Example 44.15 Find $\displaystyle\int_0^1 \frac{x+1}{x^2+2x+2}\,dx$.

Solution Let $\boxed{u = x^2+2x+2}$. Then $\frac{du}{dx} = 2x+2$, so $\boxed{\frac{1}{2}du = (x+1)\,dx}$. Then

$$\begin{aligned}
\int_0^1 \frac{x+1}{x^2+2x+2}\,dx &= \int_0^1 \frac{1}{x^2+2x+2}(x+1)\,dx \\
&= \int_{0^2+2\cdot 0+2}^{1^2+2\cdot 1+2} \frac{1}{u}\frac{1}{2}\,du = \frac{1}{2}\int_2^5 \frac{1}{u}\,du = \frac{1}{2}\Big[\ln|u|\Big]_2^5 \\
&= \frac{1}{2}\big(\ln(5)-\ln(2)\big) = \frac{1}{2}\ln\left(\frac{5}{2}\right) = \ln\sqrt{\frac{5}{2}}.
\end{aligned}$$

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Exercises for Chapter 44

Find the integrals.

1. $\displaystyle\int 6\theta\cos(3\theta^2)\,d\theta$

2. $\displaystyle\int e^{2x^2}4x\,dx$

3. $\displaystyle\int \frac{4x}{2x^2-6}\,dx$

4. $\displaystyle\int 12s^2\sqrt{4s^3+15}\,ds$

5. $\displaystyle\int \frac{3x^2+2x+1}{x^3+x^2+x}\,dx$

6. $\displaystyle\int \frac{4x}{e^{2x^2}}\,dx$

7. $\displaystyle\int \frac{\sec^2(-1/x)}{x^2}\,dx$

8. $\displaystyle\int \frac{1}{\sqrt{x}\left(1+2\sqrt{x}\right)^3}\,dx$

9. $\displaystyle\int \sin^6(x)\cos(x)\,dx$

10. $\displaystyle\int \frac{1}{x^2}\sqrt{2-\frac{1}{x}}\,dx$

11. $\displaystyle\int 2e^{-x}\,dx$

12. $\displaystyle\int 4\sin(3x)\,dx$

13. $\displaystyle\int \frac{\sin(2x)}{\cos^5(2x)}\,dx$

14. $\displaystyle\int \sin(x)e^{\cos(x)}\,dx$

15. $\displaystyle\int x\sqrt{1-x^2}\,dx$

16. $x^2\cos\left(x^3\right)\,dx$

17. $\displaystyle\int 2\sin(2x)\big(1-\cos(2x)\big)\,dx$

18. $\displaystyle\int \frac{6x^2+6}{\left(x^4+4x\right)}\,dx$

19. $\displaystyle\int \pi\sin^2(\pi x)\cos(\pi x)\,dx$

20. $\displaystyle\int \frac{\cos(6x)}{\sqrt{\sin(6x)}}\,dx$

21. $\displaystyle\int \frac{\cos\left(1/x^2\right)}{x^3}\,dx$

22. $\displaystyle\int \frac{2x^9 - e^x}{x^{10} - 5e^x}\,dx$

23. $\displaystyle\int_1^3 \frac{3x^2+2x+1}{x^3+x^2+x}\,dx$

24. $\displaystyle\int_0^1 \left(x^4+1\right)\sqrt{x^5-5x+4}\,dx$

25. $\displaystyle\int_0^3 e^{-5x}\,dx$

26. $\displaystyle\int_1^2 \frac{x+1}{\left(x^2+2x\right)^2}\,dx$

27. $\displaystyle\int_0^1 x(x^2+1)^5\,dx$

28. $\displaystyle\int_{-\pi/6}^{\pi/6} \tan(2x)\sec(2x)\,dx$

29. $\displaystyle\int_2^1 \cos\left(\frac{\pi x}{2}\right)\,dx$

30. $\displaystyle\int_0^1 x\sqrt{x^2+1}\,dx$

31. $\displaystyle\int_{-1}^0 \frac{y}{1+y^2}\,dy$

32. $\displaystyle\int_0^3 \frac{dx}{x+1}$

33. $\displaystyle\int_0^{\pi/2} \sin^4(3x)\cos(3x)\,dx$

34. $\displaystyle\int_0^1 \frac{5}{\left(5x+1\right)^2}\,dx$

35. $\displaystyle\int_0^{\sqrt{\pi/4}} \sec^2\left(x^2\right)x\,dx$

36. $\displaystyle\int_{-2}^1 \frac{3}{3x+7}\,dx$

37. Find the area under the graph of $\sec^2(2x)$ between 0 and $\pi/8$.

38. Find the area under the graph of $x\sin(x^2)$ between 0 and $\sqrt{\pi/6}$.

39. Find $\int \sin(x)\cos(x)\,dx$ using the substitution $u = \sin(x)$. Then find $\int \sin(x)\cos(x)\,dx$ using the substitution $u = \cos(x)$. Explain why it is not a contradiction that the answers look different.

