AP Calculus AB 2008 Multiple Choice Exam Solutions

PART A – (No Calculator Allowed) –

1. $\lim\limits_{x\to\infty}\dfrac{(2x-1)(3-x)}{(x-1)(x+3)}=\lim\limits_{x\to\infty}\dfrac{6x-2x^2-3+x}{x^2+2x-3}=\lim\limits_{x\to\infty}\dfrac{-2x^2+7x-3}{x^2+2x-3}$

$=\lim\limits_{x\to\infty}\dfrac{-\dfrac{2x^2}{x^2}+\dfrac{7x}{x^2}-\dfrac{3}{x^2}}{\dfrac{x^2}{x^2}+\dfrac{2x}{x^2}-\dfrac{3}{x^2}}=\lim\limits_{x\to\infty}\dfrac{-2+\dfrac{7}{x}-\dfrac{3}{x^2}}{1+\dfrac{2}{x}-\dfrac{3}{x^2}}=-2$ B

2. $\int\dfrac{1}{x^2}\,dx=\int x^{-2}\,dx=\dfrac{x^{-1}}{-1}+C=-x^{-1}+C.$ D

3. $f'(x)=(x-1)\left(3\left(x^2+2\right)^2(2x)\right)+\left(x^2+2\right)^3(1)=\left(x^2+2\right)^2\left(6x(x-1)+x^2+2\right)$

$=\left(x^2+2\right)^2\left(6x^2-6x+x^2+2\right)=\left(x^2+2\right)^2\left(7x^2-6x+2\right).$ D

4. $\int\sin(2x)+\cos(2x)\,dx=\dfrac{1}{2}\int\sin u+\cos u\,du=\dfrac{1}{2}(-\cos u+\sin u)+C$

$u=2x,\ du=2\,dx$

$=\dfrac{1}{2}\left(-\cos(2x)+\sin(2x)\right)+C.$ B

5. $\lim\limits_{x\to0}\dfrac{5x^4+8x^2}{3x^4-16x^2}=\lim\limits_{x\to0}\dfrac{x^2\left(5x^2+8\right)}{x^2\left(3x^2-16\right)}=\lim\limits_{x\to0}\dfrac{5x^2+8}{3x^2-16}=\dfrac{5(0)^2+8}{3(0)^2-16}=-\dfrac{1}{2}.$ A

6. I is true since $\lim\limits_{x\to2^-}f(x)=\lim\limits_{x\to2^+}f(x)=\lim\limits_{x\to2}\dfrac{x^2-4}{x-2}=\lim\limits_{x\to2}\dfrac{(x-2)(x+2)}{x-2}=\lim\limits_{x\to2}(x+2)=(2)+2=4.$

II is false since f is not continuous at $x = 2$ since $\lim\limits_{x\to2^-}f(x)\neq f(2).$

II is false since f is not differentiable at $x = 2$ because f is not continuous at $x = 2$. A

7. $v(t)=3t^2+6t\Rightarrow x(t)=\int v(t)\,dt=\int\left(3t^2+6t\right)dt=t^3+3t^2+x_0.$

$x(0)=2\Rightarrow x(t)=t^3+3t^2+2\Rightarrow x(1)=1^3+3(1)^2+2=6.$ B

8. $f(x)=\cos(3x)\Rightarrow f'(x)=-\sin(3x)(3)=-3\sin(3x)$

$f'\left(\dfrac{\pi}{9}\right)=-3\sin\left(3\left(\dfrac{\pi}{9}\right)\right)=-3\sin\left(\dfrac{\pi}{3}\right)=-3\left(\dfrac{\sqrt{3}}{2}\right)=-\dfrac{3\sqrt{3}}{2}.$ E

9. $g(-3)=\int_{-2}^{-3} f(t)\,dt=-\int_{-3}^{-2} f(t)\,dt=-\frac{1}{2}$; $g(-2)=\int_{-2}^{-2} f(t)\,dt=0$; $g(0)=\int_{-2}^{0} f(t)\,dt=1$;

$g(1)=\int_{-2}^{1} f(t)\,dt=2$; $g(2)=\int_{-2}^{2} f(t)\,dt=1$. Therefore $g(1)$ is greatest. D

10. The right Riemann sum of a decreasing concave down function is an underestimate of the exact area, and is under the trapezoid sum of a decreasing concave down function which is also an underestimate of the exact area. C

11. Since *f* is increasing on the far right side of the graph, f' must be positive on the far right side of the graph. That rules out C, D, and E. Since *f* has a relative maximum to the left of the origin and a relative minimum to the right of the origin, f' must have a zero to the left and to the right of the origin. B is the only graph that meets this criteria. B

