AP Calculus AB 2008 Multiple Choice Exam Solutions

PART A – (No Calculator Allowed) –

1. \( \lim_{x \to \infty} \frac{(2x-1)(3-x)}{(x-1)(x+3)} = \lim_{x \to \infty} \frac{6x-2x^2-3+x}{x^2+2x-3} = \lim_{x \to \infty} \frac{-2x^2+7x-3}{x^2+2x-3} \)
   \[= \lim_{x \to \infty} \frac{-2x^2 + 7x - 3}{x^2} = \lim_{x \to \infty} \frac{-2 + \frac{7}{x} - \frac{3}{x^2}}{1 + \frac{2}{x} - \frac{3}{x^2}} = -2 \]
   B

2. \( \int \frac{1}{x^2} \, dx = \int x^{-2} \, dx = \frac{x^{-1}}{-1} + C = -x^{-1} + C. \)  
   D

3. \( f'(x) = (x-1) \left( 3 \left( x^2 + 2 \right)^2 (2x) \right) + \left( x^2 + 2 \right)^3 (1) = \left( x^2 + 2 \right)^2 (6x(x-1) + x^2 + 2) \)
   \[= \left( x^2 + 2 \right)^2 \left( 6x^2 - 6x + x^2 + 2 \right) = \left( x^2 + 2 \right)^2 \left( 7x^2 - 6x + 2 \right). \]  
   D

4. \( \int \sin(2x) + \cos(2x) \, dx = \frac{1}{2} \int \sin u + \cos u \, du = \frac{1}{2} (-\cos u + \sin u) + C \)
   \[u = 2x, \ du = 2 \, dx \]
   \[= \frac{1}{2} (-\cos(2x) + \sin(2x)) + C. \]  
   B

5. \( \lim_{x \to 0} \frac{5x^4 + 8x^2}{3x^4 - 16x^2} = \lim_{x \to 0} \frac{x^2 \left( 5x^2 + 8 \right)}{3x^2 \left( x^2 - 16 \right)} = \lim_{x \to 0} \frac{5x^2 + 8}{3x^2 - 16} = \frac{5(0)^2 + 8}{3(0)^2 - 16} = \frac{1}{2} \)
   A

6. I is true since \( \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) = \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2} (x+2) = (2) + 2 = 4. \)
   II is false since \( f \) is not continuous at \( x = 2 \) since \( \lim_{x \to 2^-} f(x) \neq f(2) \).
   III is false since \( f \) is not differentiable at \( x = 2 \) because \( f \) is not continuous at \( x = 2. \)  
   A

7. \( v(t) = 3t^2 + 6t \Rightarrow x(t) = \int v(t) \, dt = \int (3t^2 + 6t) \, dt = t^3 + 3t^2 + x_0. \)
   \[x(0) = 2 \Rightarrow x(t) = t^3 + 3t^2 + 2 \Rightarrow x(1) = 1^3 + 3(1)^2 + 2 = 6. \]  
   B

8. \( f(x) = \cos(3x) \Rightarrow f'(x) = -\sin(3x)(3) = -3\sin(3x) \)
   \[f'(\frac{\pi}{9}) = -3\sin \left( \frac{\pi}{3} \right) = -3\sin \left( \frac{\pi}{3} \right) = -3 \left( \frac{\sqrt{3}}{2} \right) = -\frac{3\sqrt{3}}{2}. \)  
   E
9. \( g(-3) = \int_{-2}^{-3} f(t) \, dt = -\frac{1}{2}; g(-2) = \int_{-2}^{-3} f(t) \, dt = 0; g(0) = \int_{-2}^{0} f(t) \, dt = 1; \)

\( g(1) = \int_{-2}^{1} f(t) \, dt = 2; g(2) = \int_{-2}^{2} f(t) \, dt = 1. \) Therefore \( g(1) \) is greatest. D

10. The right Riemann sum of a decreasing concave down function is an underestimate of the exact area, and is under the trapezoid sum of a decreasing concave down function which is also an underestimate of the exact area. C

11. Since \( f \) is increasing on the far right side of the graph, \( f' \) must be positive on the far right side of the graph. That rules out C, D, and E. Since \( f \) has a relative maximum to the left of the origin and a relative minimum to the right of the origin, \( f' \) must have a zero to the left and to the right of the origin. B is the only graph that meets this criteria. B

12. \( f(x) = e^{x^2} = e^{2x^{-1}} \Rightarrow f'(x) = e^{2x^{-1}} (-2x^{-2}) = e^{x^2} \left(-\frac{2}{x^2}\right) = -\frac{2e^x}{x^2}. \) D

