6.4 EXPONENTIAL GROWTH AND DECAY

In many applications, the rate of change of a variable y is proportional to the value of y. If y is a function of time t, we can express this statement as

$$\frac{dy}{dt} = ky$$

Example: Find the solution to this differential equation given the initial condition that $y = y_0$ when $t = 0$. (This is the derivation of an exponential function … see notecards)

$$\int \frac{dy}{y} = \int k\,dt$$

$$\ln y = kt + C$$

$$y = e^{kt+C}$$

$$y = e^{kt} \cdot e^{C}$$

$$y = Ce^{kt}$$

$$y_0 = Ce^0$$

$$y_0 = C$$

$$\boxed{y = y_0 e^{kt}}$$

$k > 0$ (growth)

$k < 0$ (Decay)

Exponential Growth and Decay Model If y changes at a rate proportional to the amount present ($\frac{dy}{dt} = ky$) and $y = y_0$ when $t = 0$, then $$y = y_0 e^{kt}$$ where k is the **proportional constant.** Exponential **growth** occurs when $k > 0$, and exponential **decay** occurs when $k < 0$.

Example: The rate of change of y is proportional to y. When $t = 0$, $y = 2$. When $t = 2$, $y = 4$. What is the value of y when $t = 3$?

$$y = y_0 e^{kt}$$

$$2 = y_0 e^0$$

$$2 = y_0$$

$$y = 2e^{kt}$$

$$4 = 2e^{2k}$$

$$2 = e^{2k}$$

$$\ln 2 = 2k$$

$$\frac{\ln 2}{2} = k \approx .347$$

$$y = 2e^{.347t}$$

$$y = 2e^{.347(3)}$$

$$\boxed{y = 5.657}$$

Example: [1985 AP Calculus BC #33] If $\frac{dy}{dt} = -2y$ and if $y = 1$ when $t = 0$, what is the value of t for which $y = \frac{1}{2}$?

A) $-\frac{1}{2}\ln 2$ B) $-\frac{1}{4}$ C) $\frac{1}{2}\ln 2$ D) $\frac{\sqrt{2}}{2}$ E) $\ln 2$

$$\frac{dy}{dt} = -2y$$

k

$$y = y_0 e^{-2t}$$

$$1 = y_0 e^0$$

$$1 = y_0$$

$$y = e^{-2t}$$

$$\frac{1}{2} = e^{-2t}$$

$$\ln\left(\frac{1}{2}\right) = -2t$$

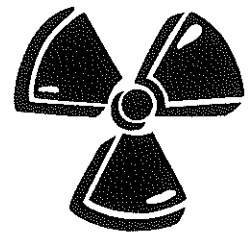
$$\frac{\ln(1/2)}{-2} = t$$

$$-\frac{1}{2}\ln\left(\frac{1}{2}\right)$$

$$\frac{1}{2}\ln\left(\frac{1}{2}\right)^{-1}$$

$$\frac{1}{2}\ln(2)$$

Example: **Radioactive Decay**: The rate at which a radioactive element decays (as measured by the number of nuclei that change per unit of time) is approximately proportional to the amount of nuclei present. Suppose that 10 grams of the plutonium isotope Pu-239 was released in the Chernobyl nuclear accident. How long will it take for the 10 grams to decay to 1 gram? [Pu-239 has a half life of 24,360 years]



$\frac{dy}{dt} = kt \rightarrow y = y_0 e^{kt}$

$y = 10e^{kt}$

$1 = 10e^{kt}$

$\frac{1}{10} = e^{kt}$

$\ln(\frac{1}{10}) = -.00002857t$

$\boxed{80589.976 \text{ years}}$

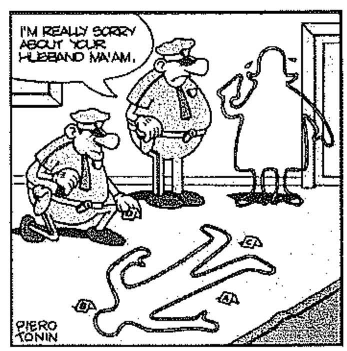
$5 = 10e^{-24360k}$

$\frac{1}{2} = e^{-24360k}$

$\ln \frac{1}{2} = -24360k$

$-.00002857 = k$

Example: **Newton's Law of Cooling**: Newton's Law of Cooling states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature in the surrounding medium. A detective finds a murder victim at 9 am. The temperature of the body is measured at 90.3 °F. One hour later, the temperature of the body is 89.0 °F. The temperature of the room has been maintained at a constant 68 °F.



