

Separation of Variables/Slope Fields

1998 AP Calculus AB Scoring Guidelines

4. Let f be a function with $f(1) = 4$ such that for all points (x, y) on the graph of f the slope is given by $\frac{3x^2 + 1}{2y}$.
- (a) Find the slope of the graph of f at the point where $x = 1$.
- (b) Write an equation for the line tangent to the graph of f at $x = 1$ and use it to approximate $f(1.2)$.
- (c) Find $f(x)$ by solving the separable differential equation $\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$ with the initial condition $f(1) = 4$.
- (d) Use your solution from part (c) to find $f(1.2)$.

<p>(a) $\frac{dy}{dx} = \frac{3x^2 + 1}{2y}$</p> $\left. \frac{dy}{dx} \right _{\substack{x=1 \\ y=4}} = \frac{3+1}{2 \cdot 4} = \frac{4}{8} = \frac{1}{2}$	<p>1: answer</p>
<p>(b) $y - 4 = \frac{1}{2}(x - 1)$</p> $f(1.2) - 4 \approx \frac{1}{2}(1.2 - 1)$ $f(1.2) \approx 0.1 + 4 = 4.1$	<p>2 {</p> <ul style="list-style-type: none"> 1: equation of tangent line 1: uses equation to approximate $f(1.2)$
<p>(c) $2y \, dy = (3x^2 + 1) \, dx$</p> $\int 2y \, dy = \int (3x^2 + 1) \, dx$ $y^2 = x^3 + x + C$ $4^2 = 1 + 1 + C$ $14 = C$ $y^2 = x^3 + x + 14$ $y = \sqrt{x^3 + x + 14} \text{ is branch with point } (1, 4)$ $f(x) = \sqrt{x^3 + x + 14}$	<p>5 {</p> <ul style="list-style-type: none"> 1: separates variables 1: antiderivative of dy term 1: antiderivative of dx term 1: uses $y = 4$ when $x = 1$ to pick one function out of a family of functions 1: solves for y 0/1 if solving a linear equation in y 0/1 if no constant of integration <p>Note: max 0/5 if no separation of variables Note: max 1/5 [1-0-0-0-0] if substitutes value(s) for x, y, or dy/dx before antidifferentiation</p>
<p>(d) $f(1.2) = \sqrt{1.2^3 + 1.2 + 14} \approx 4.114$</p>	<p>1: answer, from student's solution to the given differential equation in (c)</p>

Consider the differential equation $\frac{dy}{dx} = \frac{3x^2}{e^{2y}}$.

- (a) Find a solution $y = f(x)$ to the differential equation satisfying $f(0) = \frac{1}{2}$.
 (b) Find the domain and range of the function f found in part (a).

(a) $e^{2y} dy = 3x^2 dx$

$$\frac{1}{2}e^{2y} = x^3 + C_1$$

$$e^{2y} = 2x^3 + C$$

$$y = \frac{1}{2} \ln(2x^3 + C)$$

$$\frac{1}{2} = \frac{1}{2} \ln(0 + C); \quad C = e$$

$$y = \frac{1}{2} \ln(2x^3 + e)$$

(b) Domain: $2x^3 + e > 0$

$$x^3 > -\frac{1}{2}e$$

$$x > \left(-\frac{1}{2}e\right)^{1/3} = -\left(\frac{1}{2}e\right)^{1/3}$$

Range: $-\infty < y < \infty$

$$6 \left\{ \begin{array}{l} 1 : \text{separates variables} \\ 1 : \text{antiderivative of } dy \text{ term} \\ 1 : \text{antiderivative of } dx \text{ term} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } f(0) = \frac{1}{2} \\ 1 : \text{solves for } y \\ \text{Note: } 0/1 \text{ if } y \text{ is not a logarithmic function of } x \end{array} \right.$$

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

$$3 \left\{ \begin{array}{l} 1 : 2x^3 + e > 0 \\ 1 : \text{domain} \\ \text{Note: } 0/1 \text{ if } 0 \text{ is not in the domain} \\ 1 : \text{range} \end{array} \right.$$

Note: 0/3 if y is not a logarithmic function of x

2002 SCORING GUIDELINES (Form B)

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{3-x}{y}$.