13. \( f(x) = x^2 + 2x \Rightarrow f'(\ln x) = (\ln x)^2 + 2(\ln x) \Rightarrow f'(x) = 2(\ln x) \cdot \left(\frac{1}{x}\right) + \frac{2}{x} \cdot (\ln x) + \frac{2}{x} \cdot (\ln x + 1). \) A

14. The behavior of the graph of \( f \) is not known between the given points, so we do not know if \( f \) is increasing or decreasing on the interval \((0, 2)\). Since we do not know whether \( f \) is increasing or decreasing immediately before and after \( x = 1 \), we do not know that \( f \) has a local maximum at \( x = 1 \). Since we do not know the behavior of \( f \) in between the given points, we do not know whether \( f \) is concave up or concave down immediately before and after \( x = 1 \), so we do not know that \( f \) has an inflection point at \( x = 1 \). Since we know that \( f'' \) changes from positive to negative on the interval \((0, 2)\), the graph of \( f \) changes concavity at least once in the interval \((0, 2)\). E

15. \( \int \frac{x}{\sqrt{x^2 - 4}} \, dx = \frac{1}{2} \int \frac{1}{u} \, du = \frac{1}{2} \ln |u| + C = \frac{1}{2} \ln |x^2 - 4| + C. \)

\( u = x^2 - 4; \, du = 2x \, dx \) C

16. \( \sin (xy) = x; \) find \( \frac{dy}{dx}. \)

\( \frac{d}{dx} (\sin (xy)) = \frac{d}{dx} (x) \Rightarrow \cos (xy) \left( x \left( \frac{dy}{dx} + y \cdot 1 \right) \right) = 1 \Rightarrow x \cos (xy) \cdot \frac{dy}{dx} + y \cos (xy) = 1 \Rightarrow x \cos (xy) \cdot \frac{dy}{dx} = 1 - y \cos (xy) \Rightarrow \frac{dy}{dx} = \frac{1 - y \cos (xy)}{x \cos (xy)}. \) D
17. \( g(x) = \int_0^x f(t) \, dt \Rightarrow g'(x) = f(x) \) by the second FTC.

\[ g''(x) = f'(x) = 0 \text{ at } x = 2, x = 5. \]

\[
\begin{array}{c|ccc}
& + & - & + \\
\hline
2 & 5 & x
\end{array}
\]

\( g \) has an inflection point at \( x = 2 \) and \( x = 5 \).

18. \( x + y = k \) is tangent to the graph of \( y = x^2 + 3x + 1 \). Find \( k \).

The slope of the tangent line to the graph of \( y = x^2 + 3x + 1 = \frac{dy}{dx} = 2x + 3 \).

The equation of the tangent line is \( y = -x + k \Rightarrow \text{slope is} \ -1 \Rightarrow 2x + 3 = -1 \Rightarrow x = -2 \).

At \( x = -2 \), \( y = (-2)^2 + 3(-2) + 1 = -1 \). \( x + y = k \Rightarrow -2 + (-1) = k \Rightarrow k = -3 \).

19. \( y = \frac{5 + 2^x}{1 - 2^x} \) has horizontal asymptotes \( y = \lim_{x \to \infty} \frac{5 + 2^x}{1 - 2^x} \) and \( y = \lim_{x \to -\infty} \frac{5 + 2^x}{1 - 2^x} \).

\[
\lim_{x \to \infty} \frac{5 + 2^x}{1 - 2^x} = \lim_{x \to \infty} \frac{5}{2^x} + \lim_{x \to \infty} \frac{2^x}{2^x} = -1; \quad \lim_{x \to -\infty} \frac{5 + 2^x}{1 - 2^x} = 5 \Rightarrow y = -1 \text{ and } y = 5.
\]

20. Given \( f''(x) = x^2(x-3)(x-6), \) the critical numbers are \( x = 0, x = 3, \) and \( x = 6 \).

Consider the sign chart shown below.

\[
\begin{array}{c|ccc}
& + & + & - & + \\
\hline
0 & 3 & 6 & x
\end{array}
\]

Since \( f''(x) \) changes from positive to negative at \( x = 3 \) and from negative to positive at \( x = 6 \), there are inflection points at \( x = 3 \) and \( x = 6 \).

21. \( v(t) \) is increasing when \( v'(t) \) is positive \( \Rightarrow \) \( a(t) \) is positive, so the position function \( x(t) \) is concave up. This occurs on the interval \( 0 < t < 2 \).

