The Fundamental Theorems of Calculus

I. If f is continuous on [a, b], then the function $F(x) = \int_a^x f(t)\,dt$ has a derivative at every point in $[a, b]$, and $\frac{dF}{dx} = \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$.

II. If f is continuous on $[a, b]$, and if F is any antiderivative of f on $[a, b]$, then $\int_a^b f(t)\,dt = F(b) - F(a)$.

Note: These two theorems may be presented in reverse order. Part II is sometimes called the Integral Evaluation Theorem.

Don't overlook the obvious!

1. $\frac{d}{dx}\int_a^a f(t)\,dt = 0$, because the definite integral is a constant
2. $\int_a^b f'(x)\,dx = f(b) - f(a)$

Upgrade for part I, applying the Chain Rule

If $F(x) = \int_a^{g(x)} f(t)\,dt$, then $\frac{dF}{dx} = \frac{d}{dx}\int_a^{g(x)} f(t)\,dt = f(g(x))g'(x)$.

For example, $\frac{d}{dx}\int_2^{x^3} \sin(t^2)\,dt = \left(\left(\sin(x^3)^2\right)\right)\left(3x^2\right) = 3x^2 \sin\left(x^6\right)$

An important alternate form for part II

$F(b) = F(a) + \int_a^b f(t)\,dt$

[Think of this as: ending value = starting value plus accumulation.]

For example, given that $\int_3^{12} f'(x)\,dx = -4$ and $f(3) = 35$, find $f(12)$.

Using the alternate format, $f(12) = f(3) + \int_3^{12} f'(x)\,dx = 35 + (-4) = 31$.

Sample Problems

Multiple Choice – No Calculator

1. $\frac{d}{dx}\int_{2}^{x} \ln t \, dt =$

(A) $\ln x$ (B) $\ln 2$ (C) $\frac{1}{x}$

(D) $\frac{1}{2}$ (E) $\ln x - \ln 2$

2. If $g(x) = \int_{\pi}^{\pi x} \cos(t^2)\, dt$, then $g'(x) =$

(A) $\sin\left(\pi^2 x^2\right)$ (B) $\pi x \sin\left(\pi^2 x^2\right)$ (C) $\pi x \cos\left(\pi^2 x^2\right)$

(D) $\cos\left(\pi^2 x^2\right)$ (E) $\pi \cos\left(\pi^2 x^2\right)$

3. $\frac{d}{dx}\int_{\sin x}^{4} \sqrt{1+t^2}\, dt =$

(A) $\sqrt{1+\sin^2 x}$ (B) $-\cos x\sqrt{1+\sin^2 x}$ (C) $-\sqrt{1+\sin^2 x}$

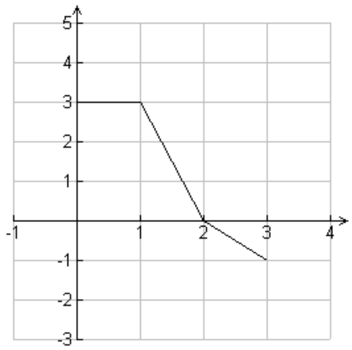
(D) $\cos x\sqrt{1+\sin^2 x}$ (E) $\sqrt{1+\cos^2 x}$

4. If f has two continuous derivatives on [5, 10], then $\int_{5}^{10} f''(t)\, dt =$

(A) $f'''(10) - f'''(5)$ (B) $f(10) - f(5)$ (C) $f'(10) - f'(5)$

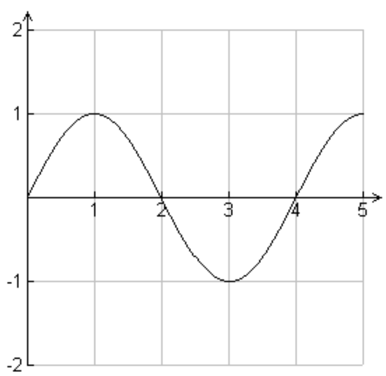
(D) $f''(10) - f''(5)$ (E) $f''(5) - f''(10)$

5. The graph of f is given, and g is an antiderivative of f. If $g(3) = 6$, find $g(0)$.



(A) 1 (B) 2 (C) 4 (D) 5 (E) 10

6. The graph of f is given. $F(x) = \int_0^x f(t)\,dt$



Which of the following statements is true?

(A) F decreases on (1, 2).
(B) F has a relative minimum at $x = 2$
(C) F decreases on (2, 4)
(D) F has a relative maximum at $x = 1$.
(E) F has a point of inflection at $x = 4$.

