form, is called the **decay rate**.

Example 3 **Given the description of the decay terms, write the exponential decay function in the form $f(t) = a(1 - r)^t$ and graph it with a graphing calculator.**

(A) The value of a truck purchased new for $28,000 decreases by 9.5% each year. Write an exponential function for this situation and graph it using a calculator. Use the graph to predict after how many years the value of the truck will be $5000.
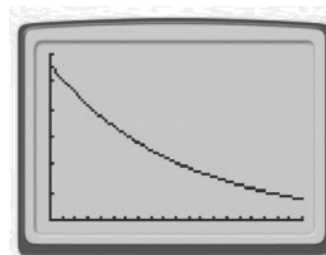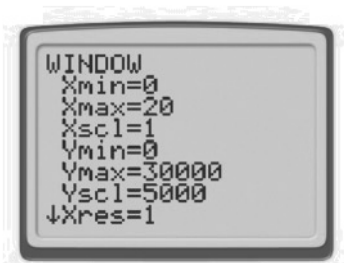
"Purchased new for $28,000..." $a = 28{,}000$

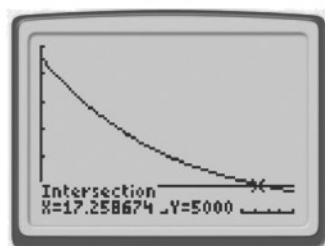"...decreases by 9.5% each year." $r = 0.095$

Substitute parameter values. $V_T(t) = 28{,}000(1 - 0.095)^t$

Simplify. $V_T(t) = 28{,}000(0.905)^t$

Graph the function with a graphing calculator. Use WINDOW to adjust the graph settings so that you can see the function and the function values that are important.


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WINDOW
 Xmin=0
 Xmax=20
 Xscl=1
 Ymin=0
 Ymax=30000
 Yscl=5000
↓Xres=1
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Find when the value reaches $5000 by finding the intersection between $V_T(t) = 28{,}000(0.905)^t$ and $V_T(t) = 5000$ on the calculator.


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CALCULATE
1:value
2:zero
3:minimum
4:maximum
5:intersect
6:dy/dx
7:∫f(x)dx
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Intersection
X=17.258674 Y=5000

The intersection is at the point $(17.26, 5000)$, which means after 17.26 years, the truck will have a value of $5000.

(B) The value of a sports car purchased new for $45,000 decreases by 15% each year. Write an exponential function for the depreciation of the sports car, and plot it along with the previous example. After how many years will the two vehicles have the same value if they are purchased at the same time?

"Purchased new for $45,000..." $\boxed{} = 45{,}000$

"...decreases by 15% each year." $r = \boxed{}$

Substitute parameter values. $V_c(t) = \boxed{}\left(1 - \boxed{}\right)^t$

Simplify. $V_c(t) = 45{,}000\left(\boxed{}\right)^t$

Add this plot to the graph for the truck value from Example A and find the intersection of the two functions to determine when the values are the same.

The intersection point is $\left(\boxed{}, \boxed{}\right)$.

After $\boxed{}$ years, the values of both vehicles will be

$\$\boxed{}$.

Reflect

7. What reference points could you use if you plotted the value function for the sports car on graph paper? Confirm that the graph passes through them using the calculate feature on a graphing calculator.

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8. Using the sports car from example B, calculate the average rate of change over the course of the first year and the second year of ownership. What happens to the absolute value of the rate of change from the first interval to the second? What does this mean in this situation?

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9. On federal income tax returns, self-employed people can depreciate the value of business equipment. Suppose a computer valued at $2765 depreciates at a rate of 30% per year. Use a graphing calculator to determine the number of years it will take for the computer's value to be $350.

💬 Elaborate

10. Which transformations of $f(x) = \left(\frac{1}{2}\right)^x$ or $f(x) = \left(\frac{1}{10}\right)^x$ change the function's end behavior?

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11. Which transformations change the location of the graph's y-intercept?