12. $f(x)=e^{\frac{2}{x}}=e^{2x^{-1}} \Rightarrow f'(x)=e^{2x^{-1}}\left(-2x^{-2}\right)=e^{\frac{2}{x}}\left(-\frac{2}{x^2}\right)=-\frac{2e^{\frac{2}{x}}}{x^2}$. D

13. $f(x)=x^2+2x \Rightarrow f(\ln x)=(\ln x)^2+2(\ln x)$

$\Rightarrow f'(x)=2(\ln x)\left(\frac{1}{x}\right)+\frac{2}{x}=\left(\frac{2}{x}\right)(\ln x)+\frac{2}{x}=\left(\frac{2}{x}\right)(\ln x+1)$. A

14. The behavior of the graph of *f* is not known between the given points, so we do not know if *f* is increasing or decreasing on the interval (0, 2). Since we do not know whether *f* is increasing or decreasing immediately before and after $x = 1$, we do not know that *f* has a local maximum at $x = 1$. Since we do not know the behavior of *f* in between the given points, we do not know whether *f* is concave up or concave down immediately before and after $x = 1$, so we do not know that *f* has an inflection point at $x = 1$. Since we know that f'' changes from positive to negative on the interval (0, 2), the graph of *f* changes concavity at least once in the interval (0, 2) E

15. $\int \frac{x}{x^2-4}\,dx=\frac{1}{2}\int\frac{1}{u}\,du=\frac{1}{2}\ln|u|+C=\frac{1}{2}\ln\left|x^2-4\right|+C.$

$u=x^2-4$; $du=2x\,dx$ C

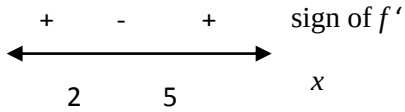
16. $\sin(xy)=x$; find $\frac{dy}{dx}$.

$\frac{d}{dx}(\sin(xy))=\frac{d}{dx}(x)\Rightarrow \cos(xy)\cdot\left(x\cdot\frac{dy}{dx}+y\cdot 1\right)=1\Rightarrow x\cos(xy)\cdot\frac{dy}{dx}+y\cos(xy)=1$

$x\cos(xy)\cdot\frac{dy}{dx}=1-y\cos(xy)\Rightarrow \frac{dy}{dx}=\frac{1-y\cos(xy)}{x\cos(xy)}$. D

17. $g(x)=\int_0^x f(t)\,dt \Rightarrow g'(x)=f(x)$ by the second FTC.

$g''(x)=f'(x)=0$ at $x=2$, $x=5$.



g has an inflection point at x = 2 and x = 5. C

18. $x+y=k$ is tangent to the graph of $y=x^2+3x+1$. Find k.

The slope of the tangent line to the graph of $y=x^2+3x+1=\dfrac{dy}{dx}=2x+3$.

The equation of the tangent line is $y=-x+k \Rightarrow$ slope is $-1 \Rightarrow 2x+3=-1 \Rightarrow x=-2$.

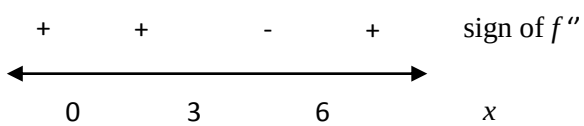
At $x=-2$, $y=(-2)^2+3(-2)+1=-1$. $x+y=k \Rightarrow -2+(-1)=k \Rightarrow k=-3$. A

19. $y=\dfrac{5+2^x}{1-2^x}$ has horizontal asymptotes $y=\lim\limits_{x\to\infty}\dfrac{5+2^x}{1-2^x}$ and $y=\lim\limits_{x\to-\infty}\dfrac{5+2^x}{1-2^x}$.

$\lim\limits_{x\to\infty}\dfrac{5+2^x}{1-2^x}=\lim\limits_{x\to\infty}\dfrac{\frac{5}{2^x}+\frac{2^x}{2^x}}{\frac{1}{2^x}-\frac{2^x}{2^x}}=\lim\limits_{x\to\infty}\dfrac{\frac{5}{2^x}+1}{\frac{1}{2^x}-1}=-1$; $\lim\limits_{x\to-\infty}\dfrac{5+2^x}{1-2^x}=5 \Rightarrow y=-1$ and $y=5$. E

20. Given $f''(x)=x^2(x-3)(x-6)$, the critical numbers are $x=0$, $x=3$, and $x=6$.

Consider the sign chart shown below.