- (a) Let $y = f(x)$ be the particular solution to the given differential equation for $1 < x < 5$ such that the line $y = -2$ is tangent to the graph of f . Find the x -coordinate of the point of tangency, and determine whether f has a local maximum, local minimum, or neither at this point. Justify your answer.
- (b) Let $y = g(x)$ be the particular solution to the given differential equation for $-2 < x < 8$, with the initial condition $g(6) = -4$. Find $y = g(x)$.

(a) $\frac{dy}{dx} = 0$ when $x = 3$

$$\left. \frac{d^2y}{dx^2} \right|_{(3,-2)} = \left. \frac{-y - y'(3-x)}{y^2} \right|_{(3,-2)} = \frac{1}{2},$$

so f has a local minimum at this point.

or

Because f is continuous for $1 < x < 5$, there is an interval containing $x = 3$ on which

$y < 0$. On this interval, $\frac{dy}{dx}$ is negative to the left of $x = 3$ and $\frac{dy}{dx}$ is positive to the

right of $x = 3$. Therefore f has a local minimum at $x = 3$.

(b) $y \, dy = (3-x) \, dx$

$$\frac{1}{2}y^2 = 3x - \frac{1}{2}x^2 + C$$

$$8 = 18 - 18 + C; C = 8$$

$$y^2 = 6x - x^2 + 16$$

$$y = -\sqrt{6x - x^2 + 16}$$

$$3 \left\{ \begin{array}{l} 1 : x = 3 \\ 1 : \text{local minimum} \\ 1 : \text{justification} \end{array} \right.$$

$$6 \left\{ \begin{array}{l} 1 : \text{separates variables} \\ 1 : \text{antiderivative of } dy \text{ term} \\ 1 : \text{antiderivative of } dx \text{ term} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition } g(6) = -4 \\ 1 : \text{solves for } y \end{array} \right.$$

Note: max 3/6 [1-1-1-0-0-0] if no constant of integration

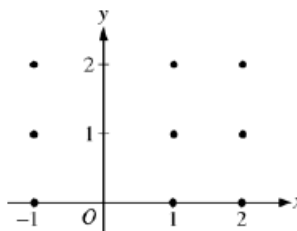
Note: 0/6 if no separation of variables

2008 SCORING GUIDELINES

Question 5

Consider the differential equation $\frac{dy}{dx} = \frac{y-1}{x^2}$, where $x \neq 0$.

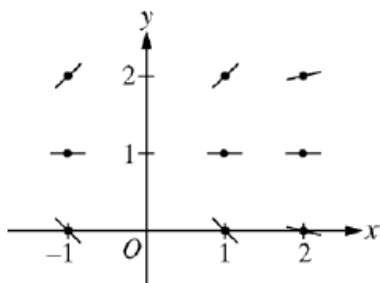
- (a) On the axes provided, sketch a slope field for the given differential equation at the nine points indicated.
(Note: Use the axes provided in the exam booklet.)



- (b) Find the particular solution $y = f(x)$ to the differential equation with the initial condition $f(2) = 0$.

- (c) For the particular solution $y = f(x)$ described in part (b), find $\lim_{x \rightarrow \infty} f(x)$.