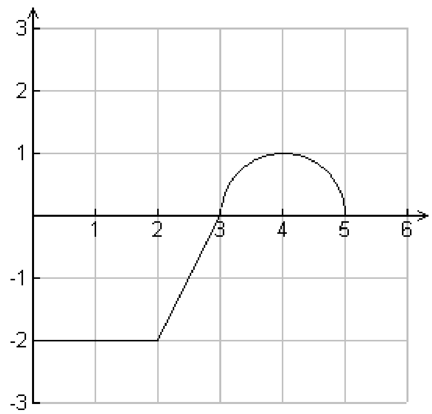
7. $\frac{d}{dx}\int_{x}^{x^2} \tan(t)\, dt =$

(A) $\tan\left(x^2\right) - \tan x$
(B) $\tan x - \tan\left(x^2\right)$
(C) $\tan x - 2x\tan\left(x^2\right)$
(D) $2x\tan\left(x^2\right) - \tan x$
(E) $\sec^2\left(x^2\right) - \sec^2 x$

8. $\int_{1}^{e} \left(x - \frac{5}{x}\right) dx =$

(A) $\frac{1}{2}e^2 - \frac{11}{2}$
(B) $\frac{1}{2}e^2 - \frac{9}{2}$
(C) $e^2 - \frac{11}{2}$
(D) $\frac{1}{2}e^2 - \frac{3}{2}$
(E) $\frac{11}{2} - \frac{1}{2}e^2$

Free Response 1 – No Calculator



The graph of f is given. It consists of two line segments and a semi-circle.

$g(x) = \int_{1}^{x} f(t)\, dt$

(a) Find $g(0)$, $g(1)$, and $g(5)$.

(b) Find $g'(2)$, $g''(2)$, and $g'''(4)$ or state that it does not exist.

(c) For what value(s) of x does the graph of g have a point of inflection? Justify your answer.

(d) Find the absolute maximum and absolute minimum values of g on [0, 5]. Justify your answer.

Key

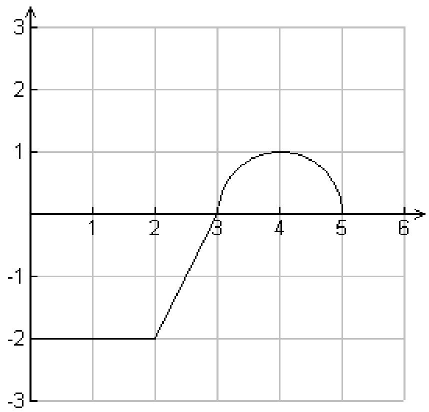
No Calculator

1. A
2. E
3. B
4. C
5. B
6. C
7. D
8. A

Calculator Allowed

1. D
2. C
3. D
4. B
5. D
6. D
7. E
8. A

Free Response 1 – No Calculator



The graph of f is given. It consists of two line segments and a semi-circle.

$g(x) = \int_1^x f(t)\,dt$

(a) Find $g(0)$, $g(1)$, and $g(5)$.

(b) Find $g'(2)$, $g''(2)$, and $g'''(4)$ or state that it does not exist.

(c) For what value(s) of x does the graph of g have a point of inflection? Justify your answer.

(d) Find the absolute maximum and absolute minimum values of g on [0, 5]. Justify your answer.

(a) $g(0) = \int_1^0 f(t)\,dt = 2$

$g(1) = \int_1^1 f(t)\,dt = 0$

$g(5) = \int_1^5 f(t)\,dt = \frac{1}{2}\pi - 3$

(b) $g'(2) = f(2) = -2$

$g''(2) = f'(2) = \text{DNE}$

$g''(4) = f'(4) = 0$

(c) g has a point of inflection at $x = 4$ because $g' = f$ changes from increasing to decreasing.

(d) Candidates are $x = 0, 3, 5$, the endpoints of the interval and the critical number.

x	$g(x)$
0	2
3	-3
5	$\frac{1}{2}\pi - 3$

The absolute minimum value is −3.
The absolute maximum value is 2.

2 pts: 1 pt $g(0)$
1 pt $g(1)$ and $g(5)$

2 pts: 1 pt $g''(2)$
1 pt $g'(2)$ and $g''(4)$

2 pts: 1 pt $x = 4$
1 pt justification

3 pts: 1 pt for candidates
1 pt evaluating candidates
1 pt for answers