Since $f''(x)$ changes from positive to negative at x = 3 and from negative to positive at x = 6, there are inflection points at x = 3 and x = 6. D

21. $v(t)$ is increasing when $v'(t)$ is positive $\Rightarrow a(t)$ is positive, so the position function $x(t)$ is concave up. This occurs on the interval $0 < t < 2$. A

22. $N=\#$people, $p=\#$ people who heard the rumor $\Rightarrow N-p=\#$ people who did not hear the rumor.

The rate at which the rumor spreads is proportional to $p(N-p) \Rightarrow \dfrac{dp}{dt}=k\cdot p(N-p)$. B

23. $\frac{dy}{dx}=\frac{x^2}{y}\Rightarrow y\,dy=x^2\,dx\Rightarrow\int y\,dy=\int x^2\,dx\Rightarrow\frac{y^2}{2}=\frac{x^3}{3}+C$

$y(3)=-2\Rightarrow\frac{(-2)^2}{2}=\frac{(3)^3}{3}+C\Rightarrow C=-7\Rightarrow\frac{y^2}{2}=\frac{x^3}{3}-7\Rightarrow y^2=\frac{2}{3}x^3-14$

$y=\pm\sqrt{\frac{2}{3}x^3-14}$ and $y(3)=-2\Rightarrow -2=\pm\sqrt{\frac{2}{3}(3)^3-14}\Rightarrow -2=\pm\sqrt{4}$

Therefore, choose the negative branch $y=-\sqrt{\frac{2}{3}x^3-14}$. E

24. $f'(2)=4\Rightarrow$ slope of the tangent line of f at $x = 2$ is $m = 4$. $f(2)=1\Rightarrow$ the point of tangency is $(x_1, y_1)=(2,1)$. The equation of the tangent line using $y-y_1=m(x-x_1)$ is as follows: $y-(1)=(4)(x-(2))\Rightarrow y=4x-7$. Using the tangent line, $f(1.9)\approx 4(1.9)-7=0.6$. B

25. $f(x)=\begin{cases} cx+d & \text{for } x\le 2\\ x^2-cx & \text{for } x>2\end{cases}$

Since f is differentiable at $x = 2$, f is continuous at $x = 2$ $\Rightarrow \lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)$.

$\lim_{x\to 2^-} f(x)=\lim_{x\to 2}(cx+d)=c(2)+d=2c+d.$

$\lim_{x\to 2^+} f(x)=\lim_{x\to 2}(x^2-cx)=(2)^2-c(2)=4-2c\Rightarrow 2c+d=4-2c.$

f is differentiable at $x = 2$, so the derivative from the left equals the derivative from the right. The derivative from the left is c and the derivative from the right is $2x-c\Rightarrow c=2x-c$. At $x=2$, $c=4-c\Rightarrow c=2$.

$2c+d=4-2c$ and $c=2\Rightarrow 2(2)+d=4-2(2)\Rightarrow d=-4\Rightarrow c+d=-2.$ B

26. Given $y=\arctan(4x)$, the slope of the tangent line at $x=\frac{1}{4}=\frac{dy}{dx}$ evaluated at $x=\frac{1}{4}$. $\frac{dy}{dx}=\frac{1}{1+(4x)^2}(4)=\frac{4}{1+16x^2}$. At $x=\frac{1}{4}$, $\frac{dy}{dx}=\frac{4}{1+16\left(\frac{1}{4}\right)^2}=2.$ A

27. Since the slopes in the slope field all equal 0 when $x = -1$, all choices except C and E are eliminated. E is not the answer since the slopes for all y would have to be the same for all choices of x (there is no y in the differential equation). C

28. $g'(x)=\frac{1}{f'(g(x))}\Rightarrow g'(3)=\frac{1}{f'(g(3))}$. $f(6)=3\Rightarrow f^{-1}(3)=6\Rightarrow g(3)=6.$

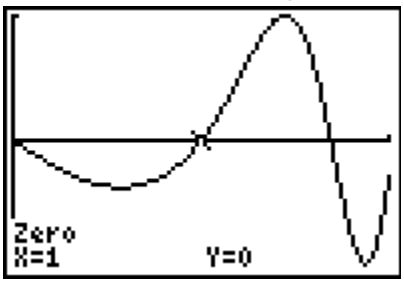
$\Rightarrow g'(3)=\frac{1}{f'(6)}=-\frac{1}{2}.$ A