(a)



2 : $\begin{cases} 1 : \text{zero slopes} \\ 1 : \text{all other slopes} \end{cases}$

(b) $\frac{1}{y-1} dy = \frac{1}{x^2} dx$

$$\ln|y-1| = -\frac{1}{x} + C$$

$$|y-1| = e^{-\frac{1}{x} + C}$$

$$|y-1| = e^C e^{-\frac{1}{x}}$$

$$y-1 = ke^{-\frac{1}{x}}, \text{ where } k = \pm e^C$$

$$-1 = ke^{-\frac{1}{2}}$$

$$k = -e^{\frac{1}{2}}$$

$$f(x) = 1 - e^{\left(\frac{1}{2} - \frac{1}{x}\right)}, x > 0$$

6 : $\begin{cases} 1 : \text{separates variables} \\ 2 : \text{antidifferentiates} \\ 1 : \text{includes constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max 3/6 [1-2-0-0-0] if no constant of integration

Note: 0/6 if no separation of variables

(c) $\lim_{x \rightarrow \infty} 1 - e^{\left(\frac{1}{2} - \frac{1}{x}\right)} = 1 - \sqrt{e}$

1 : limit

2008 SCORING GUIDELINES (Form B)

Question 6

Consider the closed curve in the xy -plane given by

$$x^2 + 2x + y^4 + 4y = 5.$$

- (a) Show that $\frac{dy}{dx} = \frac{-(x+1)}{2(y^3+1)}$.
- (b) Write an equation for the line tangent to the curve at the point $(-2, 1)$.
- (c) Find the coordinates of the two points on the curve where the line tangent to the curve is vertical.
- (d) Is it possible for this curve to have a horizontal tangent at points where it intersects the x -axis? Explain your reasoning.

(a) $2x + 2 + 4y^3 \frac{dy}{dx} + 4 \frac{dy}{dx} = 0$

$$(4y^3 + 4) \frac{dy}{dx} = -2x - 2$$

$$\frac{dy}{dx} = \frac{-2(x+1)}{4(y^3+1)} = \frac{-(x+1)}{2(y^3+1)}$$

2 : $\begin{cases} 1 : \text{implicit differentiation} \\ 1 : \text{verification} \end{cases}$

(b) $\left. \frac{dy}{dx} \right|_{(-2,1)} = \frac{-(-2+1)}{2(1+1)} = \frac{1}{4}$

Tangent line: $y = 1 + \frac{1}{4}(x + 2)$

2 : $\begin{cases} 1 : \text{slope} \\ 1 : \text{tangent line equation} \end{cases}$

- (c) Vertical tangent lines occur at points on the curve where $y^3 + 1 = 0$ (or $y = -1$) and $x \neq -1$.

On the curve, $y = -1$ implies that $x^2 + 2x + 1 - 4 = 5$, so $x = -4$ or $x = 2$.

Vertical tangent lines occur at the points $(-4, -1)$ and $(2, -1)$.

3 : $\begin{cases} 1 : y = -1 \\ 1 : \text{substitutes } y = -1 \text{ into the} \\ \text{equation of the curve} \\ 1 : \text{answer} \end{cases}$

- (d) Horizontal tangents occur at points on the curve where $x = -1$ and $y \neq -1$.

The curve crosses the x -axis where $y = 0$.

$$(-1)^2 + 2(-1) + 0^4 + 4 \cdot 0 \neq 5$$

No, the curve cannot have a horizontal tangent where it crosses the x -axis.

2 : $\begin{cases} 1 : \text{works with } x = -1 \text{ or } y = 0 \\ 1 : \text{answer with reason} \end{cases}$

2010 SCORING GUIDELINES

Question 6

Solutions to the differential equation $\frac{dy}{dx} = xy^3$ also satisfy $\frac{d^2y}{dx^2} = y^3(1 + 3x^2y^2)$. Let $y = f(x)$ be a particular solution to the differential equation $\frac{dy}{dx} = xy^3$ with $f(1) = 2$.

- (a) Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$.
- (b) Use the tangent line equation from part (a) to approximate $f(1.1)$. Given that $f(x) > 0$ for $1 < x < 1.1$, is the approximation for $f(1.1)$ greater than or less than $f(1.1)$? Explain your reasoning.
- (c) Find the particular solution $y = f(x)$ with initial condition $f(1) = 2$.

(a) $f'(1) = \left. \frac{dy}{dx} \right|_{(1,2)} = 8$

An equation of the tangent line is $y = 2 + 8(x - 1)$.