Exercise Solutions for Chapter 44

1. Let $u = 3\theta^2$, so $\frac{du}{d\theta} = 6\theta$ and $du = 6\theta\, d\theta$. Then
$$\int 6\theta\cos(3\theta^2)\,d\theta = \int \cos(3\theta^2)6\theta\,d\theta = \int \cos(u)\,du = \sin(u)+C = \sin(3\theta^2)+C$$

3. Let $u = 2x^2-6$ so $\frac{du}{dx} = 4x$ and $du = 4x\,dx$. Then
$$\int \frac{4x}{2x^2-6}\,dx = \int \frac{1}{2x^2-6}4x\,dx = \int \frac{1}{u}\,du = \ln|u|+C = \ln|2x^2-6|+C$$

5. Let $u = x^3+x^2+x$, so $\frac{du}{dx} = 3x^2+2x+1$, hence $du = \left(3x^2+2x+1\right)dx$. Then
$$\int \frac{3x^2+2x+1}{x^3+x^2+x}\,dx = \int \frac{1}{x^3+x^2+x}\left(3x^2+2x+1\right)dx = \int \frac{1}{u}\,du = \ln|u|+C = \ln|x^3+x^2+x|+C$$

7. Let $u = -1/x$ so $\frac{du}{dx} = \frac{1}{x^2}$, and $du = \frac{1}{x^2}dx$. Then
$$\int \frac{\sec^2(-1/x)}{x^2}\,dx = \int \sec^2(-1/x)\frac{1}{x^2}\,dx = \int \sec^2(u)\,du = \tan(u)+C = \tan(-1/x)+C$$

9. Let $u = \sin(x)$ so $\frac{du}{dx} = \cos(x)$ and $du = \cos(x)\,dx$ Then
$$\int \sin^6(x)\cos(x)\,dx = \int u^6\,du = \frac{u^7}{7}+C = \frac{\sin^7(x)}{7}+C$$

11. Let $u = -x$ so $\frac{du}{dx} = -1$ and $-du = dx$.
Then $\int 2e^{-x}\,dx = 2\int e^{-x}\,dx = 2\int e^u(-du)2 = -2\int e^u\,du = -2e^u+C = -2e^{-x}+C$

13. Let $u = \cos(2x)$ so $\frac{du}{dx} = -2\sin(2x)$ and $-\frac{1}{2}du = \sin(2x)\,dx$. Then
$$\int \frac{\sin(2x)}{\cos^5(2x)}\,dx = \int (\cos(2x))^{-5}\sin(2x)\,dx = \int u^{-5}\left(-\frac{1}{2}du\right) = \frac{1}{2}\int u^{-5}\,du = \frac{1}{2}\cdot\frac{u^{-4}}{-4}+C$$
$$-\frac{1}{8u^4}+C = -\frac{1}{8\cos^4(2x)}+C$$

15. Let $u = 1-x^2$, so $\frac{du}{dx} = -2x$, hence $-\frac{1}{2}du = x\,dx$. Then
$$\int x\sqrt{1-x^2}\,dx = \int \left(1-x^2\right)^{1/2}x\,dx = \int u^{1/2}\left(-\frac{1}{2}du\right) = -\frac{1}{2}\int u^{1/2}du = -\frac{1}{2}\frac{2}{3}u^{3/2}+C$$
$$-\frac{\sqrt{u}^3}{3}+C \;=\; -\frac{\sqrt{1-x^2}^3}{3}+C$$

17. Let $u = 1-\cos(2x)$, so $du = 2\sin(2x)\,dx$. Then $\displaystyle\int 2\sin(2x)\big(1-\cos(2x)\big)\,dx =$
$$\int \big(1-\cos(2x)\big)2\sin(2x)\,dx = \int u\,du = \frac{u^2}{2}+C = \frac{\big(1-\cos(2x)\big)^2}{2}+C$$

19. Let $u = \sin(\pi x)$ then $du = \cos(\pi x)\pi\,dx$. Then $\displaystyle\int \pi\sin^2(\pi x)\cos(\pi x)\,dx =$
$$\int \sin^2(\pi x)\pi\cos(\pi x)\,dx = \int u^2\,du = \frac{u^3}{3}+C = \frac{\sin^3(\pi x)}{3}+C$$

21. Let $u = 1/x^2$, so $du = -\frac{2}{x^3}\,dx$ and so $-\frac{1}{2}du = \frac{1}{x^3}\,dx$. Then $\int \frac{\cos\left(1/x^2\right)}{x^3}\,dx =$

$$\int \cos\left(1/x^2\right)\frac{1}{x^3}\,dx = \int \cos(u)\frac{-1}{2}\,du = -\frac{1}{2}\int \cos(u)\,du = -\frac{1}{2}\sin(u)+C = -\frac{1}{2}\sin\left(\frac{1}{x^2}\right)+C$$

23. Let $u = x^3+x^2+x$, so $\frac{du}{dx} = 3x^2+2x+1$, hence $du = \left(3x^2+2x+1\right)dx$. Then

$$\int_1^3 \frac{3x^2+2x+1}{x^3+x^2+x}\,dx = \int_1^3 \frac{1}{x^3+x^2+x}\left(3x^2+2x+1\right)dx = \int_{1^3+1^2+1}^{3^3+3^2+3} \frac{1}{u}du = \int_3^{39}\frac{1}{u}du$$

$$= \Big[\ln|u|\Big]_3^{39} = \ln|39| - \ln|3| = \ln\left(\frac{39}{3}\right) = \ln(13)$$

