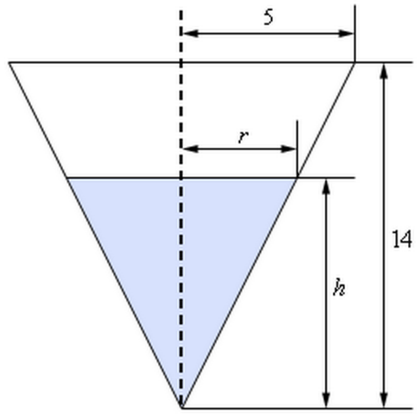
Solution
Okay, we should probably start off with a quick sketch (probably not to scale) of what is going on here.



$$\frac{r}{h} = \frac{5}{14} \qquad \Rightarrow \qquad r = \frac{5}{14}h$$

If we take this and plugit into our volume formula we have,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{5}{14}h\right)^2 h = \frac{25}{588}\pi h^3$$

This gives us a volume formula that only involved the volume and the height of the water. Note however that this volume formula is only valid for our cone, so don't be tempted to use it for other cones! If we now differentiate this we have,

$$V' = \frac{25}{196}\pi h^2 h'$$

At this point all we need to do is plug in what we know and solve for h'.

$$-2 = \frac{25}{196}\pi\left(6^2\right)h' \qquad \Rightarrow \qquad h' = \frac{-98}{225\pi} = -0.1386$$

So, it looks like the height is decreasing at a rate of 0.1386 ft/hr.

(b) At what rate is the radius of the top of the water in the tank changing when the depth of the water is 6 ft?

In this case we are asking for r' and there is an easy way to do this part and a difficult (well, more difficult than the easy way anyway….) way to do it. The "difficult" way is to redo the work in part (a) above only this time use,

$$\frac{h}{r} = \frac{14}{5} \qquad \Rightarrow \qquad h = \frac{14}{5}r$$

to get the volume in terms of V and r and then proceed as before.

That's not terribly difficult, but it is more work that we need to so. Recall from the first part that we have,

$$r = \frac{5}{14}h \qquad \Rightarrow \qquad r' = \frac{5}{14}h'$$

So, as we can see if we take the relationship that relates r and h that we used in the first part and differentiate it we get a relationship between r' and h'. At this point all we need to do here is use the result from the first part to get,

$$r' = \frac{5}{14}\left(\frac{-98}{225\pi}\right) = -\frac{7}{45\pi} = -0.04951$$

Example 5

The volume of this kind of tank is simple to compute. The volume is the area of the end times the depth. For our case the volume of the water in the tank is,

$$\begin{aligned} V &= (\text{Area of End})(\text{depth}) \\ &= \left(\tfrac{1}{2}\,\text{base} \times \text{height}\right)(\text{depth}) \\ &= \tfrac{1}{2} hw(8) \\ &= 4hw \end{aligned}$$

As with the previous example we've got an extra quantity here, w, that is also changing with time and so we need to eliminate it from the problem. To do this we'll again make use of the idea of similar triangles. If we look at the end of the tank we'll see that we again have two similar triangles. One for the tank itself and one formed by the water in the tank. Again, remember that with similar triangles ratios of sides must be equal. In our case we'll use,

$$\frac{w}{5} = \frac{h}{2} \qquad \Rightarrow \qquad w = \frac{5}{2} h$$

Plugging this into the volume gives a formula for the volume (and only for this tank) that only involved the height of the water.

$$V = 4hw = 4h\left(\frac{5}{2} h\right) = 10h^2$$

We can now differentiate this to get,

$$V' = 20hh'$$

Finally, all we need to do is plug in and solve for h'.

$$6 = 20(1.2)h' \qquad \Rightarrow \qquad h' = 0.25 \text{ m/sec}$$

So, the height of the water is rising at a rate of 0.25 m/sec.

Example 6

(a) At what rate is the tip of the shadow moving away from the pole when the person is 25 ft from the pole?

In this case we want to determine x' when $x_p = 25$ given that $x'_p = 2$.