(b) $f(1.1) \approx 2.8$

Since $y = f(x) > 0$ on the interval $1 \leq x < 1.1$,

$$\frac{d^2y}{dx^2} = y^3(1 + 3x^2y^2) > 0 \text{ on this interval.}$$

Therefore on the interval $1 < x < 1.1$, the line tangent to the graph of $y = f(x)$ at $x = 1$ lies below the curve and the approximation 2.8 is less than $f(1.1)$.

(c) $\frac{dy}{dx} = xy^3$

$$\int \frac{1}{y^3} dy = \int x dx$$

$$-\frac{1}{2y^2} = \frac{x^2}{2} + C$$

$$-\frac{1}{2 \cdot 2^2} = \frac{1^2}{2} + C \Rightarrow C = -\frac{5}{8}$$

$$y^2 = \frac{1}{\frac{5}{4} - x^2}$$

$$f(x) = \frac{2}{\sqrt{5 - 4x^2}}, \quad \frac{-\sqrt{5}}{2} < x < \frac{\sqrt{5}}{2}$$

$$2 : \begin{cases} 1 : f'(1) \\ 1 : \text{answer} \end{cases}$$

$$2 : \begin{cases} 1 : \text{approximation} \\ 1 : \text{conclusion with explanation} \end{cases}$$

$$5 : \begin{cases} 1 : \text{separation of variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables

2010 SCORING GUIDELINES (Form B)

Question 5

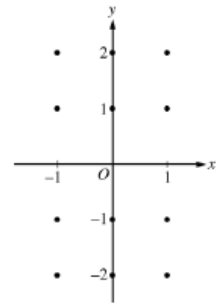
Consider the differential equation $\frac{dy}{dx} = \frac{x+1}{y}$.

- (a) On the axes provided, sketch a slope field for the given differential equation at the twelve points indicated, and for $-1 < x < 1$, sketch the solution curve that passes through the point $(0, -1)$.

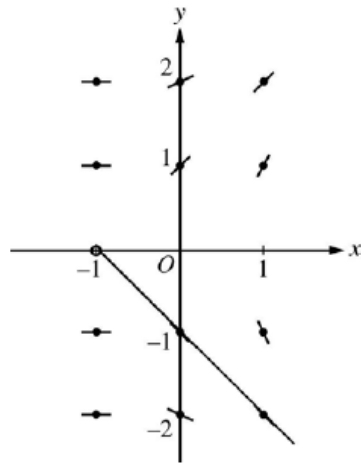
(Note: Use the axes provided in the exam booklet.)

- (b) While the slope field in part (a) is drawn at only twelve points, it is defined at every point in the xy -plane for which $y \neq 0$. Describe all points in the xy -plane, $y \neq 0$, for which $\frac{dy}{dx} = -1$.

- (c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(0) = -2$.



(a)



3 : $\begin{cases} 1 : \text{zero slopes} \\ 1 : \text{nonzero slopes} \\ 1 : \text{solution curve through } (0, -1) \end{cases}$

(b) $-1 = \frac{x+1}{y} \Rightarrow y = -x - 1$

$\frac{dy}{dx} = -1$ for all (x, y) with $y = -x - 1$ and $y \neq 0$

1 : description

(c) $\int y \, dy = \int (x+1) \, dx$

$\frac{y^2}{2} = \frac{x^2}{2} + x + C$

$\frac{(-2)^2}{2} = \frac{0^2}{2} + 0 + C \Rightarrow C = 2$

$y^2 = x^2 + 2x + 4$

Since the solution goes through $(0, -2)$, y must be negative. Therefore $y = -\sqrt{x^2 + 2x + 4}$.

5 : $\begin{cases} 1 : \text{separates variables} \\ 1 : \text{antiderivatives} \\ 1 : \text{constant of integration} \\ 1 : \text{uses initial condition} \\ 1 : \text{solves for } y \end{cases}$

Note: max 2/5 [1-1-0-0-0] if no constant of integration

Note: 0/5 if no separation of variables