(a) Assuming the temperature, T, of the body obeys Newton's Law of Cooling, write a differential equation for T.

$\frac{dT}{dt} = k(T - T_a)$

(b) Solve the differential equation to estimate the time the murder occurred.

98.6

$\frac{dT}{dt} = k(T-68)$

$\int \frac{dT}{T-68} = \int k\,dt$

$\ln|T-68| = kt + C$

$T - 68 = e^{kt+C}$

$T = 68 + Ce^{kt}$

$90.3 = 68 + Ce^{0}$

$22.3 = C$

$T = 68 + 22.3e^{kt}$

$89 = 68 + 22.3e^{k}$

$21 = 22.3e^{k}$

$.9417 = e^{k}$

$-.060 = k$

$T = 68 + 22.3e^{-.06t}$

$98.6 = 68 + 22.3e^{-.06t}$

$30.6 = 22.3e^{-.06t}$

$1.372 = e^{-.06t}$

$\ln(1.372) = -.06t$

$-5.274 = t$

5.274 hrs before 9

$\frac{.274}{1} = \frac{x}{60}$

16 min.

5 hr 16 min →

$\boxed{3:44 \text{ am}}$

Example: [1988 AP Calculus BC #43] Bacteria in a certain culture increase at rate proportional to the number present. If the number of bacteria doubles in three hours, in how many hours will the number of bacteria triple?

A) $\frac{3\ln 3}{\ln 2}$ B) $\frac{2\ln 3}{\ln 2}$ C) $\frac{\ln 3}{\ln 2}$ D) $\ln\left(\frac{27}{2}\right)$ E) $\ln\left(\frac{9}{2}\right)$

$\frac{dy}{dt} = ky \rightarrow y = y_0 e^{kt}$

$2y_0 = y_0 e^{3k}$

$2 = e^{3k}$

$\ln 2 = 3k$

$\frac{\ln 2}{3} = k$

$y = y_0 e^{\frac{1}{3}\ln 2\, t}$

$3y_0 = y_0 e^{\frac{1}{3}\ln 2\, t}$

$3 = e^{\frac{1}{3}\ln 2\, t}$

$3 = e^{\ln 2^{1/3} t}$

$3 = 2^{1/3} e^{t}$

$\frac{3}{2^{1/3}} = e^{t}$

$\ln\left(\frac{3}{2^{1/3}}\right) = t$

$\frac{\ln 3}{\ln 2^{1/3}} = t$

$\frac{\ln 3}{\frac{1}{3}\ln 2} = t$

$\boxed{\frac{3\ln 3}{\ln 2} = t}$

Example: [AP Calculus 1993 AB #42] A puppy weighs 2.0 pounds at birth and 3.5 pounds two months later. If the weight of the puppy during its first 6 months is increasing at a rate proportional to its weight, then how much will the puppy weigh when it is 3 months old?

A) 4.2 pounds B) 4.6 pounds C) 4.8 pounds D) 5.6 pounds E) 6.5 pounds

$\frac{dy}{dt} = ky \rightarrow y = y_0 e^{kt}$

$y = 2e^{kt}$

$3.5 = 2e^{2k}$

$1.75 = e^{2k}$

$\frac{\ln 1.75}{2} = \frac{2k}{2}$

$.280 \approx k$

$y = 2e^{.28t}$

$y = 2e^{.28(3)}$

$y = 4.630$

Example: [1993 AP Calculus BC #38] During a certain epidemic, the number of people that are infected at any time increases at rate proportional to the number of people that are infected at that time. If 1,000 people are infected when the epidemic is first discovered, and 1,200 are infected 7 days later, how many people are infected 12 days after the epidemic is first discovered?