The equation we'll need here is,

$$x = x_p + x_s$$

but we'll need to eliminate x_s from the equation in order to get an answer. To do this we can again make use of the fact that the two triangles are similar to get,

$$\frac{5.5}{12} = \frac{x_s}{x} = \frac{x_s}{x_p + x_s} \qquad \text{Note}: \frac{5.5}{12} = \frac{\frac{11}{2}}{12} = \frac{11}{24}$$

We'll need to solve this for x_s.

$$\frac{11}{24}\left(x_p + x_s\right) = x_s$$

$$\frac{11}{24}x_p = \frac{13}{24}x_s$$

$$\frac{11}{13}x_p = x_s$$

Our equation then becomes,

$$x = x_p + \frac{11}{13}x_p = \frac{24}{13}x_p$$

Now all that we need to do is differentiate this, plug in and solve for x'.

$$x' = \frac{24}{13}x'_p \qquad \Rightarrow \qquad x' = \frac{24}{13}(2) = 3.6923 \text{ ft/sec}$$

The tip of the shadow is then moving away from the pole at a rate of 3.6923 ft/sec. Notice as well that we never actually had to use the fact that $x_p = 25$ for this problem.

(b) At what rate is the tip of the shadow moving away from the person when the person is 25 ft from the pole?

This part is actually quite simple if we have the answer from (a) in hand, which we do of course. In this case we know that x_s represents the length of the shadow, or the distance of the tip of the shadow from the person so it looks like we want to determine x'_s when $x_p = 25$.

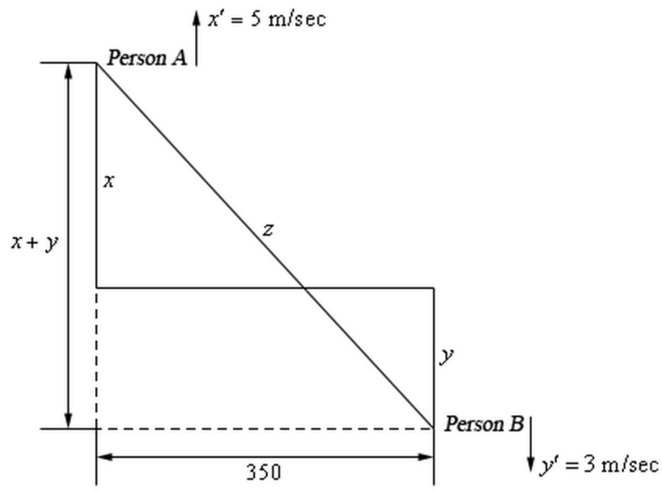
Again, we can use $x = x_p + x_s$, however unlike the first part we now know that $x'_p = 2$ and $x' = 3.6923$ ft/sec so in this case all we need to do is differentiate the equation and plug in for all the known quantities.

$$x' = x'_p + x'_s$$

$$3.6923 = 2 + x'_s \qquad x'_s = 1.6923 \text{ ft/sec}$$

The tip of the shadow is then moving away from the person at a rate of 1.6923 ft/sec.

Solution
There is a lot to digest here with this problem. Let's start off with a sketch of the situation.



Now we are after z' and we know that $x' = 5$ and $y' = 3$. We want to know z' after Person A had been riding for 25 minutes and Person B has been riding for $25 - 7 = 18$ minutes. After converting these times to seconds (because our rates are all in m/sec) this means that at the time we're interested in each of the bike riders has rode,

$$x = 5(25 \times 60) = 7500 \text{ m} \qquad\qquad y = 3(18 \times 60) = 3240 \text{ m}$$

Next, the Pythagorean theorem tells us that,

$$z^2 = (x+y)^2 + 350^2 \qquad \textbf{(2)}$$

Therefore, 25 minutes after Person A starts riding the two bike riders are

$$z = \sqrt{(x+y)^2 + 350^2} = \sqrt{(7500+3240)^2 + 350^2} = 10745.7015 \text{ m}$$

apart.

To determine the rate at which the two riders are moving apart all we need to do then is differentiate **(2)** and plug in all the quantities that we know to find z'.

$$2zz' = 2(x+y)(x'+y')$$

$$2(10745.7015)z' = 2(7500+3240)(5+3)$$

$$z' = 7.9958 \text{ m/sec}$$

So, the two riders are moving apart at a rate of 7.9958 m/sec.