Worksheet 2.7 Logarithms and Exponentials

Section 1 Logarithms

The mathematics of logarithms and exponentials occurs naturally in many branches of science. It is very important in solving problems related to growth and decay. The growth and decay may be that of a plant or a population, a crystalline structure or money in the bank. Therefore we need to have some understanding of the way in which logs and exponentials work.

Definition: If x and b are positive numbers and $b \neq 1$ then the logarithm of x to the base b is the power to which b must be raised to equal x. It is written $\log_b x$. In algebraic terms this means that

$$\boxed{\begin{aligned} \text{if } y &= \log_b x \quad \text{then} \\ x &= b^y \end{aligned}}$$

The formula $y = \log_b x$ is said to be written in logarithmic form and $x = b^y$ is said to be written in exponential form. In working with these problems it is most important to remember that $y = \log_b x$ and $x = b^y$ are equivalent statements.

Example 1 : If $\log_4 x = 2$ then

$$\begin{aligned} x &= 4^2 \\ x &= 16 \end{aligned}$$

Example 2 : We have $25 = 5^2$. Then $\log_5 25 = 2$.

Example 3 : If $\log_9 x = \frac{1}{2}$ then

$$\begin{aligned} x &= 9^{\frac{1}{2}} \\ x &= \sqrt{9} \\ x &= 3 \end{aligned}$$

Example 4 : If $\log_2 \frac{y}{3} = 4$ then

$$\begin{aligned} \frac{y}{3} &= 2^4 \\ \frac{y}{3} &= 16 \\ y &= 16 \times 3 \\ y &= 48 \end{aligned}$$

Exercises:

1. Write the following in exponential form:
   (a) $\log_3 x = 9$
   (b) $\log_2 8 = x$
   (c) $\log_3 27 = x$
   (d) $\log_4 x = 3$
   (e) $\log_2 y = 5$
   (f) $\log_5 y = 2$
2. Write the following in logarithm form:
   (a) $y = 3^4$
   (b) $27 = 3^x$
   (c) $m = 4^2$
   (d) $y = 3^5$
   (e) $32 = x^5$
   (f) $64 = 4^x$
3. Solve the following:
   (a) $\log_3 x = 4$
   (b) $\log_m 81 = 4$
   (c) $\log_x 1000 = 3$
   (d) $\log_2 \frac{x}{2} = 5$
   (e) $\log_3 y = 5$
   (f) $\log_2 4x = 5$

Section 2 Properties of Logs

Logs have some very useful properties which follow from their definition and the equivalence of the logarithmic form and exponential form. Some useful properties are as follows:

$$\begin{aligned} \log_b mn &= \log_b m + \log_b n \\ \log_b \frac{m}{n} &= \log_b m - \log_b n \\ \log_b m^a &= a \log_b m \\ \log_b m &= \log_b n \quad \text{if and only if} \quad m = n \end{aligned}$$

Note that for all of the above properties we require that $b > 0$, $b \neq 1$, and $m, n > 0$. Note also that $\log_b 1 = 0$ for any $b \neq 0$ since $b^0 = 1$. In addition, $\log_b b = 1$ since $b^1 = b$. We can apply these properties to simplify logarithmic expressions.

Example 1 :

$$\begin{aligned} \log_b \frac{xy}{z} &= \log_b xy - \log_b z \\ &= \log_b x + \log_b y - \log_b z \end{aligned}$$

Example 2 :

$$\begin{aligned} \log_5 5^p &= p \log_5 5 \\ &= p \times 1 \\ &= p \end{aligned}$$

Example 3 :

$$\begin{aligned} \log_2 (8x)^{\frac{1}{3}} &= \frac{1}{3} \log_2 8x \\ &= \frac{1}{3} [\log_2 8 + \log_2 x] \\ &= \frac{1}{3} [3 + \log_2 x] \\ &= 1 + \frac{1}{3} \log_2 x \end{aligned}$$

Example 4 : Find x if

$$2 \log_b 5 + \frac{1}{2} \log_b 9 - \log_b 3 = \log_b x$$

$$\begin{aligned} \log_b 5^2 + \log_b 9^{\frac{1}{2}} - log_b 3 &= \log_b x \\ \log_b 25 + \log_b 3 - \log_b 3 &= \log_b x \\ \log_b 25 &= \log_b x \\ x &= 25 \end{aligned}$$

Example 5 :

$$\begin{aligned}
\log_2 \frac{8x^3}{2y} &= \log_2 8x^3 - \log_2 2y \\
&= \log_2 8 + \log_2 x^3 - [\log_2 2 + \log_2 y] \\
&= 3 + 3\log_2 x - [1 + \log_2 y] \\
&= 3 + 3\log_2 x - 1 - \log_2 y \\
&= 2 + 3\log_2 x - \log_2 y
\end{aligned}$$

Exercises:

1. Use the logarithm laws to simplify the following:
   (a) $\log_2 xy - \log_2 x^2$
   (b) $\log_2 \frac{8x^2}{y} + \log_2 2xy$
   (c) $\log_3 9xy^2 - \log_3 27xy$
   (d) $\log_4 (xy)^3 - \log_4 xy$
   (e) $\log_3 9x^4 - \log_3 (3x)^2$
2. Find x if:
   (a) $2\log_b 4 + \log_b 5 - \log_b 10 = \log_b x$
   (b) $\log_b 30 - \log_b 5^2 = \log_b x$
   (c) $\log_b 8 + \log_b x^2 = \log_b x$
   (d) $\log_b (x+2) - \log_b 4 = \log_b 3x$
   (e) $\log_b (x-1) + \log_b 3 = \log_b x$

Section 3 The Natural Logarithm and Exponential

The natural logarithm is often written as ln which you may have noticed on your calculator.

$$\ln x = \log_e x$$

The symbol e symbolizes a special mathematical constant. It has importance in growth and decay problems. The logarithmic properties listed above hold for all bases of logs. If you see $\log x$ written (with no base), the natural log is implied. The number e can not be written exactly in decimal form, but it is approximately 2.718. Of course, all the properties of logs that we have written down also apply to the natural log. In particular,

$$e^y = x \quad \text{and} \quad \ln x = y$$

are equivalent statements. We also have $e^0 = 1$ and $\ln 1 = 0$.

