Math 124 Practice With the Product and Quotient Rules

1. $f(x)=(1+\sqrt{x})(x^3)$ $f'(x)=\frac{d}{dx}\left(1+\sqrt{x}\right)\cdot(x^3)+\left(1+\sqrt{x}\right)\cdot\frac{d}{dx}(x^3)$

$\frac{d}{dx}\left(1+\sqrt{x}\right)=0+\frac{1}{2}x^{-1/2}$ (since $\sqrt{x}=x^{1/2}$)

$\frac{d}{dx}(x^3)=3x^2$

$$f'(x)=\left(\frac{1}{2}x^{-1/2}\right)\cdot(x^3)+\left(1+\sqrt{x}\right)\cdot 3x^2$$

$$f'(x)=\frac{x^3}{2\sqrt{x}}+3x^2+3\sqrt{x^5}$$

2. $g(t)=\left(\frac{2}{t}+t^5\right)\left(t^3+1\right)$ $g'(t)=\frac{d}{dt}\left(\frac{2}{t}+t^5\right)\cdot(t^3+1)+\left(\frac{2}{t}+t^5\right)\cdot\frac{d}{dt}(t^3+1)$

$\frac{d}{dt}\left(\frac{2}{t}+t^5\right)=-2t^{-2}+5t^4$ (since $\frac{2}{t}=2t^{-1}$)

$\frac{d}{dt}(t^3+1)=3t^2+0$

$$g'(t)=(-2t^{-2}+5t^4)\cdot(t^3+1)+\left(\frac{2}{t}+t^5\right)\cdot(3t^2)$$

$$g'(t)=(-2t-2t^{-2}+5t^7+5t^4)+6t+3t^7$$

$$g'(t)=8t^7+5t^4+4t-\frac{2}{t^2}$$

3. $h(y)=\frac{1}{y^3+2y+1}$

$$h'(y)=\frac{(y^3+2y+1)\cdot\frac{d}{dy}(1)-1\cdot\frac{d}{dy}\left(y^3+2y+1\right)}{\left(y^3+2y+1\right)^2}$$

$\frac{d}{dy}(1)=0;\quad \frac{d}{dy}\left(y^3+2y+1\right)=3y^2+2$

$$h'(y)=\frac{0-(3y^2+2)}{\left(y^3+2y+1\right)^2}=\frac{-3y^2-2}{(y^3+2y+1)^2}$$

Note: once we've learned Chain Rule, that may be more efficient here (try it and decide for yourself)

4. $f(x) = \dfrac{1}{x+\sqrt{x}}$

$$f'(x) = \frac{\left(x+\sqrt{x}\right)\cdot \frac{d}{dx}(1) - 1\cdot \frac{d}{dx}\left(x+\sqrt{x}\right)}{\left(x+\sqrt{x}\right)^2}$$

$$\frac{d}{dx}(1) = 0; \quad \frac{d}{dx}\left(x+\sqrt{x}\right) = 1 + \tfrac{1}{2}x^{-1/2}$$

$$f'(x) = \frac{-\left(1+\frac{1}{2}x^{-1/2}\right)}{\left(x+\sqrt{x}\right)^2} = \frac{-1-\frac{1}{2\sqrt{x}}}{(x+\sqrt{x})^2} = \frac{-2\sqrt{x}-1}{2\sqrt{x}\,(x+\sqrt{x})^2}$$

Again, try Chain Rule here, to see if you like that better!

5. $y = 2^x e^x$

$$\frac{dy}{dx} = \left(2^x\right)\cdot\frac{d}{dx}\left(e^x\right) + \frac{d}{dx}\left(2^x\right)\cdot\left(e^x\right)$$

$$\frac{d}{dx}\left(e^x\right) = e^x; \quad \frac{d}{dx}\left(2^x\right) = (\ln 2)\left(2^x\right)$$

$$\frac{dy}{dx} = \left(2^x\right)\cdot\left(e^x\right) + (\ln 2)\left(2^x\right)\cdot\left(e^x\right) = \left(2^x e^x\right)(1+\ln 2)$$

