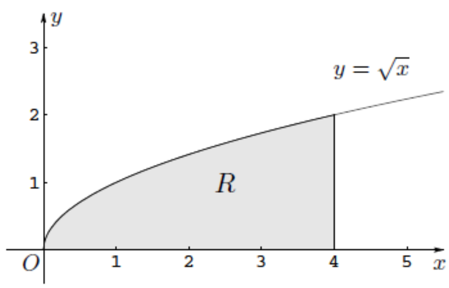
Area / Volume

1998 AP Calculus AB Scoring Guidelines

1. Let R be the region bounded by the x–axis, the graph of $y = \sqrt{x}$, and the line $x = 4$.
   (a) Find the area of the region R.
   (b) Find the value of h such that the vertical line $x = h$ divides the region R into two regions of equal area.
   (c) Find the volume of the solid generated when R is revolved about the x–axis.
   (d) The vertical line $x = k$ divides the region R into two regions such that when these two regions are revolved about the x–axis, they generate solids with equal volumes. Find the value of k.

(a)



$$A = \int_0^4 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big|_0^4 = \frac{16}{3} \quad \text{or} \quad 5.333$$

2 { 1: $A = \int_0^4 \sqrt{x}\,dx$; 1: answer }

(b)
$$\int_0^h \sqrt{x}\,dx = \frac{8}{3} \qquad \text{–or–} \qquad \int_0^h \sqrt{x}\,dx = \int_h^4 \sqrt{x}\,dx$$

$$\frac{2}{3}h^{3/2} = \frac{8}{3} \qquad\qquad \frac{2}{3}h^{3/2} = \frac{16}{3} - \frac{2}{3}h^{3/2}$$

$h = \sqrt[3]{16}$ or 2.520 or 2.519

2 { 1: equation in h ; 1: answer }

(c)
$$V = \pi \int_0^4 (\sqrt{x})^2\,dx = \pi \frac{x^2}{2}\Big|_0^4 = 8\pi$$

or 25.133 or 25.132

3 { 1: limits and constant ; 1: integrand ; 1: answer }

(d)
$$\pi \int_0^k (\sqrt{x})^2\,dx = 4\pi \qquad \text{–or–} \qquad \pi \int_0^k (\sqrt{x})^2\,dx = \pi \int_k^4 (\sqrt{x})^2\,dx$$

$$\pi\frac{k^2}{2} = 4\pi \qquad\qquad \pi\frac{k^2}{2} = 8\pi - \pi\frac{k^2}{2}$$

$k = \sqrt{8}$ or 2.828

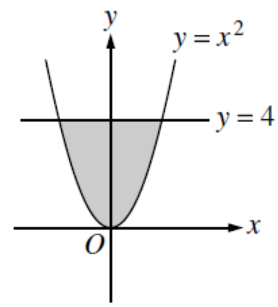
2 { 1: equation in k ; 1: answer }

2. The shaded region, R, is bounded by the graph of $y = x^2$ and the line $y = 4$, as shown in the figure above.

   (a) Find the area of R.

   (b) Find the volume of the solid generated by revolving R about the x–axis.

   (c) There exists a number k, $k > 4$, such that when R is revolved about the line $y = k$, the resulting solid has the same volume as the solid in part (b). Write, but do not solve, an equation involving an integral expression that can be used to find the value of k.



(a) $\text{Area} = \int_{-2}^{2} (4 - x^2)\, dx$

$= 2 \int_0^2 (4 - x^2)\, dx$

$= 2 \left[4x - \frac{x^3}{3} \right]_0^2$

$= \frac{32}{3} = 10.666$ or 10.667

2 { 1: integral; 1: answer }

(b) $\text{Volume} = \pi \int_{-2}^{2} \left(4^2 - (x^2)^2 \right) dx$

$= 2\pi \int_0^2 (16 - x^4)\, dx$

$= 2\pi \left[16x - \frac{x^5}{5} \right]_0^2$

$= \frac{256\pi}{5} = 160.849$ or 160.850

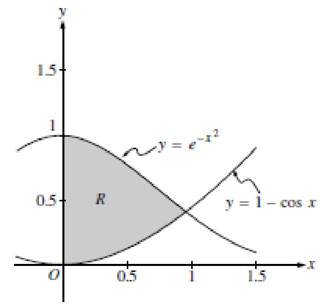
3 { 1: limits and constant; 1: integrand; 1: answer }

(c) $\pi \int_{-2}^{2} \left[(k - x^2)^2 - (k - 4)^2 \right] dx = \frac{256\pi}{5}$

4 { 1: limits and constant; 2: integrand < −1 > each error; 1: equation }

AP Calculus AB–1 / BC–1 **2000**

Let R be the shaded region in the first quadrant enclosed by the graphs of $y = e^{-x^2}$, $y = 1 - \cos x$, and the y-axis, as shown in the figure above.



