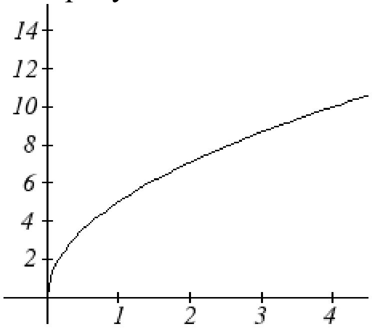
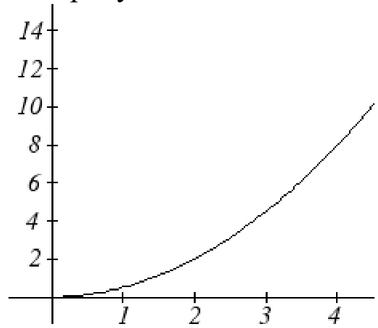
Concavity

The total sales, in thousands of dollars, for two companies over 4 weeks are shown.

Company A



Company B



As you can see, the sales for each company are increasing, but they are increasing in very different ways. To describe the difference in behavior, we can investigate how the average rate of change varies over different intervals. Using tables of values,

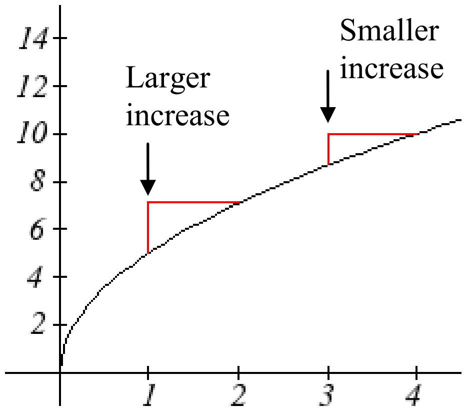
Company A

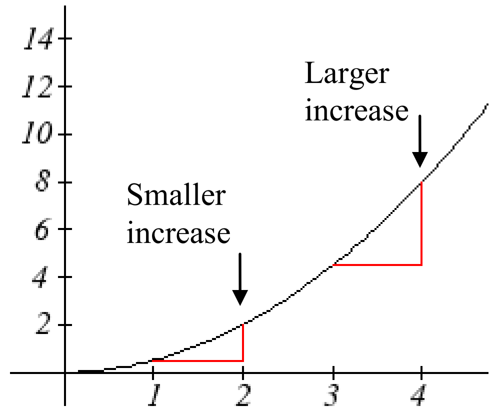
Week	Sales	Rate of Change
0	0	
		5
1	5	
		2.1
2	7.1	
		1.6
3	8.7	
		1.3
4	10	

Company B

Week	Sales	Rate of Change
0	0	
		0.5
1	0.5	
		1.5
2	2	
		2.5
3	4.5	
		3.5
4	8	

From the tables, we can see that the rate of change for company A is *decreasing*, while the rate of change for company B is *increasing*.





When the rate of change is getting smaller, as with Company A, we say the function is **concave down**. When the rate of change is getting larger, as with Company B, we say the function is **concave up**.

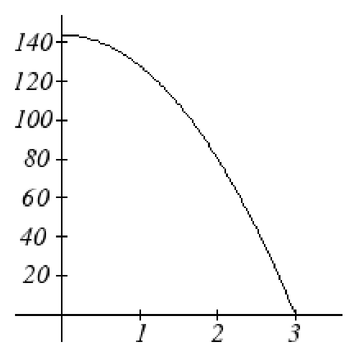
Concavity

A function is **concave up** if the rate of change is increasing.
A function is **concave down** if the rate of change is decreasing.
A point where a function changes from concave up to concave down or vice versa is called an **inflection point**.

Example 10

An object is thrown from the top of a building. The object's height in feet above ground after t seconds is given by the function $h(t) = 144 - 16t^2$ for $0 \le t \le 3$. Describe the concavity of the graph.

Sketching a graph of the function, we can see that the function is decreasing. We can calculate some rates of change to explore the behavior



t	*h(t)*	Rate of Change
0	144	
		-16
1	128	
		-48
2	80	
		-80
3	0	

Notice that the rates of change are becoming more negative, so the rates of change are *decreasing*. This means the function is concave down.

Example 11

The value, V, of a car after t years is given in the table below. Is the value increasing or decreasing? Is the function concave up or concave down?

t	0	2	4	6	8
V(t)	28000	24342	21162	18397	15994

Since the values are getting smaller, we can determine that the value is decreasing. We can compute rates of change to determine concavity.

t	0	2	4	6	8
V(t)	28000	24342	21162	18397	15994
Rate of change		-1829	-1590	-1382.5	-1201.5