6. $g(x) = \dfrac{x^2+1}{x^3-5}$

$$g'(x) = \frac{(x^3-5)\frac{d}{dx}(x^2+1) - (x^2+1)\frac{d}{dx}(x^3-5)}{(x^3-5)^2}$$

$$\frac{d}{dx}(x^2+1) = 2x; \quad \frac{d}{dx}(x^3-5) = 3x^2$$

$$g'(x) = \frac{(x^3-5)(2x) - (x^2+1)(3x^2)}{(x^3-5)^2} = \frac{x[(x^3-5)\cdot 2 - (x^2+1)\cdot 3x]}{(x^3-5)^2}$$

$$g'(x) = \frac{x(-x^3-3x-10)}{(x^3-5)^2} = \frac{-x(x^3+3x+10)}{(x^3-5)^2}$$

Note that factored form will be nicer when we get to chapter 4 (which is why I’m showing it here).

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7. $y = \dfrac{\sqrt{x}}{x^3+1}$

$$\frac{dy}{dx} = \frac{(x^3+1)\frac{d}{dx}(x^{1/2}) - x^{1/2}\frac{d}{dx}(x^3+1)}{(x^3+1)^2}$$

$$\tfrac{d}{dx}(x^{1/2}) = \tfrac{1}{2}x^{-1/2};\ \tfrac{d}{dx}(x^3+1) = 3x^2$$

$$\frac{dy}{dx} = \frac{(x^3+1)\frac{1}{2}x^{-1/2} - x^{1/2}(3x^2)}{(x^3+1)^2} = \frac{\frac{1}{2}x^{-1/2}\left((x^3+1)-6x^3\right)}{(x^3+1)^2} = \frac{\frac{1}{2}x^{-1/2}\left(-5x^3+1\right)}{2\sqrt{x}(x^3+1)^2}$$

$$\frac{dy}{dx} = \frac{-5x^3+1}{2\sqrt{x}(x^3+1)^2}$$

Factoring out $(1/2)x^{-1/2}$ is a nice approach to simplification here.

8. $z = \dfrac{t^2}{(t-4)(2-t^3)}$

$$z' = \frac{(t-4)(2-t^3)\frac{d}{dt}(t^2) - t^2\frac{d}{dt}\left[(t-4)(2-t^3)\right]}{(t-4)^2(2-t^3)^2}$$

$$\tfrac{d}{dt}(t^2) = 2t$$

$$\tfrac{d}{dt}\left[(t-4)(2-t^3)\right] = (t-4)\tfrac{d}{dt}(2-t^3) + \tfrac{d}{dt}(t-4)(2-t^3) = (t-4)(-3t^2) + 1(2-t^3)$$

$$= -3t^3 + 12t^2 + 2 - t^3 = -4t^3 + 12t^2 + 2$$

$$z' = \frac{(t-4)(2-t^3)(2t) - t^2(-4t^3+12t^2+2)}{(t-4)^2(2-t^3)^2} = \frac{2t\left[(t-4)(2-t^3) - t(-2t^3+6t^2+1)\right]}{(t-4)^2(2-t^3)^2}$$

$$z' = \frac{2t(-t^4+4t^3+2t-8+2t^4-6t^3-t)}{(t-4)^2(2-t^3)^2} = \frac{2t(t^4-2t^3+t-8)}{(t-4)^2(2-t^3)^2}$$

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9. $h(x) = \dfrac{(x^3+1)\sqrt{x}}{x^2}$

$$h'(x) = \frac{(x^2)\frac{d}{dx}\left[(x^3+1)\sqrt{x}\right] - (x^3+1)\sqrt{x}\,\frac{d}{dx}(x^2)}{x^4}$$

$$\frac{d}{dx}\left[(x^3+1)\sqrt{x}\right] = \frac{d}{dx}(x^3+1)\sqrt{x} + (x^3+1)\frac{d}{dx}(x^{1/2}) = (3x^2)\sqrt{x} + (x^3+1)(\tfrac{1}{2}x^{-1/2})$$
$$= 3x^{5/2} + \tfrac{1}{2}x^{5/2} + \tfrac{1}{2}x^{-1/2} = \tfrac{7}{2}x^{5/2} + \tfrac{1}{2}x^{-1/2}$$