(a) Find the area of the region R.

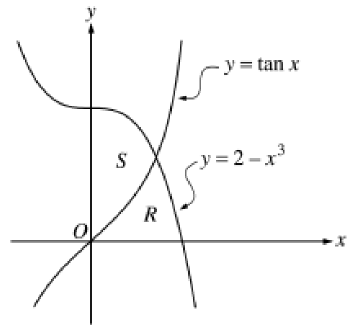
(b) Find the volume of the solid generated when the region R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

Region R $e^{-x^2} = 1 - \cos x$ at $x = 0.941944 = A$	1 : Correct limits in an integral in (a), (b), or (c).
(a) Area $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right) dx$ $= 0.590$ or 0.591	2 { 1 : integrand 1 : answer
(b) Volume $= \pi \int_0^A \left(\left(e^{-x^2}\right)^2 - (1 - \cos x)^2 \right) dx$ $= 0.55596\pi = 1.746$ or 1.747	3 { 2 : integrand and constant < −1> each error 1 : answer
(c) Volume $= \int_0^A \left(e^{-x^2} - (1 - \cos x)\right)^2 dx$ $= 0.461$	3 { 2 : integrand < −1> each error Note: 0/2 if not of the form $k \int_c^d (f(x) - g(x))^2 \, dx$ 1 : answer

Question 1

Let R and S be the regions in the first quadrant shown in the figure above. The region R is bounded by the x-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$. The region S is bounded by the y-axis and the graphs of $y = 2 - x^3$ and $y = \tan x$.



(a) Find the area of R.

(b) Find the area of S.

(c) Find the volume of the solid generated when S is revolved about the x-axis.

Point of intersection

$2 - x^3 = \tan x$ at $(A, B) = (0.902155, 1.265751)$

(a) Area $R = \int_0^A \tan x\,dx + \int_A^{\sqrt[3]{2}} (2 - x^3)\,dx = 0.729$

or

Area $R = \int_0^B \left((2-y)^{1/3} - \tan^{-1} y\right) dy = 0.729$

or

Area $R = \int_0^{\sqrt[3]{2}} (2 - x^3)\,dx - \int_0^A (2 - x^3 - \tan x)\,dx = 0.729$

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(b) Area $S = \int_0^A (2 - x^3 - \tan x)\,dx = 1.160$ or 1.161

or

Area $S = \int_0^B \tan^{-1} y\,dy + \int_B^2 (2-y)^{1/3}\,dy = 1.160$ or 1.161

or

Area S

$= \int_0^2 (2-y)^{1/3}\,dy - \int_0^B \left((2-y)^{1/3} - \tan^{-1} y\right) dy$

$= 1.160$ or 1.161

$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \pi \int_0^A \left((2 - x^3)^2 - \tan^2 x\right) dx$

$= 2.652\pi$ or 8.331 or 8.332

$3 : \begin{cases} 1 : \text{limits and constant} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$

Question 1

Let f and g be the functions given by $f(x) = e^x$ and $g(x) = \ln x$.

(a) Find the area of the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$.

(b) Find the volume of the solid generated when the region enclosed by the graphs of f and g between $x = \frac{1}{2}$ and $x = 1$ is revolved about the line $y = 4$.

(c) Let h be the function given by $h(x) = f(x) - g(x)$. Find the absolute minimum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$, and find the absolute maximum value of $h(x)$ on the closed interval $\frac{1}{2} \le x \le 1$. Show the analysis that leads to your answers.

(a) Area $= \int_{1/2}^{1} (e^x - \ln x)\,dx = 1.222$ or 1.223

2 { 1 : integral
1 : answer }

(b) Volume $= \pi \int_{1/2}^{1} \left((4 - \ln x)^2 - (4 - e^x)^2\right) dx$

$= 7.515\pi$ or 23.609

4 { 1 : limits and constant
2 : integrand
$<-1>$ each error
Note: 0 / 2 if not of the form
$k \int_a^b \left(R(x)^2 - r(x)^2\right) dx$
1 : answer }

(c) $h'(x) = f'(x) - g'(x) = e^x - \frac{1}{x} = 0$

$x = 0.567143$

Absolute minimum value and absolute maximum value occur at the critical point or at the endpoints.