25. Let $u = -5x$ so $\frac{du}{dx} = -5$ and $-\frac{1}{5}du = dx$. Then

$$\int_0^3 e^{-5x}\,dx = \int_{-5\cdot 0}^{-5\cdot 3} e^u\left(-\frac{1}{5}du\right) = -\frac{1}{5}\int_0^{-15} e^u\,du = -\frac{1}{5}\Big[e^u\Big]_0^{-15} = -\frac{1}{5}\left(e^{-15}-e^0\right) = \frac{1}{5}-\frac{1}{5e^{15}}$$

27. Let $u = x^2+1$, so $\frac{du}{dx} = 2x$ and $\frac{1}{2}du = x\,dx$. Then

$$\int_0^1 x(x^2+1)^5\,dx = \int_0^1 (x^2+1)^5 x\,dx = \int_{0^2+1}^{1^2+1} u^5\frac{1}{2}du = \frac{1}{2}\int_1^2 u^5 du = \frac{1}{2}\left[\frac{u^6}{6}\right]_1^2 = \frac{1}{2}\left(\frac{2^6}{6}-\frac{1^6}{6}\right) = \frac{21}{4}$$

29. $\int_2^1 \cos\left(\frac{\pi x}{2}\right)dx = \left[\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_2^1 = \frac{2}{\pi}\sin\left(\frac{\pi}{2}\right) - \frac{2}{\pi}\sin(\pi) = \frac{2}{\pi}\cdot 1 - \frac{2}{\pi}\cdot 0 = \frac{2}{\pi}$

31. Let $u = 1+y^2$. Then $du = 2y\,dy$, so $\frac{1}{2}du = y\,dy$ and $\int_{-1}^0 \frac{y}{1+y^2}\,dy = \int_{-1}^0 \frac{1}{1+y^2}y\,dy$

$$= \int_{1+(-1)^2}^{1+0^2} \frac{1}{u}\frac{1}{2}du = \frac{1}{2}\int_2^1 \frac{1}{u}du = \frac{1}{2}\Big[\ln|u|\Big]_1^2 = \frac{1}{2}\Big(\ln|2| - \ln|1|\Big) = \frac{1}{2}\ln(2)$$

33. Let $u = \sin(3x)$. Then $du = \cos(3x)3\,dx$, so $\frac{1}{3}du = \cos(3x)\,dx$ and

$$\int_0^{\pi/2} \sin^4(3x)\cos(3x)\,dx = \int_{\sin(3\cdot 0)}^{\sin(3\cdot\pi/2)} u^4\frac{1}{3}du = \frac{1}{3}\int_0^{-1} u^4 du = \frac{1}{3}\left[\frac{u^5}{5}\right]_0^{-1} = -\frac{1}{15}$$

35. Let $u = x^2$. Then $du = 2x\,dx$, so $\frac{1}{2}du = x\,dx$ and $\int_0^{\sqrt{\pi/4}} \sec^2\left(x^2\right)x\,dx =$

$$\int_{0^2}^{\sqrt{\pi/4}^2} \sec^2(u)\frac{1}{2}du = \frac{1}{2}\int_0^{\pi/4}\sec^2(u)\,du = \frac{1}{2}\Big[\tan(u)\Big]_0^{\pi/4} = \frac{1}{2}\Big(\tan(\pi/4)-\tan(0)\Big) = \frac{1}{2}.$$

37. Find the area under the graph of $\sec^2(2x)$ between 0 and $\pi/8$.

The answer will be $\int_0^{\pi/8}\sec^2(2x)\,dx$. Let $u = 2x$, so $du = 2\,dx$ and $\frac{1}{2}du = dx$. Then

$$\int_0^{\pi/8}\sec^2(2x)\,dx = \int_{2\cdot 0}^{2\cdot\pi/8}\sec^2(u)\frac{1}{2}du = \frac{1}{2}\int_0^{\pi/4}\sec^2(u)\,du = \frac{1}{2}\Big(\tan(\pi/4)-\tan(0)\Big) = \frac{1}{2}.$$

39. If $u = \sin(x)$, then $du = \cos(x)\,dx$, so $\int \sin(x)\cos(x)\,dx = \int u\,du = \frac{u^2}{2}+C = \frac{\sin^2(x)}{2}+C$.

If $u = \cos(x)$, then $-du = \sin(x)\,dx$, so $\int \sin(x)\cos(x)\,dx = -\int u\,du = -\frac{u^2}{2}+C = -\frac{\cos^2(x)}{2}+C$.

These two answers look different, but this is not a contradiction. Add a constant of 1/2 to the second answer and you get $\frac{1}{2} - \frac{\cos^2(x)}{2} + C = \frac{1-\cos^2(x)}{2} + C = \frac{\sin^2(x)}{2} + C$. So the first answer is just 1/2 plus the second answer, and the 1/2 gets absorbed into the C.