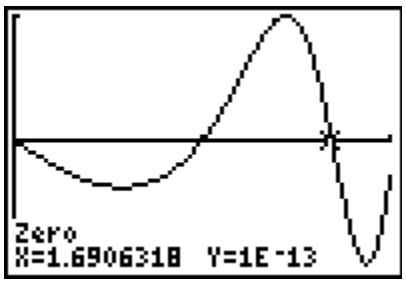
PART B – (Graphing Calculator Required) –

76. f is increasing on the intervals where f' is positive which is [-2, 3]. There is some controversy on whether to include x = -2, x = 1, and x = 3 in the answer, but they are included this time. B

77. Since $\lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$, $\lim_{x\to 2} f(x)$ does not exist. C

78. Consider the graph of f' shown below.



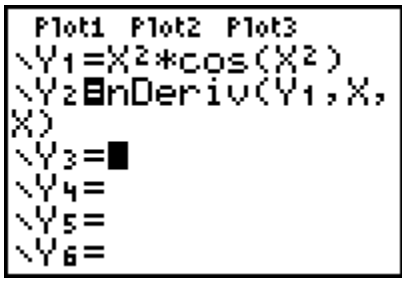


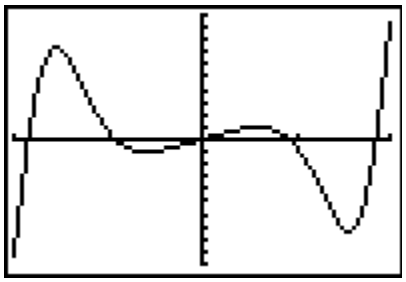
f is increasing on the intervals where f' is positive. This is on the interval [1, 1.691]. B

79. $\int_{-5}^{2} f(x)\,dx = -17$ and $\int_{5}^{2} f(x)\,dx = -4 \Rightarrow \int_{2}^{5} f(x)\,dx = -\int_{5}^{2} f(x)\,dx = -(-4) = 4$

$\Rightarrow \int_{2}^{5} f(x)\,dx + \int_{5}^{2} f(x)\,dx = -17 + 4 = -13.$ B

80. The graph of f'' is shown below. The points of inflection are where f'' changes sign.

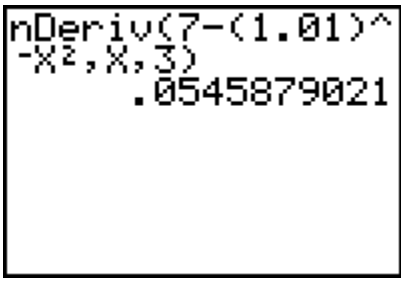




This occurs five times. E

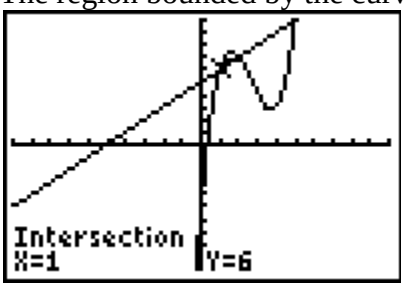
81. $\int_{2}^{4} f(x)\,dx = G(4) - G(2) \Rightarrow G(4) = G(2) + \int_{2}^{4} f(x)\,dx = -7 + \int_{2}^{4} f(x)\,dx.$ E

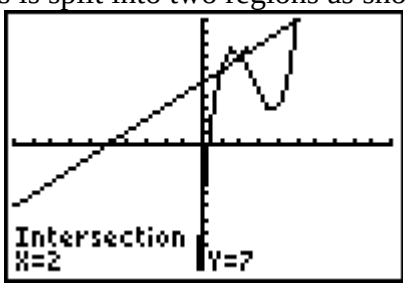
82. $a(3) = v'(3) \approx 0.055.$

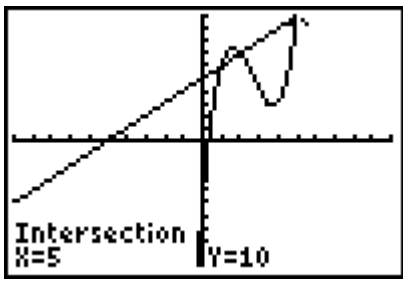


B

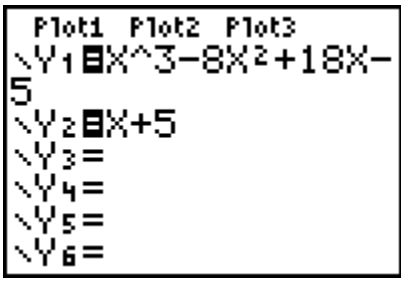
83. The region bounded by the curves is split into two regions as shown below.