$h(0.567143) = 2.330$

$h(0.5) = 2.3418$

$h(1) = 2.718$

The absolute minimum is 2.330.

The absolute maximum is 2.718.

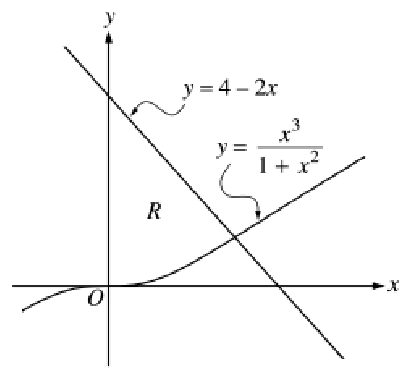
3 { 1 : considers $h'(x) = 0$
1 : identifies critical point and endpoints as candidates
1 : answers }

Note: Errors in computation come off the third point.

2002 SCORING GUIDELINES (Form B)

Question 1

Let R be the region bounded by the y-axis and the graphs of $y = \dfrac{x^3}{1+x^2}$ and $y = 4 - 2x$, as shown in the figure above.



(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

Region R

$$\frac{x^3}{1+x^2} = 4 - 2x \text{ at } x = 1.487664 = A$$

1 : Correct limits in an integral in (a), (b), or (c).

(a) $$\text{Area} = \int_0^A \left(4 - 2x - \frac{x^3}{1+x^2}\right) dx$$

$$= 3.214 \text{ or } 3.215$$

2 { 1 : integrand; 1 : answer }

(b) Volume

$$= \pi \int_0^A \left((4-2x)^2 - \left(\frac{x^3}{1+x^2}\right)^2\right) dx$$

$$= 31.884 \text{ or } 31.885 \text{ or } 10.149\pi$$

3 { 2 : integrand and constant; $< -1 >$ each error; 1 : answer }

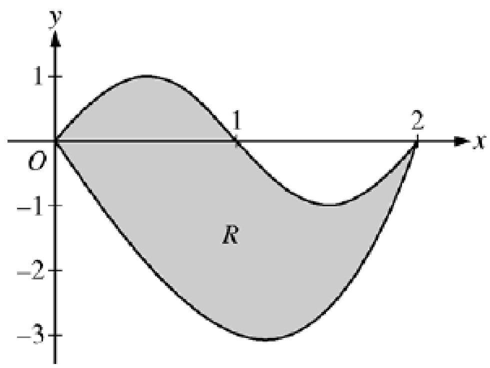
(c) $$\text{Volume} = \int_0^A \left(4 - 2x - \frac{x^3}{1+x^2}\right)^2 dx$$

$$= 8.997$$

3 { 2 : integrand; $< -1 >$ each error; note: 0/2 if not of the form $k\int_c^d (f(x) - g(x))^2\, dx$; 1 : answer }

2008 SCORING GUIDELINES

Question 1



Let R be the region bounded by the graphs of $y = \sin(\pi x)$ and $y = x^3 - 4x$, as shown in the figure above.

(a) Find the area of R.

(b) The horizontal line $y = -2$ splits the region R into two parts. Write, but do not evaluate, an integral expression for the area of the part of R that is below this horizontal line.

(c) The region R is the base of a solid. For this solid, each cross section perpendicular to the x-axis is a square. Find the volume of this solid.

(d) The region R models the surface of a small pond. At all points in R at a distance x from the y-axis, the depth of the water is given by $h(x) = 3 - x$. Find the volume of water in the pond.

(a) $\sin(\pi x) = x^3 - 4x$ at $x = 0$ and $x = 2$ Area $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 4$	3 : { 1 : limits 1 : integrand 1 : answer
(b) $x^3 - 4x = -2$ at $r = 0.5391889$ and $s = 1.6751309$ The area of the stated region is $\int_r^s \left(-2 - \left(x^3 - 4x\right)\right) dx$	2 : { 1 : limits 1 : integrand
(c) Volume $= \int_0^2 \left(\sin(\pi x) - \left(x^3 - 4x\right)\right)^2 dx = 9.978$	2 : { 1 : integrand 1 : answer
(d) Volume $= \int_0^2 (3 - x)\left(\sin(\pi x) - \left(x^3 - 4x\right)\right) dx = 8.369$ or 8.370	2 : { 1 : integrand 1 : answer

2008 SCORING GUIDELINES (Form B)

Question 1

Let R be the region in the first quadrant bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is rotated about the vertical line $x = -1$.

(c) The region R is the base of a solid. For this solid, the cross sections perpendicular to the y-axis are squares. Find the volume of this solid.