22. \( N = \#\text{people}, \ p = \#\text{people who heard the rumor} \Rightarrow N - p = \#\text{people who did not hear the rumor} \).

The rate at which the rumor spreads is proportional to \( p(N-p) \Rightarrow \frac{dp}{dt} = k \cdot p(N-p) \).
23. \[ \frac{dy}{dx} = \frac{x^2}{y} \Rightarrow y \, dy = x^2 \, dx \Rightarrow \int y \, dy = \int x^2 \, dx \Rightarrow \frac{y^2}{2} = \frac{x^3}{3} + C \]

\[ y(3) = -2 \Rightarrow \frac{(-2)^2}{2} = \frac{(3)^3}{3} + C \Rightarrow C = -7 \Rightarrow \frac{y^2}{2} = \frac{x^3}{3} - 7 \Rightarrow y^2 = \frac{2}{3}x^3 - 14 \]

\[ y = \pm \sqrt[3]{\frac{2}{3}}x^3 - 14 \] and \[ y(3) = -2 \Rightarrow -2 = \pm \sqrt[3]{\frac{2}{3}}(3)^3 - 14 \Rightarrow -2 = \pm \sqrt[3]{4} \]

Therefore, choose the negative branch \[ y = -\sqrt[3]{\frac{2}{3}}x^3 - 14. \]

24. \[ f'(2) = 4 \Rightarrow \text{slope of the tangent line of } f \text{ at } x = 2 \text{ is } m = 4. \quad f(2) = 1 \Rightarrow \text{the point of tangency is } (x_1, y_1) = (2, 1). \] The equation of the tangent line using \[ y - y_1 = m(x - x_1) \] is as follows:

\[ y - 1 = 4(x - 2) \Rightarrow y = 4x - 7. \] Using the tangent line, \[ f(1.9) \approx 4(1.9) - 7 = 0.6. \]

25. \[ f(x) = \begin{cases} cx + d & \text{for } x \leq 2 \\ x^3 - cx & \text{for } x > 2 \end{cases} \]

Since \( f \) is differentiable at \( x = 2, f \) is continuous at \( x = 2 \Rightarrow \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x). \)

\[ \lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} (cx + d) = c(2) + d = 2c + d. \]

\[ \lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x^3 - cx) = (2)^3 - c(2) = 4 - 2c \Rightarrow 2c + d = 4 - 2c. \]

\( f \) is differentiable at \( x = 2, \) so the derivative from the left equals the derivative from the right. The derivative from the left is \( c \) and the derivative from the right is \( 2x - c \Rightarrow c = 2x - c. \) At \( x = 2, c = 4 - c \Rightarrow c = 2. \)

\[ 2c + d = 4 - 2c \] and \( c = 2 \Rightarrow 2(2) + d = 4 - 2(2) \Rightarrow d = -4 \Rightarrow c + d = -2. \]

26. Given \( y = \arctan(4x), \) the slope of the tangent line at \( x = \frac{1}{4}, \) evaluated at \( x = \frac{1}{4}, \)

\[ \frac{dy}{dx} = \frac{1}{1 + (4x)^2} = \frac{4}{1 + 16x^2}. \] At \( x = \frac{1}{4}, \)

\[ \frac{dy}{dx} = \frac{4}{1 + 16\left(\frac{1}{4}\right)^2} = 2. \]

27. Since the slopes in the slope field all equal 0 when \( x = -1, \) all choices except \( C \) and \( E \) are eliminated. \( E \) is not the answer since the slopes for all \( y \) would have to be the same for all choices of \( x \) (there is no \( y \) in the differential equation).

28. \[ g'(x) = \frac{1}{f'(g(x))} \Rightarrow g'(3) = \frac{1}{f'(g(3))}. \quad f(6) = 3 \Rightarrow f^{-1}(3) = 6 \Rightarrow g(3) = 6. \]

\[ \Rightarrow g'(3) = \frac{1}{f'(6)} = -\frac{1}{2}. \]
PART B – (Graphing Calculator Required) –

76. \( f \) is increasing on the intervals where \( f' \) is positive which is \([-2, 3]\). There is some controversy on whether to include \( x = -2, x = 1, \) and \( x = 3 \) in the answer, but they are included this time.