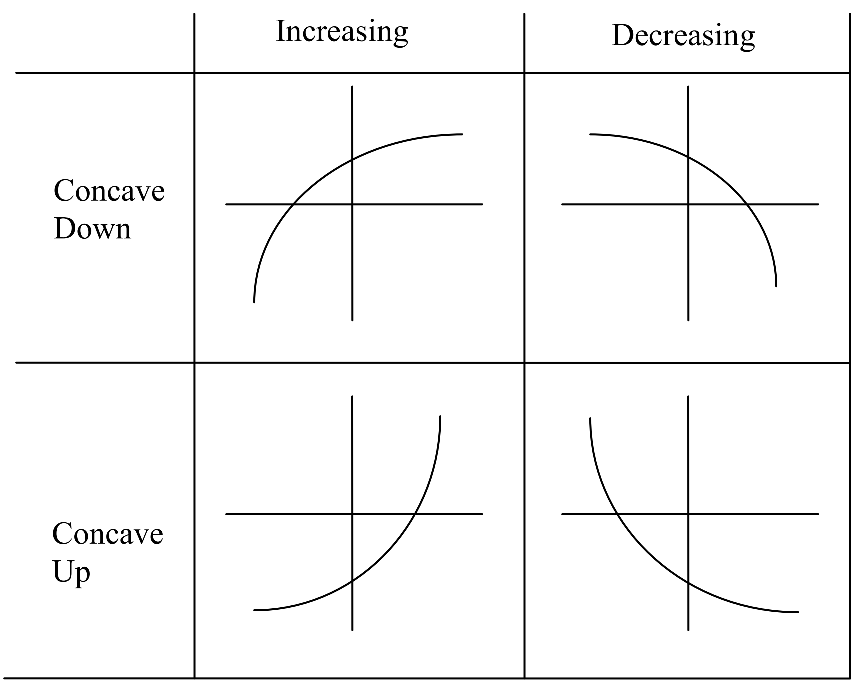
Since these values are becoming less negative, the rates of change are *increasing*, so this function is concave up.

Try it Now

5. Is the function described in the table below concave up or concave down?

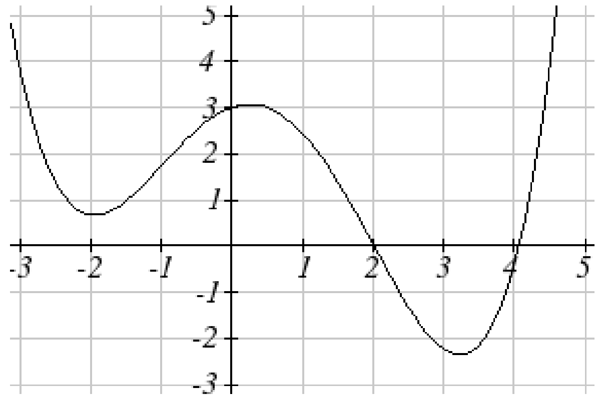
x	0	5	10	15	20
g(x)	10000	9000	7000	4000	0

Graphically, concave down functions bend downwards like a frown, and concave up function bend upwards like a smile.



Example 12

Estimate from the graph shown the intervals on which the function is concave down and concave up.



On the far left, the graph is decreasing but concave up, since it is bending upwards. It begins increasing at x = -2, but it continues to bend upwards until about x = -1.

From x = -1 the graph starts to bend downward, and continues to do so until about x = 2. The graph then begins curving upwards for the remainder of the graph shown.

From this, we can estimate that the graph is concave up on the intervals $(-\infty,-1)$ and $(2,\infty)$, and is concave down on the interval $(-1,2)$. The graph has inflection points at x = -1 and x = 2.

Try it Now

6. Using the graph from Try it Now 4, $f(x)=x^3-6x^2-15x+20$, estimate the intervals on which the function is concave up and concave down.

Behaviors of the Toolkit Functions

We will now return to our toolkit functions and discuss their graphical behavior.

Function	Increasing/Decreasing	Concavity
Constant Function $f(x)=c$	Neither increasing nor decreasing	Neither concave up nor down
Identity Function $f(x)=x$	Increasing	Neither concave up nor down
Quadratic Function $f(x)=x^2$	Increasing on $(0,\infty)$ Decreasing on $(-\infty,0)$ Minimum at $x = 0$	Concave up $(-\infty,\infty)$
Cubic Function $f(x)=x^3$	Increasing	Concave down on $(-\infty,0)$ Concave up on $(0,\infty)$ Inflection point at (0,0)
Reciprocal $f(x)=\frac{1}{x}$	Decreasing $(-\infty,0)\cup(0,\infty)$	Concave down on $(-\infty,0)$ Concave up on $(0,\infty)$

Function	Increasing/Decreasing	Concavity
Reciprocal squared $f(x)=\frac{1}{x^2}$	Increasing on $(-\infty,0)$ Decreasing on $(0,\infty)$	Concave up on $(-\infty,0)\cup(0,\infty)$
Cube Root $f(x)=\sqrt[3]{x}$	Increasing	Concave down on $(0,\infty)$ Concave up on $(-\infty,0)$ Inflection point at (0,0)
Square Root $f(x)=\sqrt{x}$	Increasing on $(0,\infty)$	Concave down on $(0,\infty)$
Absolute Value $f(x)=\lvert x\rvert$	Increasing on $(0,\infty)$ Decreasing on $(-\infty,0)$	Neither concave up or down