Example 1 : $e^{\log_e a} = a$

Example 2 : $e^{a \log_e x} = e^{\log_e x^a} = x^a$

Example 3 :

$$\begin{aligned} \log_e e^{2y} &= 2y \log_e e \\ &= 2y \end{aligned}$$

Example 4 : $\log_e \frac{x^2}{5} = 2 \log_e x - \log_e 5$

Exercises:

1. Use your calculator to find the following:
   (a) $\ln 1.4$
   (b) $\ln 0.872$
   (c) $\ln \frac{6.4 \times 3.8}{10}$
   (d) $e^{0.62}$
   (e) $e^{3.8}$
   (f) $(e^{0.24})^2$
   (g) $e^{1.4} \times e^{0.8}$
   (h) $6e^{-4.1}$
   (i) $\frac{e^{8.2}}{1068}$
   (j) $e^{-2.4} \times e^{6.1} \div (8 + \ln 2)$

2. Simplify the following
   (a) $\log x^2 - \log xy + 4 \log y$
   (b) $\ln(8x)^{\frac{1}{2}} + \ln 4x^2 - \ln(16x)^{\frac{1}{2}}$
   (c) $e^6 e^{-6}$
   (d) $12e^7 \div 6e^2$
   (e) $\ln e^2$
   (f) $\ln(e^2 \ln e^3)$

3. Find x in each of the following:

   (a) $\ln x = 2.7$

   (b) $\ln(x + 1) = 1.86$

   (c) $x = e^{9.8} \div e^{7.6}$

   (d) $6.27 = e^{x}$

   (e) $4.12 = e^{-2x}$

Exercises 2.7 Logarithms and Exponentials

1. Evaluate
   (a) $\log_{10} 1000$
   (b) $\log_4 1$
   (c) $\log_3 27$
   (d) $\log_2 \frac{1}{4}$
   (e) $\log_a a^x$

2. Solve for x
   (a) $\log_4 x = 2$
   (b) $\log_{\frac{1}{3}} x = 4$
   (c) $\log_{10}(2x + 1) = 2$
   (d) $\log_2 64 = x$
   (e) $\log_b 81 = 4$

3. (a) Use log laws to solve $\log_3 x = \log_3 7 + \log_3 3$.
   (b) Without tables, simplify $2\log_{10} 5 + \log_{10} 8 - \log_{10} 2$.
   (c) If $\log_{10} 8 = x$ and $\log_{10} 3 = y$, express the following in terms of x and y only:

   i. $\log_{10} 24$ ii. $\log_{10} \frac{9}{8}$ iii. $\log_{10} 720$

4. (a) The streptococci bacteria population N at time t (in months) is given by $N = N_0 e^{2t}$ where N_0 is the initial population. If the initial population was 100, how long does it take for the population to reach one million?
   (b) The formula for the amount of energy E (in joules) released by an earthquake is

$$E = 1.74 \times 10^{19} \times 10^{1.44M}$$

   where M is the magnitude of the earthquake on the Richter scale.

   i. The Newcastle earthquake in 1989 had a magnitude of 5 on the Richter scale. How many joules were released?
   ii. In an earthquake in San Francisco in the 1900's the amount of energy released was double that of the Newcastle earthquake. What was its Richter magnitude?

Answers 2.7

Section 1

1. (a) $x = 3^9$ (b) $8 = 2^x$ (c) $27 = 3^x$ (d) $x = 4^3$ (e) $y = 2^5$ (f) $y = 5^2$

2. (a) $4 = \log_3 y$ (b) $x = \log_3 27$ (c) $2 = \log_4 m$ (d) $5 = \log_3 y$ (e) $5 = \log_x 32$ (f) $x = \log_4 64$

3. (a) 81 (b) 3 (c) 10 (d) 64 (e) 243 (f) 8

Section 2

1. (a) $\log_2 \frac{y}{x}$ (b) $4 + 3\log_2 x$ (c) $\log_3 y - 1$ (d) $2\log_4(xy)$ (e) 0

2. (a) 8 (b) $\frac{6}{5}$ (c) $\frac{1}{8}$ (d) $\frac{2}{11}$ (e) $1\frac{1}{2}$

Section 3

1. (a) 0.34 (b) -0.14 (c) 0.89 (d) 1.86 (e) 44.70 (f) 1.62 (g) 9.03 (h) 0.10 (i) 3.41 (j) 4.65

2. (a) $\ln xy^3$ (b) $\frac{1}{2}\ln 8 + 2\ln x$ (c) 1 (d) $2e^5$ (e) 2 (f) $2 + \ln 3$

3. (a) 14.88 (b) 5.42 (c) 9.03 (d) 1.84 (e) -0.71

Exercises 2.7

1. (a) 3 (b) 0 (c) 3 (d) -2 (e) x

2. (a) 16 (b) $\frac{1}{81}$ (c) 49.5 or $\frac{99}{2}$ (d) 6 (e) 3

3. (a) 21
   (b) 2
   (c) i. $x+y$ ii. $2y-x$ iii. $2y+x+1$

4. (a) 4.6054 months
   (b) i. 2.76×10^{26} Joules ii. 5.2 on the Richter scale.