77. Since \( \lim_{x \to 2^-} f(x) \neq \lim_{x \to 2^+} f(x) \), \( \lim_{x \to 2} f(x) \) does not exist.

78. Consider the graph of \( f' \) shown below.

\[
\begin{align*}
\text{Graph of } f' &\quad \text{Graph of } f''
\end{align*}
\]

\( f \) is increasing on the intervals where \( f' \) is positive. This is on the interval \([1, 1.691]\).

79. \[
\int_{-5}^{2} f(x) \, dx = -17 \text{ and } \int_{5}^{2} f(x) \, dx = -4 \implies \int_{5}^{2} f(x) \, dx = -\int_{5}^{2} f(x) \, dx = -(4) = 4
\]

\[
\implies \int_{5}^{2} f(x) \, dx + \int_{5}^{2} f(x) \, dx = -17 + 4 = -13.
\]

80. The graph of \( f'' \) is shown below. The points of inflection are where \( f'' \) changes sign.

\[
\begin{align*}
\text{Graph of } f'' &\quad \text{Graph of } f''
\end{align*}
\]

This occurs five times.

81. \[
\int_{2}^{4} f(x) \, dx = G(4) - G(2) \implies G(4) = G(2) + \int_{2}^{4} f(x) \, dx = -7 + \int_{2}^{4} f(x) \, dx.
\]
82. $a(3) = v'(3) \approx 0.055$.

$$\text{nDeriv}((7-(1.01)^{-2}),x,3) = 0.05545879021$$

83. The region bounded by the curves is split into two regions as shown below.

Using the calculator, the area is shown below.

Formally, the area is

$$\int_{1}^{2} (x^3 - 8x^2 + 18x - 5 - (x + 5)) \, dx + \int_{2}^{5} (x^3 - 8x^2 + 18x - 5 - (x + 5)) \, dx.$$

84. $f$ has a relative maximum where $f'$ changes from positive to negative. This occurs at $x = 4$.  C

85. $\int_{-4}^{1} f'(x) \, dx = f(-1) - f(-4) = -1.5 - 0.75 = 2.25$.  B

86. The particle starts at the origin, which eliminates B and D. The velocity is negative at $t = 0$, so the particle is moving down at that point, which eliminates A. The velocity is positive and increasing at $t = 1$ and $t = 2$, so the particle is moving up at an increasing rate. The velocity is 0 at $t = 3$, so the answer is C.  C
87. \( x(3) = x(0) + \int_0^3 v(t) \, dt = 2 + \int_0^3 \sqrt{1 + t^2} \, dt \approx 6.512. \)

88. The radius of the sphere is decreasing at a rate of 2 centimeters per second \( \Rightarrow \frac{dr}{dt} = -2 \text{ cm/sec}. \)

The rate of change of the surface area of the sphere is \( \frac{dS}{dt} \) given \( S = 4\pi r^2. \)

Differentiating with respect to \( t, \) \( \frac{d}{dt} (S) = \frac{d}{dt} (4\pi r^2) \Rightarrow \frac{dS}{dt} = 4\pi \left( 2r \frac{dr}{dt} \right). \)

At \( r = 3, \) \( \frac{dS}{dt} = 4\pi \left( 2(3 \text{ cm}) \left( -2 \text{ cm/sec} \right) \right) = -48\pi \text{ cm}^2/\text{sec}. \)

89. \( f \) could be decreasing on the interval \((-2, 0), \) reach a minimum value at \( x = 0 \) at a cusp so \( f \) is not differentiable at \( x = 0 \) (which rules out A, B, C, and D), and then be decreasing on the interval \((0, 2). \) An example is shown in the graph below.

90. Approximate \( f'(3) \) using a symmetric difference quotient \( f'(3) \approx \frac{f(4) - f(2)}{4 - 2} \) Since \( f'(3) = 2, \)

we can rule out B and C. Since \( f''(3) < 0, \) \( f \) is concave down and increasing at a decreasing rate.

The function in D is increasing at a constant rate, the function in E is increasing at an increasing rate, and the function in A is increasing at a decreasing rate.

91. The average value of \( y \) on the interval \((-1, 3) \) is \( \frac{1}{3 - (-1)} \int_{-1}^3 \frac{\cos x}{x^2 + x + 2} \, dx \approx 0.183. \)
92. \( \int_{0}^{4} f(x) \, dx \) gives the population along a horizontal strip with a width of one mile perpendicular to the river. Since the population density is consistent at any point along a strip \( x \) miles from the river’s edge, the same integral applies for all 7 miles of the city, so the total population is \( 7 \int_{0}^{4} f(x) \, dx \).