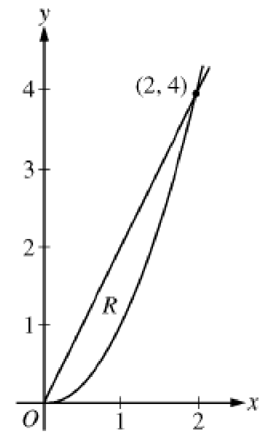
The graphs of $y = \sqrt{x}$ and $y = \frac{x}{3}$ intersect at the points $(0, 0)$ and $(9, 3)$.

(a) $\int_0^9 \left(\sqrt{x} - \frac{x}{3}\right) dx = 4.5$ OR $\int_0^3 \left(3y - y^2\right) dy = 4.5$	$3 : \begin{cases} 1 : \text{limits} \\ 1 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(b) $\pi \int_0^3 \left((3y+1)^2 - \left(y^2+1\right)^2\right) dy$ $= \frac{207\pi}{5} = 130.061 \text{ or } 130.062$	$4 : \begin{cases} 1 : \text{constant and limits} \\ 2 : \text{integrand} \\ 1 : \text{answer} \end{cases}$
(c) $\int_0^3 \left(3y - y^2\right)^2 dy = 8.1$	$2 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{limits and answer} \end{cases}$

2009 SCORING GUIDELINES

Question 4

Let R be the region in the first quadrant enclosed by the graphs of $y = 2x$ and $y = x^2$, as shown in the figure above.



(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, at each x the cross section perpendicular to the x-axis has area $A(x) = \sin\left(\frac{\pi}{2}x\right)$. Find the volume of the solid.

(c) Another solid has the same base R. For this solid, the cross sections perpendicular to the y-axis are squares. Write, but do not evaluate, an integral expression for the volume of the solid.

(a) Area $= \int_0^2 \left(2x - x^2\right)\,dx$

$$= x^2 - \frac{1}{3}x^3 \Big|_{x=0}^{x=2}$$

$$= \frac{4}{3}$$

3 : { 1 : integrand; 1 : antiderivative; 1 : answer }

(b) Volume $= \int_0^2 \sin\left(\frac{\pi}{2}x\right)\,dx$

$$= -\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right)\Big|_{x=0}^{x=2}$$

$$= \frac{4}{\pi}$$

3 : { 1 : integrand; 1 : antiderivative; 1 : answer }

(c) Volume $= \int_0^4 \left(\sqrt{y} - \frac{y}{2}\right)^2\,dy$

3 : { 2 : integrand; 1 : limits }

2009 SCORING GUIDELINES (Form B)

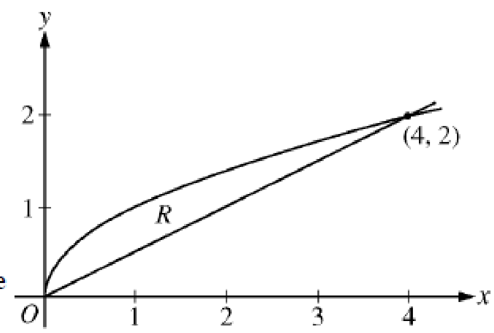
Question 4

Let R be the region bounded by the graphs of $y = \sqrt{x}$ and $y = \frac{x}{2}$, as shown in the figure above.

(a) Find the area of R.

(b) The region R is the base of a solid. For this solid, the cross sections perpendicular to the x-axis are squares. Find the volume of this solid.

(c) Write, but do not evaluate, an integral expression for the volume of the solid generated when R is rotated about the horizontal line $y = 2$.



(a) Area $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right) dx = \frac{2}{3}x^{3/2} - \frac{x^2}{4}\Big|_{x=0}^{x=4} = \frac{4}{3}$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(b) Volume $= \int_0^4 \left(\sqrt{x} - \frac{x}{2}\right)^2 dx = \int_0^4 \left(x - x^{3/2} + \frac{x^2}{4}\right) dx$

$$= \frac{x^2}{2} - \frac{2x^{5/2}}{5} + \frac{x^3}{12}\Big|_{x=0}^{x=4} = \frac{8}{15}$$

$3 : \begin{cases} 1 : \text{integrand} \\ 1 : \text{antiderivative} \\ 1 : \text{answer} \end{cases}$

(c) Volume $= \pi \int_0^4 \left(\left(2 - \frac{x}{2}\right)^2 - \left(2 - \sqrt{x}\right)^2 \right) dx$

$3 : \begin{cases} 1 : \text{limits and constant} \\ 2 : \text{integrand} \end{cases}$