



Exponential & Logarithmic Equations

Until now, the equations you've been asked to solve have looked like $x^2 - x + 6 = 0$, where x has been in the base of any exponential expressions. With logarithms, you now have the ability to solve equations like $10^{x+2} = 50$, where the x is in the exponent instead. This kind of problem is called an **exponential equation**. The way to solve most of these equations is to turn them into logarithms. We'll also look at **logarithmic equations** in this worksheet.

Example 1: Solve $4^{x+2} = 64$.

Solution: This is the easier sort of exponential equation. It is possible to write both sides of the equation with the same base. In that case, we can conclude that the resulting exponents must be equal. So:

$$\begin{aligned} 4^{x+2} &= 64 \\ 4^{x+2} &= 4^3 \\ \therefore x + 2 &= 3 \\ x &= 1 \end{aligned}$$

It is not always possible to solve an exponential equation this way.

For the rest of the exponential equations, and for logarithmic equations, it is useful to know the Laws of Logarithms, particularly these three:

$$\log_a (xy) = \log_a x + \log_a y \qquad \log_{10} 100x = \log_{10} 100 + \log_{10} x = 2 + \log_{10} x$$

$$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y \qquad \log_2 \frac{3}{8} = \log_2 3 - \log_2 8 = \log_2 3 - 3$$

$$\log_a (x^y) = y \log_a x \qquad \log_5 4^6 = 6 \log_5 4; \ln e^x = x \ln e = x \cdot 1 = x$$

Example 2: Solve $2^x = 7$.

Solution: Since 7 cannot be expressed as a power of 2, we will have to take the log of both sides to bring the x down from the exponent so we can work with it. (You can use any base for the logarithm, but a common log, with base 10, will be most convenient, since you'll need to evaluate with your calculator.)

$$\begin{aligned} 2^x &= 7 \\ \log 2^x &= \log 7 \\ x \log 2 &= \log 7 \\ x &= \frac{\log 7}{\log 2} = \frac{0.8451 \dots}{0.3010 \dots} = 2.8074 \dots \end{aligned}$$

So the answer is approximately 2.81.

Example 3: Solve $\log x + \log (x - 3) = 1$.

Solution: This is a logarithmic equation. Just as we use a logarithm to solve the exponential equation, we'll use an exponent to solve the logarithmic equation:

$$\begin{aligned}\log x + \log (x - 3) &= 1 \\ \log x(x - 3) &= 1 \\ 10^{\log x(x-3)} &= 10^1 \\ x(x - 3) &= 10 && \text{Since } a^{\log_a b} = b. \\ x^2 - 3x &= 10 \\ x^2 - 3x - 10 &= 0 \\ (x - 5)(x + 2) &= 0 \\ x &= 5 \text{ or } -2\end{aligned}$$

We have two answers, but in problems like this, we need to check our solutions against the original question:

$$\begin{aligned}x = 5? \quad \text{L.H.S.} &= \log x + \log (x - 3) = \log 5 + \log (5 - 3) \\ &= \log 5 + \log 2 \\ &= \log (5 \times 2) \\ &= \log 10 = 1 = \text{R.H.S.} \checkmark \\ x = -2? \quad \text{L.H.S.} &= \log x + \log (x - 3) = \log (-2) + \log (-2 - 3)\end{aligned}$$

However, $\log (-2)$ is undefined, so $x \neq -2$. The solution is $x = 5$.

EXERCISES

A. Solve by rewriting each side with the same base:

1) $2^x = 16$

7) $5^x = \frac{1}{125}$

2) $4^x = 128$

8) $5^{4x-7} = 125$

3) $3^x = 1$

9) $4^{3x+5} = 16$

4) $5^{x-2} = 1$

10) $4^x = 8^{x-1}$

5) $2^{2x+3} - 1 = 0$

11) $3^{2x^2} \cdot 3^{5x} = 27$

6) $9^{2-x} = 81^{6x}$

12) $2^{x^2} \cdot 2^{6x} = \frac{1}{32}$

B. Solve, and round to three decimal places when necessary:

1) $3^{4-x} - 7 = 74$

6) $2^{x+3} = 5$

2) $2^x = 11$

7) $4^{2x-1} = 5^{x+2}$

3) $2^x = 44$

8) $5^{2x+2} = 3^{5x-1}$

4) $2.8^x = 41$

9) $3^{x-1} = 4 \cdot 5^{1-3x}$

5) $1.7^x = 20$

10) $6^{x-1} = 3^{2x-3}$

C. Solve:

1) $\log(x-1) = 2$

3) $\log(x-9) + \log x = 1$

2) $\log x + \log(x+9) = 1$

4) $\log(x+3) - \log x = 1$

5) $\log(x + 4) - \log x = 2$

11) $\log_3 \sqrt{x} = 1$

6) $\log(3x^2 + 2x - 4) = 0$

12) $4 \log \sqrt{x} - 5 \sqrt{\log x} - 3 = 0$

7) $\log(x + 3) = \log x + \log 3$

13) $\log(x^2 - 1) = 1 + \log(x - 1)$

8) $-\log(x - 2) = 1 - \log(x^2 - 4)$

14) $\log_2(2 - 2x) + \log_2(1 - x) = 5$

9) $\log_4 x = -\frac{3}{2}$

15) $\log_x \frac{9}{4} = -\frac{2}{3}$

10) $(\log x)^2 - \log x^2 + 1 = 0$

16) $\log_{x-1}(4x - 4) = 2$

SOLUTIONS

A. (1) $x = 4$ (2) $x = \frac{7}{2}$ (3) $x = 0$ (4) $x = 2$ (5) $x = -\frac{3}{2}$ (6) $x = \frac{2}{13}$ (7) $x = -3$

(8) $x = \frac{5}{2}$ (9) $x = -1$ (10) $x = 3$ (11) $x = \frac{1}{2}$ or -3 (12) $x = -1$ or -5

B. (1) $x = 0$ (2) $x \approx 3.459$ (3) $x \approx 5.459$ (4) $x \approx 3.607$ (5) $x \approx 5.646$ (6) $x \approx -0.678$

(7) $x = 2 \div \log \frac{16}{5} \approx 3.959$ (8) $x = -\log 75 \div \log \frac{25}{243} \approx 1.898$

(9) $x = \log 60 \div \log 375 \approx 0.691$ (10) $x = \log \frac{2}{9} \div \log \frac{2}{3} \approx 3.710$

C. (1) $x = 101$ (2) $x = 1$ (not -10) (3) $x = 10$ (not -1) (4) $x = \frac{1}{3}$ (5) $x = \frac{4}{99}$

(6) $x = -\frac{5}{3}$ or 1 (7) $x = \frac{3}{2}$ (8) $x = 8$ (9) $x = \frac{1}{8}$ (10) $x = 10$ (11) $x = 9$

(12) $x = 10^9$ (13) $x = 8$ (14) $x = -3$ (15) $x = \frac{8}{27}$ (16) $x = 5$