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12. **Discussion** How are reference points and asymptotes helpful when graphing transformations of $f(x) = \left(\frac{1}{2}\right)^x$ or $f(x) = \left(\frac{1}{10}\right)^x$ or when writing equations for transformed graphs?

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13. Give the general form of an exponential decay function based on a known decay rate and describe its parameters.

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14. **Essential Question Check-In** How is the graph of $f(x) = b^x$ used to help graph the function $g(x) = ab^{x-h} + k$?

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Describe the transformation(s) from each parent function and give the domain and range of each function.

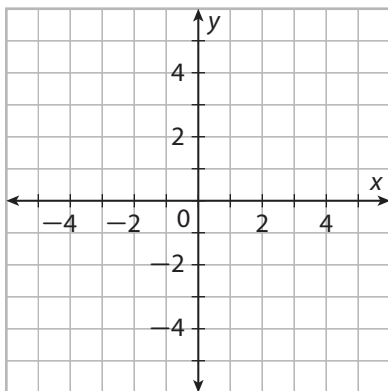**1.** $g(x) = \left(\dfrac{1}{2}\right)^x + 3$

2. $g(x) = \left(\dfrac{1}{10}\right)^{x+4}$

3. $g(x) = -\left(\dfrac{1}{10}\right)^{x-1} + 2$
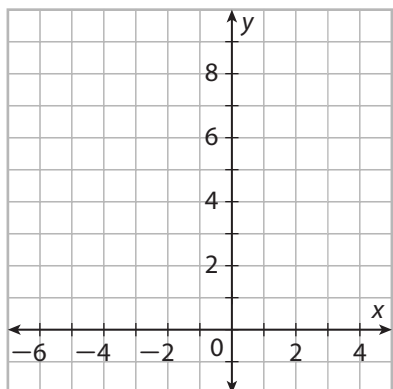
4. $g(x) = 3\left(\dfrac{1}{2}\right)^{x+3} - 6$

Graph the given transformed function. Then describe the domain and range of the transformed function using set notation.
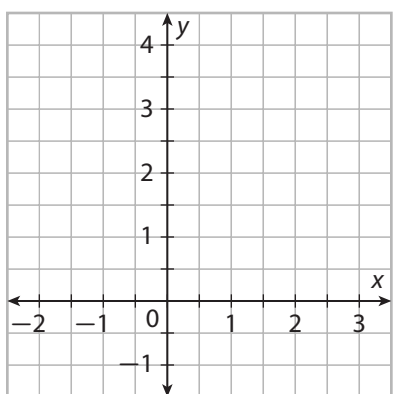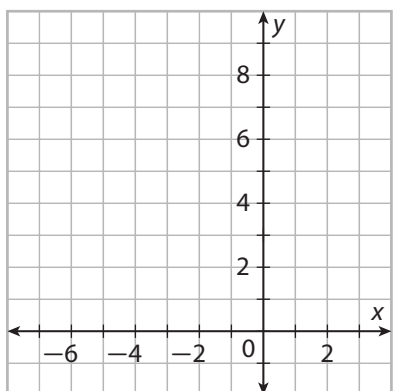
5. $g(x) = -2\left(\dfrac{1}{2}\right)^{x-1} + 2$

6. $g(x) = \left(\frac{1}{4}\right)^{x+2} + 3$

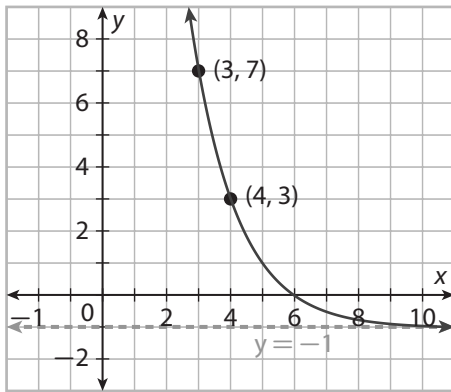**7.** $g(x) = \frac{1}{2}\left(\frac{1}{3}\right)^{x-\frac{1}{2}} + 2$

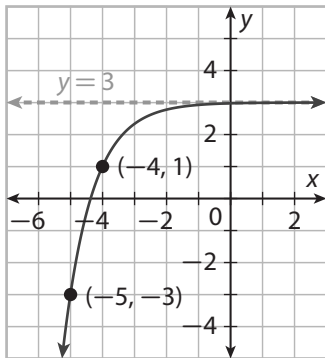**8.** $g(x) = -3\left(\frac{1}{2}\right)^{x+2} + 7$

Write the function represented by each graph and state the domain and range using set notation.

