Solution

Since $t = 12$ is not one of the times given in the table, we should approximate the derivative by using a difference quotient with the best (closest) data available. Because 12 lies between 10 and 13, the best approximation for $y'(12)$ is found by:

$$y'(12) \approx \frac{y(13) - y(10)}{13 - 10} = \frac{3697 - 3108}{3} = 196.333 \text{ people per year.}$$

When $t = 12$ years, the population of the town is increasing at a rate of approximately 196.333 people per year.

Notice that since the difference quotient is equal to the change in population divided by the change in years, the units for the answer are people per year. Notice, too, that the difference quotient is also the average rate of change of the population function from time $t = 10$ to $t = 13$ years.

Solution

The average value of $y(t)$ from $t = 0$ to $t = 20$ years is given by:

$$\frac{1}{20}\int_0^{20} y(t)\,dt$$

$$\approx \frac{1}{20}\left(\frac{1}{2}(4)(2500+2724)+\frac{1}{2}(6)(2724+3108)+\frac{1}{2}(3)(3108+3697)+\frac{1}{2}(7)(3697+4283)\right)$$

= 3304.075 people.

The average population over the 20-year period is approximately 3304.075 people.

Notice that $y(t)$ is measured in people, and t is measured in years so the units of:

$$\frac{1}{20}\int_0^{20} y(t)\,dt \text{ are } \left(\frac{1}{\text{yrs}}\right)(\text{people})(\text{yrs}) = \text{people.}$$

This makes sense because the average population should have the number of people as the units.

Notice also that the function given in this problem represents the *population* of the town. Therefore the *average population* is the *average value of the population function*, which is found by using the formula:

$$\text{Average Value of } f \text{ on } [a, b] = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Solution

Since $\int_0^{12} R(t)dt \approx 4(R(2)+R(6)+R(10)) = 4(13.4+14.3+14.8) = 170$ gallons, approximately 170 gallons of water flowed into the tank between $t = 0$ and $t = 12$ hours.

Notice that the integral $\int_0^{12} R(t)dt$ does not have a coefficient of $\frac{1}{12}$ so the integral gives the *total change* or *net change*, not the *average value of the rate of change.* The integral of a rate of change is the total change or net change. Another way to recognize that the value of the integral is the total change is to notice the units in the problem. Since $R(t)$ is measured in gallons per hour, and t is measured in hours, the integral has units of:

$$\left(\frac{\text{gal}}{\text{hr}}\right)(\text{hr}) = \text{gal}.$$

Solution

The average rate of water flow is given by $\frac{1}{12}\int_0^{12} P(t)dt = \frac{1}{12}(169.2) = 14.1$ gal/hr.

Notice that the value of $\int_0^{12} P(t)dt$ is 169.2 gallons, which is very close to the answer to (1).

Notice also that since the function given in this problem represents the rate of water flow, then the average rate of water flow is just the average value of the given function, which is found by the formula

$$\text{Average Value of } f \text{ on } [a, b] = \frac{1}{b-a}\int_a^b f(x)dx.$$

In this case, the given function is a rate of change function, and the average rate of change is the average value of the given rate of change function. If the function given in the problem had represented the amount of water in the tank, then the average rate of change would have been found by the formula

$$\text{Average Rate of Change of } f \text{ on } [a, b] = \frac{f(b)-f(a)}{b-a}.$$

As in part (1), the units are helpful in determining the meaning of the answer.

$\frac{1}{12}\int_0^{12} P(t)dt$ has units of $\left(\frac{1}{\text{hr}}\right)\left(\frac{\text{gal}}{\text{hr}}\right)(\text{hr}) = \frac{\text{gal}}{\text{hr}}$

which are the units for the average value of the rate of change function.

Solution

Distance $\approx (2)(6.8)+(3)(7.4)+(2)(15.6)+(3)(24.9)=141.7$ cm.

Notice that this quantity approximates the integral $\int_0^{10} v_A(t)\,dt$ and that the units are:

$$(\text{sec})\left(\frac{\text{cm}}{\text{sec}}\right)=\text{cm}.$$

Solution

Let $v_B(t)$ be the velocity of particle *B* at time *t*. Then:

$$v_B(t)=\int(2t-7)\,dt=t^2-7t+C.$$

At $t = 1$, we have $13=1-7+C$, so that $C=19$ and $v_B(t)=t^2-7t+19$.

Hence, $v_B(5)=9>7.4=v_A(5)$, and we conclude that particle *B* is traveling faster at time $t = 5$ seconds.