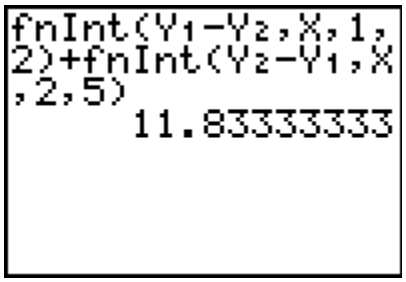






Using the calculator, the area is shown below.





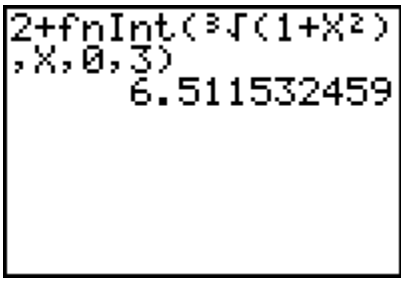
Formally, the area is $\int_1^2 \left(\left(x^3 - 8x^2 + 18x - 5\right) - \left(x + 5\right)\right) dx + \int_2^5 \left(\left(x^3 - 8x^2 + 18x - 5\right) - \left(x + 5\right)\right) dx.$ B

84. f has a relative maximum where f' changes from positive to negative. This occurs at $x = 4$. C

85. $\int_{-4}^{-1} f'(x)\, dx = f(-1) - f(-4) = -1.5 - 0.75 = 2.25.$ B

86. The particle starts at the origin, which eliminates B and D. The velocity is negative at $t = 0$, so the particle is moving down at that point, which eliminates A. The velocity is positive and increasing at $t = 1$ and $t = 2$, so the particle is moving up at an increasing rate. The velocity is 0 at $t = 3$, so the answer is C. C

87. $x(3) = x(0) + \int_0^3 v(t)\,dt = 2 + \int_0^3 \sqrt[3]{1+t^2}\,dt \approx 6.512.$



D

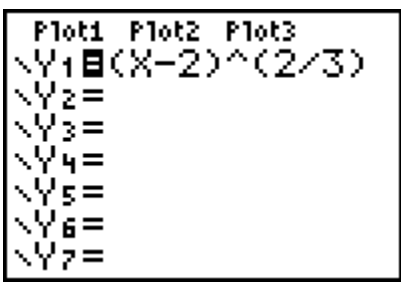
88. The radius of the sphere is decreasing at a rate of 2 centimeters per second $\Rightarrow \frac{dr}{dt} = -2\,\text{cm}/\text{sec}$.

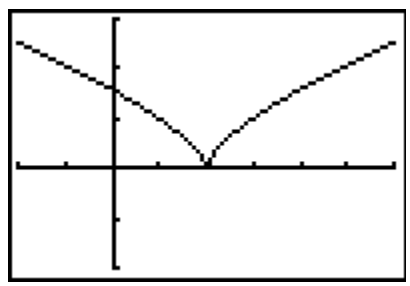
The rate of change of the surface area of the sphere is $\frac{dS}{dt}$ given $S = 4\pi r^2$.

Differentiating with respect to t, $\frac{d}{dt}(S) = \frac{d}{dt}(4\pi r^2) \Rightarrow \frac{dS}{dt} = 4\pi\left(2r\frac{dr}{dt}\right)$.

At $r = 3$, $\frac{dS}{dt} = 4\pi\left(2(3\text{ cm})\left(-2\,\text{cm}/\text{sec}\right)\right) = -48\pi\,\text{cm}^2/\text{sec}$. C

89. f could be decreasing on the interval (–2, 0), reach a minimum value at $x = 0$ at a cusp so f is not differentiable at $x = 0$ (which rules out A, B, C, and D), and then be decreasing on the interval (0, 2). An example is shown in the graph below.





E

90. Approximate $f'(3)$ using a symmetric difference quotient $f'(3) \approx \frac{f(4) - f(2)}{4-2}$ Since $f'(3) = 2$, we can rule out B and C. Since $f''(3) < 0$, f is concave down and increasing at a decreasing rate. The function in D is increasing at a constant rate, the function in E is increasing at an increasing rate, and the function in A is increasing at a decreasing rate. A

91. The average value of y on the interval (–1, 3) is $\frac{1}{3-(-1)}\int_{-1}^{3} \frac{\cos x}{x^2 + x + 2}\,dx \approx 0.183.$ C

92. $\int_0^4 f(x)\,dx$ gives the population along a horizontal strip with a width of one mile perpendicular to the river. Since the population density is consistent at any point along a strip x miles from the river's edge, the same integral applies for all 7 miles of the city, so the total population is $7\int_0^4 f(x)\,dx$. B