A) 343 B) 1,343 C) 1,367 D) 1,400 E) 2,057

$\frac{dy}{dt} \rightarrow y = y_0 e^{kt}$

$y = 1000e^{kt}$

$1200 = 1000e^{7k}$

$1.2 = e^{7k}$

$\ln(1.2) = 7k$

$.026 \approx k$

$y = 1000e^{.026t}$

$y = 1000e^{.026(12)}$

$y = 1367$

Example: [1998 AP Calculus AB #84] Population y grows according to the equation $\frac{dy}{dt} = ky$, where k is a constant and t is measured in years. If the population doubles every 10 years, then the value of k is

A) 0.069 B) 0.200 C) 0.301 D) 3.322 E) 5.000

$y = y_0 e^{kt}$

$2 = e^{10k}$

$\ln 2 = 10k$

$.069 \approx k$

Notecards from Section 6.4: Derivation of an exponential function

Key

Calculus **Newton's Law of Cooling**

1) When an object is removed from a furnace and placed in an environment with a constant temperature of $90°F$, its core temperature is $1500°F$. One hour after it is removed, the core temperature is $1120°F$. Find the core temperature 5 hours after the object is removed from the furnace.

$\frac{dT}{dt} = k(T - T_a) \rightarrow T = A + Ce^{kt}$

$1500 = 90 + Ce^0$

$1410 = C$

$T = 90 + 1410e^{kt}$

$1120 = 90 + 1410e^k$

$1030 = 1410e^k$

$.730 = e^k$

$-.314 = k$

$T = 90 + 1410e^{-.314t}$

$T = 90 + 1410e^{-.314(5)}$

$\boxed{T = 383.298°F}$

2) A cup of coffee is poured from a pot whose contents are $95°C$ into a non-insulated cup in a room at $20°C$. After a minute, the coffee has cooled to $90°C$. How much time is required before the coffee reaches a drinkable temperature of $65°C$?

$\frac{dT}{dt} = k(T - T_a) \rightarrow T = A + Ce^{kt}$

$T = 20 + Ce^{kt}$

$95 = 20 + C$

$75 = C$

$T = 20 + 75e^{kt}$

$90 = 20 + 75e^k$

$\frac{70}{75} = e^k$

$-.069 \approx k$

$T = 20 + 75e^{-.069t}$

$65 = 20 + 75e^{-.069t}$

$45 = 75e^{-.069t}$

$\frac{3}{5} = e^{-.069t}$

$\ln(3/5) = -.069t$

$\boxed{7.403 = t}$

3) Suppose that a corpse was discovered in a motel room at midnight and its temperature was $80°F$. The temperature of the room is kept constant at $60°F$. Two hours later the temperature of the corpse dropped to $75°F$. Find the time of death.

$\frac{dT}{dt} = k(T - T_a) \rightarrow T = A + Ce^{kt}$

$T = 60 + Ce^{kt}$

$80 = 60 + C$

$20 = C$

$T = 60 + 20e^{kt}$

$75 = 60 + 20e^{2k}$

$15 = 20e^{2k}$

$3/4 = e^{2k}$

$\ln(3/4) = 2k$

$-.144 \approx k$

$T = 60 + 20e^{-.144t}$

$98.6 = 60 + 20e^{-.144t}$

$38.6 = 20e^{-.144t}$

$1.93 = e^{-.144t}$

$\ln(1.93) = -.144t$

$-4.566 = t$

$\frac{.566}{100} = \frac{x}{60}$

34 minutes

4 hrs, 34min. before midnight

$\boxed{7:26 \text{ pm}}$

4) A homicide victim was found in a room that is kept at a constant temperature of $70°F$. A body temperature measurement was made at time t and another was made one hour later. The results were: $T(t) = 80°F$ and $T(t+1) = 75°F$, where time is measured in hours. Assuming that the victim's temperature was $98.6°F$ just before death, determine the time of death relative to point t.

$\frac{dT}{dt} = k(T - T_a) \rightarrow T = A + Ce^{kt}$

$T = 70 + Ce^{kt}$

$80 = 70 + Ce^0$

$10 = C$

$T = 70 + 10e^{kt}$

$75 = 70 + 10e^{k(1)}$

$5 = 10e^k$

$\frac{1}{2} = e^k$

$\ln(1/2) = k$

$-.693 = k$

$T = 70 + 10e^{-.693t}$

$98.6 = 70 + 10e^{-.693t}$

$28.6 = 10e^{-.693t}$

$2.86 = e^{-.693t}$

$\ln(2.86) = -.693t$

$-1.516 = t$

$\frac{51.6}{100} = \frac{x}{60}$

$31 = x$

$\boxed{1 \text{ hr } 31 \text{ min before } t}$