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Write the exponential decay function described in the situation and use a graphing calculator to answer each question asked.

11. **Medicine** A quantity of insulin used to regulate sugar in the bloodstream breaks down by about 5% each minute after the injection. A bodyweight-adjusted dose is generally 10 units. How long does it take for the remaining insulin to be half of the original injection?

12. **Paleontology** Carbon-14 is a radioactive isotope of carbon that is used to date fossils. There are about 1.5 atoms of carbon-14 for every trillion atoms of carbon in the atmosphere, which known as 1.5 ppt (parts per trillion). Carbon in a living organism has the same concentration as carbon-14. When an organism dies, the carbon-14 content decays at a rate of 11.4% per millennium (1000 years). Write the equation for carbon-14 concentration (in ppt) as a function of time (in millennia) and determine how old a fossil must be that has a measured concentration of 0.2 ppt.

13. **Music** Stringed instruments like guitars and pianos create a note when a string vibrates back and forth. The distance that the middle of the string moves from the center is called the amplitude (a), and for a guitar, it starts at 0.75 mm when a note is first struck. Amplitude decays at a rate that depends on the individual instrument and the note, but a decay rate of about 25% per second is typical. Calculate the time it takes for an amplitude of 0.75 mm to reach 0.1 mm.

14. Analyze Relationships Compare the graphs of $f(x) = \left(\frac{1}{2}\right)^x$ and $g(x) = x^{\frac{1}{2}}$. Which of the following properties are the same? Explain.

   a. Domain

   b. Range

   c. End behavior as x increases

   d. End behavior as x decreases

15. Communicate Mathematical Ideas A quantity is reduced to half of its original amount during each given time period. Another quantity is reduced to one quarter of its original amount during the same given time period. Determine each decay rate, state which is greater, and explain your results.

16. Multiple Representations Exponential decay functions are written as transformations of the function $f(x) = b^x$, where $0 < b < 1$. However, it is also possible to use negative exponents as the basis of an exponential decay function. Use the properties of exponents to show why the function $f(x) = 2^{-x}$ is an exponential decay function.

17. Represent Real-World Problems You buy a video game console for $500 and sell it 5 years later for $100. The resale value decays exponentially over time. Write a function that represents the resale value, R, in dollars over the time, t, in years. Explain how you determined your function.

Lesson Performance Task

Sodium-24 is a radioactive isotope of sodium used as a diagnostic aid in medicine. It undergoes radioactive decay to form the stable isotope magnesium-24 and has a half-life of about 15 hours. This means that, in this time, half the amount of a sample mass of sodium-24 decays to magnesium-24. Suppose we start with an initial mass of 100 grams sodium-24.

a. Use the half-life of sodium-24 to write an exponential decay function of the form $m_{Na}(t) = m_0(1 - r)^t$, where m_0 is the initial mass of sodium-24, r is the decay rate, t is the time in hours, and $m_{Na}(t)$ is the mass of sodium-24 at time t. What is the meaning of r?

b. The combined amounts of sodium-24 and magnesium-24 must equal m_0, or 100, for all possible values of t. Show how to write a function for $m_{Mg}(t)$, the mass of magnesium-24 as a function of t.

c. Use a graphing calculator to graph $m_{Na}(t)$ and $m_{Mg}(t)$. Describe the graph of $m_{Mg}(t)$ as a series of transformations of $m_{Na}(t)$. What does the intersection of the graphs represent?

Mass of Na/Mg (g) vs. Time in Minutes (t)