$$h'(x) = \frac{(x^2)(\frac{7}{2}x^{5/2} + \frac{1}{2}x^{-1/2}) - (x^3+1)x^{1/2}(2x)}{x^4} = \frac{(\frac{7}{2}x^{9/2} + \frac{1}{2}x^{3/2}) - (2x^{9/2} + 2x^{3/2})}{x^4}$$

$$h'(x) = \frac{\frac{1}{2}x^{3/2}\left(7x^3 + 1 - 4x^3 - 4\right)}{x^4} = \frac{\frac{1}{2}x^{3/2}\left(3x^3 - 3\right)}{x^4} = \frac{3\sqrt{x^3}\left(x^3 - 1\right)}{2x^4}$$

Again, factored form will be nicer when we start setting these derivatives equal to zero and solving.

10. $y = \dfrac{(e^m)(\sqrt[3]{m})}{m^2+3}$

$$y' = \frac{(m^2+3)\frac{d}{dm}\left[e^m\sqrt[3]{m}\right] - em\sqrt[3]{m}\,\frac{d}{dm}(m^2+3)}{(m^2+3)^2}$$

$$\frac{d}{dm}\left[e^m\sqrt[3]{m}\right] = \frac{d}{dm}(e^m)(\sqrt[3]{m}) + e^m\frac{d}{dm}(\sqrt[3]{m}) = e^m\sqrt[3]{m} + e^m\,\tfrac{1}{3}m^{-2/3}$$

$$y' = \frac{(m^2+3)(e^m\sqrt[3]{m} + e^m\,\frac{1}{3}m^{-2/3}) - e^m\sqrt[3]{m}(2m)}{(m^2+3)^2} = \frac{e^m(m^{7/3}) + \frac{1}{3}e^m m^{4/3} + 3e^m m^{1/3} + e^m m^{-2/3} - 2e^m m^{4/3}}{(m^2+3)^2}$$

$$y' = \frac{e^m m^{-2/3}\left((m^2+3)(m+\frac{1}{3}) - m(2m)\right)}{(m^2+3)^2} = \frac{e^m\left(m^3 + \frac{1}{3}m^2 + 3m + 1 - 2m^2\right)}{\sqrt[3]{m^2}\,(m^2+3)^2}$$

$$y' = \frac{e^m(3m^3 - 5m^2 + 9m + 3)}{3\sqrt[3]{m^2}\,(m^2+3)^2}$$

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11. $g(x) = (x+\sqrt{x})(3^x)$

$$g'(x) = \frac{d}{dx}(x + x^{1/2})(3^x) + (x + x^{1/2})\frac{d}{dx}(3^x)$$
$$= (1 + \tfrac{1}{2}x^{-1/2})(3^x) + (x + x^{1/2})(\ln 3)(3^x)$$
$$= 3^x(1 + x\ln 3 + \sqrt{x}\ln 3 + \frac{1}{2\sqrt{x}})$$

12. Find $f'(10)$ where $f(x) = g(x)h(x)$, $g(10) = -4$, $h(10) = 560$, $g'(10) = 0$, and $h'(10) = 35$

Since $f(x)$ is defined as a product, use the product rule for f':

$f'(x) = g'(x)h(x) + g(x)h'(x)$

at $x = 10$: $f'(10) = g'(10)h(10) + g(10)h'(10)$

$= 0(560) - 4(35) = -140$

13. Find $y'(-3)$ where $y(x) = \dfrac{z(x)}{1+x^2}$, $z(-3) = 6$, and $z'(-3) = 15$

Since $y(x)$ is defined as a quotient, use the quotient rule:

$$y'(x) = \frac{(1+x^2)z'(x) - z(x)\frac{d}{dx}(1+x^2)}{(1+x^2)^2} = \frac{(1+x^2)z'(x) - z(x)(2x)}{(1+x^2)^2}$$

$$y'(-3) = \frac{(1+(-3)^2)z'(-3) - z(-3)2\cdot(-3)}{(1+(-3)^2)^2}$$

$$= \frac{(1+(-3)^2)(15) - 6\cdot 2\cdot(-3)}{(1+(-3)^2)^2}$$

$$= \frac{(1+9)15 + 36}{10^2} = \frac{186}{100} = \frac{93}{50}$$